22/10/21

MA 633 - Partition Theory - Lec. 29

We have shown that

$$-\frac{1}{2}\frac{d^2}{dz^2}(zq)_\infty(z^{-1}q)_\infty\Big|_{z=1} = \frac{-q}{3(q)_\infty}\frac{d}{dq}(q)_\infty^3$$

Using the fact that

$f'(x) = f(x)\frac{d}{dx}(\log f(x))$, we see that

$$\frac{d}{dq}(q)_\infty^3 = (q)_\infty^3 \frac{d}{dq}\log(q)_\infty^3$$

$$= 3(q)_\infty^3 \frac{d}{dq}\log(q)_\infty$$

$$= 3(q)_\infty^3 \frac{d}{dq}\log\prod_{n=1}^{\infty}(1-q^n)$$

$$= 3(q)_\infty^3 \frac{d}{dq}\sum_{n=1}^{\infty}\log(1-q^n)$$

$$= 3(q)_\infty^3 \sum_{n=1}^{\infty}\frac{-nq^{n-1}}{1-q^n}.$$

Therefore,

$$\frac{-q}{3(q)_\infty}\frac{d}{dq}(q)_\infty^3 = (q)_\infty^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

This proves claim 1.

Claim 2: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2} N_2(n) q^n$

$$= \frac{-1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}} (1+q^n)}{(1-q^n)^2}.$$

Proof: LHS $= \displaystyle\sum_{n=0}^{\infty} \frac{1}{2} N_2(n) q^n$

$$= \sum_{n=0}^{\infty} \frac{N_2(n) - N_1(n)}{2} q^n.$$

because $N_1(n) = 0$.

Reason: $N_1(n) = \displaystyle\sum_{m=-\infty}^{\infty} m N(m,n)$

$$\overset{m \to -m}{=} \sum_{m=-\infty}^{\infty} -m N(-m,n)$$

$$= \sum_{m=-\infty}^{\infty} -m N(m,n) \quad \left(\because N(-m,n) = N(m,n) \right)$$

$$= -N_1(n).$$

$$\Rightarrow N_1(n) = 0.$$

$$\Rightarrow \text{LHS} = \sum_{n=0}^{\infty} \frac{N_2(n) - N_1(n)}{2} q^n$$

$$= \frac{1}{2} \frac{d^2}{dz^2} R(z;q)\Big|_{z=1}.$$

rank gen. fn.

Reason: $R(z,q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}$

$$= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) z^m q^n$$

Hence $\frac{d^2}{dz^2} R(z;q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} m(m-1) N(m,n) z^m q^n$

$\Rightarrow$

$$\frac{d^2}{dz^2} R(z;q)\Big|_{z=1} = \sum_{n=0}^{\infty} \left(N_2(n) - N_1(n)\right) q^n.$$

$$= \frac{1}{2(q)_\infty} \frac{d^2}{dz^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}} (1-z)}{1-zq^n}\Bigg|_{z=1} \quad \text{—} (\star)$$

Reason:

From Thm. 43, we have

$$-1 + \frac{1}{1-z} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n} = \frac{z}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1-zq^n}$$

RHS

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} \frac{zq^n}{1-zq^n}$$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} \frac{1-(1-zq^n)}{1-zq^n}$$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1-zq^n} - \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}}$$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1-zq^n} - 1 \quad (\text{by Euler's PNT})$$

Hence $\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} \frac{(1-z)}{(1-zq^n)}.$$

Hence LHS of claim 2 (from (☆))

$$= \frac{1}{2(q)_\infty} \frac{d^2}{dz^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}} (1-z)}{1-zq^n} \Bigg|_{z=1}$$

Note that

$$\frac{d^2}{dz^2} \frac{1-z}{1-zq^n}\Bigg|_{z=1} = \frac{-2q^n}{(1-q^n)^2}.$$

Hence

$$= \frac{-1}{(q)_\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}+n}}{(1-q^n)^2}$$

$$= \frac{-1}{(q)_\infty}\left\{ \sum_{n=-\infty}^{-1} \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{(1-q^n)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{(1-q^n)^2} \right\}$$

$\downarrow n \to -n$

$$= \frac{-1}{(q)_\infty}\left\{ \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n(n-1)}{2}+2n}}{(1-q^n)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{(1-q^n)^2} \right\}$$

$$= \frac{-1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{(1-q^n)^2}(1+q^n).$$

This proves claim 2.

Hence from $\star\star$, claim 1 & claim 2, we have

$$\sum_{n=0}^{\infty} spt(n)\, q^n$$

$$= \frac{-1}{2(q)_\infty^3} \frac{d^2}{dz^2} (zq)_\infty (z^{-1}q)_\infty \Big|_{z=1}$$

$$+ \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} \frac{(1+q^n)}{(1-q^n)^2}$$

$$= \frac{(q)_\infty^2}{(q)_\infty^3} \sum_{n=1}^{\infty} \frac{n q^n}{1-q^n} + \sum_{n=1}^{\infty} \frac{-1}{2} N_2(n)\, q^n$$

$$= \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{n q^n}{1-q^n} + \sum_{n=1}^{\infty} \frac{-1}{2} N_2(n)\, q^n .$$

$(\star\star\star)$

The only thing remaining to be proved is

$$\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} = \sum_{n=1}^{\infty} np(n)q^n . \quad -(\cancel{*}_4)$$

We now prove this. We know that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q)_\infty}$$

Differentiating both sides w.r.t. q, we have

$$\sum_{n=1}^{\infty} np(n)q^{n-1} = \frac{d}{dq}\frac{1}{(q)_\infty}$$

$$= \frac{1}{(q)_\infty}\frac{d}{dq}\log\left(\frac{1}{(q)_\infty}\right) \qquad \left[f'(x) = f(x)\frac{d}{dx}(\log f(x))\right]$$

$$= \frac{-1}{(q)_\infty}\frac{d}{dq}\log\prod_{n=1}^{\infty}(1-q^n)$$

$$= \frac{-1}{(q)_\infty}\frac{d}{dq}\sum_{n=1}^{\infty}\log(1-q^n)$$

$$= \frac{-1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{-nq^{n-1}}{1-q^n}$$

$$= \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^n} .$$

Multiply both sides by q to complete the proof.

From $(\star\star\star)$ & $(\star_4)$, we have

$$\sum_{n=1}^{\infty} spt(n)\, q^n = \sum_{n=1}^{\infty} n\,p(n)\, q^n + \sum_{n=1}^{\infty} -\frac{1}{2} N_2(n)\, q^n$$

$$= \sum_{n=1}^{\infty} \left(n\,p(n) - \frac{1}{2} N_2(n) \right) q^n .$$

This proves that

$$spt(n) = n\,p(n) - \frac{1}{2} N_2(n).$$

$\blacksquare$