27/10/21

MA 633 - Partition Theory - Lec. 32

Thm. 47 (Andrews)

Let $S_n(q) = (-q;q)_n$ & let $S(q) = (-q;q)_\infty$. Then

$$\sum_{n=0}^{\infty} \left(S(q) - S_n(q)\right) = \sum_{k=1}^{\infty} k q^k (-q;q)_{k-1}.$$

Proof: Take a partition of a positive integer into distinct parts with largest part $= k$.

These are counted on LHS by $S(q) - S_0(q)$, $S(q) - S_1(q), \ldots, S(q) - S_{k-1}(q)$. But such partitions are not counted by $S(q) - S_n(q)$, where $n \geq k$.

Example: $k = 2$.

$(-q;q)_\infty - (-q;q)_2 =$

$(-q;q)_2 \left((-q^3;q)_\infty - 1\right)$

$= (1+q)(1+q^2)\left((1+q^3)(1+q^4)\cdots - 1\right)$

Note that this power series starts with q^3. Hence it cannot enumerate partitions with largest part $= 2$.

Hence such partitions are counted on LHS with weight $= k$.

This proves the identity as the RHS does the same.

Thm. 4.8 (Andrews)

Let $S_n^*(q) = \dfrac{1}{(q;q^2)_{n+1}}$ & $S^*(q) = \dfrac{1}{(q;q^2)_\infty}$.

Then

$$\sum_{n=0}^{\infty} \left(S^*(q) - S_n^*(q) \right) = \sum_{k=0}^{\infty} \frac{k q^{2k+1}}{(q;q^2)_{k+1}}.$$

Proof: Take a partition of a positive integer into odd parts with largest part $= 2k+1$.

This partition is counted by $S^*(q) - S_0^*(q)$, ... $S^*(q) - S_{k-1}^*(q)$.

But the terms $S^*(q) - S_n^*(q)$, $n \geq k$, do not count such partitions.

Example: $k = 2$

$$\frac{1}{(q;q^2)_\infty} - \frac{1}{(q;q^2)_2} = \frac{1}{(q;q^2)_2} \left(\frac{1}{(q^5;q^2)_\infty} - 1 \right)$$

$$= \frac{1}{(1-q)(1-q^3)} \left\{ \frac{1}{(1-q^5)(1-q^7)\cdots\cdots} - 1 \right\}$$

$$= (1+q+q^2+\cdots)(1+q^3+q^6+\cdots\cdots)$$

$$\times \left\{ (1+q^5+q^{10}+\cdots\cdots)(1+q^7+q^{14}+\cdots)\cdots - 1 \right\}$$

$$= q^5 + \cdots\cdots\cdots$$

So the ptn. into odd parts with largest part $= 5$ $\underline{\underline{\text{is}}}$ getting counted, in the difference $S^*(q) - S^*(q)$.

So such ptns. are counted with weight k.

This proves the idty. as the RHS counts the same.

$$* \; D(q) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}.$$

$$= -\frac{1}{2} + \sum_{n=1}^{\infty} d(n) q^n \qquad (d(n): \text{divisor fn.})$$

$D(q)$ is the generating fn. of partitions into non-distinct parts.

Example: $n = 6$:
partitions of 6 into non-distinct parts.

$$\left.\begin{array}{c} 6 \\ 3+3 \\ 2+2 \\ 1+1+1+1+1+1 \end{array}\right\}$$ There are $d(6)$ many such partitions.

We will prove sum-of-tails identities in Theorem 46 in a tutorial session.

ROGERS – RAMANUJAN IDENTITIES

Thm. 49 (i) $$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

(ii) $$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

Euler's theorem: The number of partitions of an integer n into odd parts equals the number of partitions of n into parts which differ by at least one,

$$(-q;q)_\infty = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n} = \frac{1}{(q;q^2)_\infty}$$