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MA 633 - Partition Theory - Lec. 27

The smallest parts partition function $(spt(n))$

— introduced by George Andrews in 2008

$spt(n)$ = the number of smallest parts in all partitions of n.

Eg. Consider the 5 partitions of 4,

4
$3+1$
$2+2$
$2+1+1$
$1+1+1+1$

$\Rightarrow spt(4) = 10.$

Andrews showed that

$$spt(5n+4) \equiv 0 \pmod 5$$
$$spt(7n+5) \equiv 0 \pmod 7$$
$$spt(13n+6) \equiv 0 \pmod{13}$$

key identity for proving these congruences is:

$$\boxed{spt(n) = np(n) - \frac{1}{2} N_2(n).}$$

$N_2(n)$ is the second Atkin-Garvan rank moment defined by

$$N_2(n) = \sum_{m=-\infty}^{\infty} m^2 N(m,n).$$

More generally, $N_j(n) = \sum_{m=-\infty}^{\infty} m^j N(m,n)$.

Later on, Dyson showed

$$n\,p(n) = \frac{1}{2} M_2(n),$$

where $M_j(n) = \sum_{m=-\infty}^{\infty} m^j M(m,n)$

$$\Longrightarrow \quad \operatorname{spt}(n) = \frac{1}{2}\left(M_2(n) - N_2(n)\right)$$

Since $\operatorname{spt}(n) > 0$, we have $M_2(n) > N_2(n)$.

The generating function for $\operatorname{spt}(n)$:

$$\sum_{n=1}^{\infty} \operatorname{spt}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \cdot \frac{1}{(q^{n+1};q)_\infty}$$

smallest part

because

For $|x|<1$,

$$\frac{x}{(1-x)^2} = x\frac{d}{dx}\frac{1}{(1-x)} = x\frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$= x\sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} n x^n = \sum_{n=1}^{\infty} n x^n.$$

Hence

$$\frac{q^n}{(1-q^n)^2} = \sum_{m=1}^{\infty} m q^{mn}$$

number of times the smallest part appears

smallest part

Hence $\displaystyle spt(n) = \sum_{\pi \vdash n} \omega(\pi)$,

where $\omega(\pi)$ = the number of appearances of the smallest part in π.

So $$\sum_{n=1}^{\infty} spt(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \cdot \frac{(q)_n}{(q)_n (q^{n+1})_\infty}$$

$$= \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(q)_{n-1}\, q^n}{1-q^n}.$$

Deriving Andrews' identity:

We will prove it using generating functions.

Fact: If f is atleast twice differentiable at $z=1$, then

$$\frac{d^2}{dz^2}\left[(1-z)(1-z^{-1})f(z)\right]\bigg|_{z=1} = -2f(1).$$

Using this, we can see that

$$\sum_{n=1}^{\infty} \mathrm{spt}(n)q^n = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(q)_{n-1}q^n}{1-q^n}$$

$$= \frac{-1}{2(q)_\infty}\frac{d^2}{dz^2}\sum_{n=0}^{\infty}\frac{(z)_n(z^{-1})_n q^n}{(q)_n}\bigg|_{z=1}$$

The ${}_8\phi_7$ transformation of Watson is given by

$${}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-N} \\ & \sqrt{a}, & -\sqrt{a}, & \frac{aq}{b}, & \frac{aq}{c}, & \frac{aq}{d}, & \frac{aq}{e}, & aq^{N+1} \end{matrix}; \frac{a^2q^{N+2}}{bcde}\right]$$

$$= \frac{(aq)_N\left(\frac{aq}{de}\right)_N}{\left(\frac{aq}{d}\right)_N\left(\frac{aq}{e}\right)_N}\,{}_4\phi_3\left[\begin{matrix} \frac{aq}{bc}, & d, & e, & q^{-N} \\ \frac{deq^{-N}}{a}, & \frac{aq}{b}, & \frac{aq}{c} \end{matrix}; q\right]$$

Let $d = z$, $e = z^{-1}$ & then let $a = 1$, and then let $b, c, N \to \infty$. This gives

$$\sum_{n=0}^{\infty} \frac{(z)_n (z^{-1})_n q^n}{(q)_n}$$

$$= \frac{(zq)_\infty (z^{-1}q)_\infty}{(q)_\infty^2} \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} (1+q^n) \times \frac{(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} \right\}$$