23/10/21

MA-633 – Partition theory: - LEC-30

① spt(n) : it count's the number of appearances of the smallest part in each partition of n.

- $$\sum_{n=1}^{\infty} spt(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \times \frac{1}{(q^{n+1};q)_\infty}$$

② $$spt(n) = np(n) - \frac{1}{2} N_2(n). \quad \text{---} (*)$$

- $$1+\sum_{n=1}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}$$

- $N_2(n)$: Atkin-Garvan 2nd moment of Rank.

- $$\underbrace{N_j(n) = \sum_{m=-\infty}^{\infty} m^j N(m,n)}_{j^{th} \text{ moment of Rank.}}$$

③ $N(m,q,n)$:= # of partitions of 'n' with rank $\equiv m \ (q)$

$\equiv m \mod q$

$$\Rightarrow N(0,5,5n+4) = N(1,5,5n+4)$$

$= N(2, 5, 5n+4) = N(3, 5, 5n+4)$
$= N(4, 5, 5n+4)$. $\rightarrow$ R_1

$$\Rightarrow \boxed{N(m, \alpha, n) = \sum_{r=-\infty}^{\infty} N(m + r\alpha; n)}$$

- $N(m, n) = N(-m, n)$ — (1)
- $N(m, \alpha, n) = N(\alpha - m, \alpha, n)$.

$$\sum_{r=-\infty}^{\infty} N(m - \alpha + r\alpha, n)$$
$\downarrow$ (1)
$$\sum_{r=-\infty}^{\infty} N(\alpha - m - r\alpha, n)$$
$\downarrow$ $r \longrightarrow -r$
$$\sum_{-\infty}^{\infty} N(\alpha - m + r\alpha, n)$$
$$N(\alpha - m, \alpha, n)$$

Claim: (1) $spt(5n+4) \equiv 0 \pmod 5$
(2) $spt(7n+5) \equiv 0 \pmod 7$
(3) $spt(13n+6) \equiv 0 \pmod{13}$

proof of (1):

$$spt(n) = n\,p(n) - \frac{1}{2} N_2(n)$$

$$\cdot \quad spt(5n+4) = (5n+4)\cancelto{0}{p(5n+4)} - \frac{1}{2} N_2(5n+4)$$

$$spt(5n+4) \equiv -\frac{1}{2} N_2(5n+4) \pmod 5$$

$$= -\frac{1}{2} \sum_{m=-\infty}^{\infty} m^2 N(m, 5n+4)$$

$$= -\frac{1}{2}\Bigg(\sum_{m=-\infty}^{\infty} (5m)^2 N(5m, 5n+4)$$

$$+ \sum_{m=-\infty}^{\infty} (5m+1)^2 N(5m+1, 5n+4) + \sum_{m=-\infty}^{\infty} (5m+2)^2 N(5m+2, 5n+4)$$

$$+ \sum_{m=-\infty}^{\infty} (5m+3)^2 N(5m+3, 5n+4)$$

$$+ \sum_{m=-\infty}^{\infty} (5m+4)^2 N(5m+4, 5n+4)\Bigg)$$

$$\equiv -\frac{1}{2}\Big(N(1,5,5n+4) - N(2,5,5n+4)$$

$$- N(3,5,5n+4) + N(4,5,5n+4)\Big)$$

use R_1

$$\equiv 0 \pmod 5.$$

proof of ② $spt(7n+5) \equiv 0\ (7)$

$$spt(7n+5) = (7n+5)p(7n+5) - \frac{1}{2} N_2(7n+5)$$

$$\equiv -\frac{1}{2} N_2(7n+5) \pmod 7$$

$$= -\frac{1}{2}\Big(2N(1,7,7n+5) + 8N(2,7,7n+5) + 4N(3,7,7n+5)\Big)$$

$$= -\Big(N(1, \_\_) + 4N(2, \_\_) + 2N(3,7,7n+5)\Big)$$

$$\equiv 0 \pmod 7$$

Above we used the fact that

$$N(1,7,7n+5) = N(2, \_\_) \cdots = N(7,7,7n+5)$$

$$* \quad \sigma(q) := \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n}.$$

$$= \sum_{n=1}^{\infty} S(n) q^n$$

- $\limsup_{n\to\infty} |S(n)| = \infty$!

- $S(n) = 0$ for infinitely many 'n'.

[Partition's & indefinite quadratic forms]

(Ramanujan)

$$* \sum_{n=0}^{\infty} \left((-q;q)_\infty - (-q;q)_n \right) \quad (\text{S.O.T})$$

$$= (-q;q)_\infty \left(-\frac{1}{2} + \sum \frac{q^n}{1-q^n} \right) + \frac{1}{2}\sigma(q).$$

- Zagier obtained

$$\sum_{n=0}^{\infty} \left((q)_\infty - (q)_n \right) = \frac{1}{2} \sum n \chi(n) q^{\frac{(n^2-1)}{24}}$$

$$-(q)_\infty\left(\frac{1}{2}-\sum\frac{q^n}{1-q^n}\right)\ldots$$