$$\xrightarrow[\to\infty]{d,e,N} \sum_{n=0}^{\infty} \frac{\left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^2 q^n}{(zq)_n (z^{-1}q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}$$

13/10/21

MA 633 - Partition Theory - Lec. 26

$$\Rightarrow \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}$$

$$= \frac{1}{(q)_\infty}\left\{1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} (1+q^n)\left[1 - \frac{(1-q^n)}{1+q^n}\left\{\frac{1}{1-zq^n} + \frac{z^{-1}q^n}{1-z^{-1}q^n}\right\}\right]\right\}$$

$$= \frac{1}{(q)_\infty}\left\{1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} (1+q^n) - \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} \frac{(1-q^n)}{1-zq^n} - \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} \frac{(1-q^n) z^{-1} q^n}{1-z^{-1}q^n}\right\}$$

Note that

$$1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} (1+q^n)$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}}$$

$$(n \longrightarrow -n)$$

$$= 1 + \text{—— " ——} \sum_{n=-\infty}^{-1} (-1)^n q^{\frac{n(3n-1)}{2}}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty$$

(Euler's pent. no. thm.)

$$= \frac{1}{(q)_\infty} \Big\{ (q)_\infty - \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} (1-q^n) \sum_{m=0}^{\infty} z^m q^{mn} \qquad \text{(A)}$$

$$- \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} (1-q^n) \sum_{m=1}^{\infty} z^{-m} q^{mn} \Big\}$$

But $$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N(m,n) z^m q^n. \qquad \text{(B)}$$

Now compare the coeff. of z^m, $m > 0$, on both sides to complete the proof. $\blacksquare$

$$\sum_{n=0}^{\infty} N(m,n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n(3n-1)}{2} + mn} (1-q^n)$$

Thm. 4.3 (Garvan) For $|q|<1$, $|z|<1$ & $|q|<|z|<|q|^{-1}$, we have

$$-1+\frac{1}{1-z}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}=\frac{z}{(q)_\infty}\underbrace{\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1-zq^n}}_{\text{Level-3 Appell-Lerch sum}}$$

Proof:

From previous theorem, we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}$$

$$=\frac{1}{(q)_\infty}\left\{1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}}(1+q^n)\left[1-\frac{(1-q^n)}{1+q^n}\left\{\frac{1}{1-zq^n}+\frac{z^{-1}q^n}{1-z^{-1}q^n}\right\}\right]\right\}$$

$$=1+\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{\frac{n(3n-1)}{2}}(1-q^n)\left\{\frac{1}{1-zq^n}+\frac{z^{-1}q^n}{1-z^{-1}q^n}\right\}$$

$$=1+\frac{z^{-1}}{(q)_\infty}\sum_{n=-\infty}^{\infty}{}'(-1)^{n-1}q^{\frac{n(3n+1)}{2}}\frac{1-q^n}{1-z^{-1}q^n}\quad (*)$$

($\sum'$ implies the $n=0$ term is omitted.)

since $\displaystyle\sum_{n=-\infty}^{-1}(-1)^{n-1}q^{\frac{n(3n+1)}{2}}\frac{(1-q^n)}{1-z^{-1}q^n}$

$n\to -n$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}q^{\frac{n(3n-1)}{2}}\frac{(1-q^{-n})}{1-z^{-1}q^{-n}}$$

But $\dfrac{1-q^{-n}}{1-z^{-1}q^{-n}} = \dfrac{1-\dfrac{1}{q^n}}{1-\dfrac{z^{-1}}{q^n}} = \dfrac{q^n-1}{q^n-z^{-1}}$

$$= \frac{1-q^n}{z^{-1}(1-zq^n)}$$

Hence. $\dfrac{z^{-1}}{(q)_\infty}\displaystyle\sum_{n=-\infty}^{-1}(-1)^{n-1}q^{\frac{n(3n+1)}{2}}\dfrac{1-q^n}{1-z^{-1}q^n}$

$$= \frac{\cancel{z^{-1}}}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{\frac{n(3n-1)}{2}}\frac{1-q^n}{\cancel{z^{-1}}(1-zq^n)}$$

$$= \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{\frac{n(3n-1)}{2}}\frac{(1-q^n)}{1-zq^n}.$$

Replacing z by z^{-1} in (*), we get

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}$$

$$= 1+\frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}{}'(-1)^{n-1}q^{\frac{n(3n+1)}{2}}\frac{1-q^n}{1-zq^n}$$

$$= 1+\frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}{}'\frac{(-1)^{n-1}q^{\frac{n(3n+1)}{2}}}{1-zq^n}+\frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}{}'(-1)^n\frac{q^{\frac{3n(n+1)}{2}}}{1-zq^n}$$

$$= 1 + \frac{z}{(q)_\infty}\left\{ \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^{n-1} q^{\frac{n(3n+1)}{2}}}{(1-zq^n)} + \overbrace{\sum_{n=-\infty}^{\infty}{}' (-1)^n q^{\frac{n(3n+1)}{2}} + 1 - (q)_\infty}^{(=0 \text{ by Euler's PNT})} + \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1-zq^n} \right\}$$

$$= 1 + \frac{z}{(q)_\infty}\left\{ 1 + \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1-zq^n} - \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1-zq^n}\left(1-(1-zq^n)\right) - (q)_\infty \right\}$$

$$= 1 + \frac{z}{(q)_\infty}\left\{ 1 + (1-z) \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{(1-zq^n)} - (q)_\infty \right\}$$

$$= 1 - z + \frac{z}{(q)_\infty}\left(1 + (1-z) \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1-zq^n} \right)$$

Dividing both sides by $1-z$, we get

$$\frac{1}{1-z}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}$$

$$= 1+\frac{z}{(q)_\infty}\left(\frac{1}{1-z}+\sum_{n=-\infty}^{\infty}{}'\frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1-zq^n}\right)$$

$$\Rightarrow -1+\frac{1}{1-z}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}$$

$$= \frac{1}{(q)_\infty}\left\{\frac{z}{1-z}+z\sum_{n=-\infty}^{\infty}{}'\frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1-zq^n}\right\}$$

$$= \frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1-zq^n}$$

$\blacksquare$