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# MA 633 - Partition Theory - Lec. 31

Ramanujan's function:

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n}.$$

Let $\sigma(q) = \sum_{n=0}^{\infty} s(n) q^n$. Then Andrews showed that

(i) $\limsup_{n\to\infty} |s(n)| = \infty$.

(ii) $s(n) = 0$ for infinitely many $n$.

- These conjectures were proved by Andrews, Dyson & Hickerson (Inventiones Mathematicae).

- $\sigma(q)$ is a prototypical example of a quantum modular form.

- $\sigma(q) = 1 + q - q^2 + 2q^3 - 2q^4 + \dots + 4q^{45} + \dots + 8q^{3288}$

Combinatorics of $\sigma(q)$:

(*) $\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n}$ is the generating fn. of the number of partitions of $n$ into distinct parts with even rank minus those with odd rank.

Thm. 45 (Andrews) Let $r(m,n)$ be the number of partitions into distinct parts with rank $m$. Then

$$\sum_{m,n=0}^{\infty} r(m,n)\, t^m q^n = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(tq;q)_n}$$

$$= 1 + \sum_{n=1}^{\infty} t^{n-1} q^n (-t^{-1}q;q)_{n-1}.$$

Remark: Letting $t=-1$ in Thm. 45 proves (*).

Proof:
$$\frac{q^{\frac{n(n+1)}{2}}}{(tq;q)_n} = \frac{q^{1+2+3+\cdots+n}}{(1-tq)(1-tq^2)\cdots(1-tq^n)}$$

$$= \sum_{k_1,k_2,\ldots k_n=0}^{\infty} t^{k_1+k_2+\cdots+k_n} q^{1+2+3+\cdots+n+1\cdot k_1+2\cdot k_2+\cdots nk_n}$$

Take the following partition of the number $1+2+3+\cdots+n+1\cdot k_1+2\cdot k_2+\cdots+n\cdot k_n$

$(k_1+1)+(k_2+1)\cdots+(k_{n-1}+1)+(k_n+1)$
$(k_2+1)+(k_3+1)+\cdots\cdots+(k_n+1)$
$(k_3+1)+(k_4+1)+\cdots\cdots+(k_n+1)$
$\vdots$
$(k_n+1)$

Note that this is a partition into distinct parts.

Similarly, each partition into distinct parts can be uniquely represented in terms of $k$'s.

(1) Start with the smallest part in such a partition & figure out $k_n$.
(2) Then figure out $k_{n-1}$, & so on.

This establishes a 1-1 correspondence.

Note that rank of such a partition
$= (k_1+1)+(k_2+1)+\cdots+(k_n+1) - n$
$= k_1+k_2+\cdots+k_n$
$=$ power of $t$.

This proves the first part of the theorem.

Example: Consider $4+1$ as a partition of 5 into distinct parts.

Note that rank $= 2$, $n=2$.

We need to figure out only $k_1$ & $k_2$.

$k_2+1=1 \Rightarrow k_2=0$. Also, $(k_1+1)+(k_2+1)=4$
$\Rightarrow k_1+2=4 \Rightarrow k_1=2$.

Hence the expression in the power series corresponding to this partition is

$$q^{\frac{2\cdot(2+1)}{2}}(tq)^{k_1}(tq^2)^{k_2}$$

$$= q^3 (tq)^2 (tq^2)^0 = t^2 q^5 .$$

rank

number being partitioned

To prove the second part, note that in the sum $1 + \sum_{n=1}^{\infty} t^{n-1} q^n (-t^{-1}q;q)_{n-1}$,

$n$ denotes the largest part of a partition into distinct parts. Note that the power of $t$

$= n -$ number of parts

($\because$ whenever a part appears, the corresponding power of $q$ has a $t^{-1}$ attached to it.)

$=$ rank,

Hence $1 + \sum_{n=1}^{\infty} t^{n-1} q^n (-t^{-1}q;q)_{n-1}$

$$= \sum_{m,n=0}^{\infty} r(m,n) t^m q^n .$$

$\blacksquare$

## Two identities of Ramanujan from the Lost Notebook

(i) $\sum_{n=0}^{\infty} \left( (-q;q)_\infty - (-q;q)_n \right) = (-q;q)_\infty D(q) + \frac{1}{2}\sigma(q)$, where $D(q) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}$.

Note that $\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q^{mn}$

$$\sum_{k=1}^{\infty} \Big( \sum_{n|k} 1 \Big) q^k = \sum_{k=1}^{\infty} \underbrace{d(k)}_{\text{number of positive divisors of } k,} q^k .$$

(ii) $\sum_{n=0}^{\infty} \left( \frac{1}{(q;q^2)_\infty} - \frac{1}{(q;q^2)_n} \right)$

$$= \frac{1}{(q;q^2)_\infty} D(q^2) + \frac{1}{2} \sigma(q).$$