20/10/21

MA 633 – Partition Theory – Lec. 28

Fact: If f is atleast twice differentiable at $z=1$, then

$$\frac{d^2}{dz^2}\left[(1-z)(1-z^{-1})f(z)\right]\Big|_{z=1} = -2f(1).$$

Using this, we can see that

$$\sum_{n=1}^{\infty} \operatorname{spt}(n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(q)_{n-1}\, q^n}{1-q^n}$$

$$= \frac{-1}{2(q)_\infty} \frac{d^2}{dz^2} \sum_{n=0}^{\infty} \frac{(z)_n (z^{-1})_n\, q^n}{(q)_n}\Bigg|_{z=1} \longrightarrow A$$

The ${}_8\phi_7$ transformation of Watson is given by

$${}_8\phi_7\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ q^{-N} \\ \sqrt{a},\ -\sqrt{a},\ \frac{aq}{b},\ \frac{aq}{c},\ \frac{aq}{d},\ \frac{aq}{e},\ aq^{N+1} \end{matrix};\ \frac{a^2q^{N+2}}{bcde}\right]$$

$$= \frac{(aq)_N \left(\frac{aq}{de}\right)_N}{\left(\frac{aq}{d}\right)_N \left(\frac{aq}{e}\right)_N}\ {}_4\phi_3\left[\begin{matrix} \frac{aq}{bc},\ d,\ e,\ q^{-N} \\ \frac{deq^{-N}}{a},\ \frac{aq}{b},\ \frac{aq}{c} \end{matrix};\ q\right]$$

Let $d = z$, $e = z^{-1}$ & then let $a = 1$, and then let $b, c, N \to \infty$. This gives

$$\sum_{n=0}^{\infty} \frac{(z)_n (z^{-1})_n q^n}{(q)_n}$$

$$= \frac{(zq)_\infty (z^{-1}q)_\infty}{(q)_\infty^2} \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} (1+q^n) \times \frac{(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} \right\} \quad \text{B}$$

Substituting B in A, we get

$$\sum_{n=1}^{\infty} spt(n) q^n$$

$$= \frac{-1}{2(q)_\infty} \frac{d^2}{dz^2} \frac{(zq)_\infty (z^{-1}q)_\infty}{(q)_\infty^2} \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} (1+q^n) \times \frac{(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} \right\} \Bigg|_{z=1}$$

Now use Leibnitz's rule and note that

$$\frac{d}{dz}\left(\frac{(zq)_\infty (z^{-1}q)_\infty}{(q)_\infty^2}\right) \frac{d}{dz}\left\{ 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} (1+q^n) \times \frac{(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} \right\} \Bigg|_{z=1} = 0.$$

Tut. prob.

Hence $\displaystyle\sum_{n=1}^{\infty} spt(n) q^n$

$$= \frac{-1}{2(q)_\infty^3} \frac{d^2}{dz^2} (zq)_\infty (z^{-1}q)_\infty \Big|_{z=1}$$

$$\frac{-1}{2(q)_\infty} \frac{d^2}{dz^2}\left[1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} (1+q^n)\left\{\frac{(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)}\right\}\right]_{z=1}$$

$$= \frac{-1}{2(q)_\infty^3} \frac{d^2}{dz^2} (zq)_\infty (z^{-1}q)_\infty \Big|_{z=1}$$

$$-\frac{1}{2(q)_\infty} \cdot (-2) \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} \frac{(1+q^n)}{(1-q^n)^2}$$

$$= \frac{-1}{2(q)_\infty^3} \frac{d^2}{dz^2} (zq)_\infty (z^{-1}q)_\infty \Big|_{z=1}$$

$$+ \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} \frac{(1+q^n)}{(1-q^n)^2}.$$

Next, we show

Claim 1: $\displaystyle \frac{-1}{2} \frac{d^2}{dz^2} (zq)_\infty (z^{-1}q)_\infty \Big|_{z=1} = (q)_\infty^2 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.$

Proof: Jacobi triple product identity:

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty,$$

where $\displaystyle f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab|<1.$

$$= \frac{-1}{2(q)_\infty} \frac{d^2}{dz^2}\left[\sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}} \sum_{j=0}^{2n} z^{-n+j}\right]_{z=1}$$

$$= \frac{-1}{2(q)_\infty} \sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}} \sum_{j=0}^{2n}(-n+j)(-n+j-1)$$

Now $\sum_{j=0}^{2n}(-n+j)(-n+j-1)$

$$= \sum_{j=0}^{2n}\left(n^2 - nj + n - nj + j^2 - j\right)$$

$$= \sum_{j=0}^{2n}\left(n^2 + n + j^2 - (2n+1)j\right)$$

$$= (n^2+n)\sum_{j=0}^{2n}(1) + \sum_{j=0}^{2n} j^2 - (2n+1)\sum_{j=0}^{2n}(j)$$

$$= (n^2+n)(2n+1) + \frac{(2n)(2n+1)(4n+1)}{6} - \frac{(2n+1)(2n)(2n+1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{3}.$$

Hence $-\frac{1}{2}\frac{d^2}{dz^2}(zq)_\infty(z^{-1}q)_\infty\Big|_{z=1}$

$$= \frac{-1}{2(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}}\frac{n(n+1)(2n+1)}{3}$$

$$= -\frac{q}{3(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}-1}\left(\frac{n(n+1)}{2}\right)\cdot(2n+1)$$

$$= -\frac{q}{3(q)_\infty}\frac{d}{dq}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}$$

Jacobi's idty.

$$-\frac{q}{3(q)_\infty}\frac{d}{dq}(q)_\infty^3 .$$