28/10/21

MA 633 - Partition Theory - Lec. 33

ROGERS - RAMANUJAN IDENTITIES

Thm. 49 (i) $$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

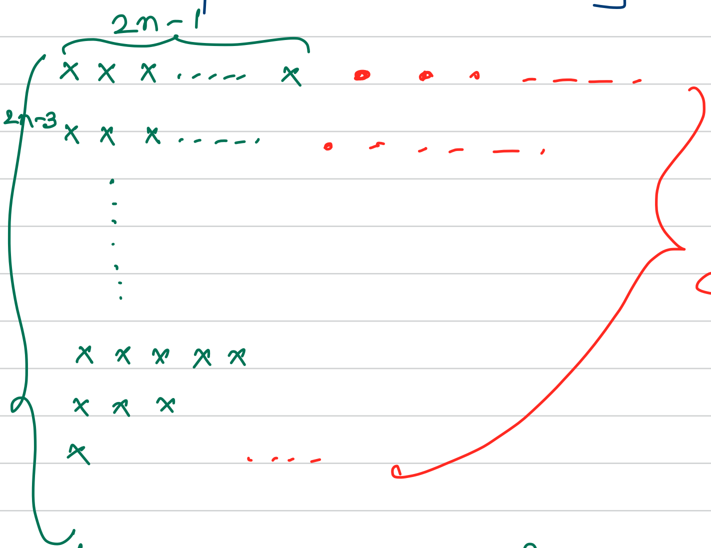
ii) $$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

Euler's theorem: The number of partitions of an integer n into odd parts equals the number of partitions of n into parts which differ by at least one,

$$(-q;q)_\infty = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n} = \frac{1}{(q;q^2)_\infty}$$

Consider $\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{q^{1+3+5+\cdots+2n-1}}{(q;q)_n}$

This is a generating function for the number of partitions of a positive integer whose parts differ by at least 2.



Comes from $\dfrac{1}{(q;q)_n}$

comes from q^{n^2}.

Similarly $\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}$ is the gen. fn. of ptns. of a positive integer into parts differing by at least 2, and with no 1's.

(Hint:
Write $n^2+n = 2+4+6+\cdots+(2n-2)+2n$.)

Experimentally discovering the 1st Rogers – Ramanujan identity.

n	# of such partitions	partitions whose parts differ by at least 2
1	1	1
2	1	2
3	1	3
4	2	4, 3+1
5	2	5, 4+1
6	3	6, 5+1, 4+2
7	3	7, 6+1, 5+2
8	4	8, 7+1, 6+2, 5+3
9	5	9, 8+1, 7+2, 6+3, 5+3+1
10	6	10, 9+1, 8+2, 7+3, 6+4, 6+3+1
11	7	11, 10+1, 9+2, 8+3, 7+4, 6+4+1, 7+3+1

1, 4, 6, 9

11 1's

9+1+1

6+4+1

6+1+1+1+1+1

4+4+1+1+1

4+7 1's.

Goal: For all $n \in \mathbb{N}$, Construct a set N s.t.

$$p(n \mid \text{parts in } N) = p(n \mid \text{parts differing by at least } 2). \quad (*)$$

① Start with $N = \phi$.

② Add 1 to N, so that $N = \{1\}$, and thus holds for $n = 1$.

③ 2 will not be added to N because 2 can be written as $\underbrace{1+1}_{\text{parts in } N}$

④ 3 will not be added.

(5) 4 will be added to N, for, otherwise with the existing elements in N, we get only one partition of 4 $(4 = 1+1+1+1)$,
$N = \{1, 4\}$.

(6) 5 will not be added to N.

(7) 6 will be added to N,
$N = \{1, 4, 6\}$.

(8) 7 will not be added to N

(9) 8 will not be added to N.

(10) 9 will be added to N,
$N = \{1, 4, 6, 9\}$.

(11) 10 will not be added to N.

(12) 11 will be added to N.

$N = \{1, 4, 6, 9, 11\}$.

Continuing like this, we will see that N consists of numbers $\equiv 1, 4 \pmod 5$.

This experimentally leads us to the 1st Rogers- Ramanujan identity.

Proofs of the Rogers–Ramanujan identities due to G.N. Watson

Consider the following $_8\phi_7$ to $_4\phi_3$ transf. of Watson,

$$
{}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-N} \\ \sqrt{a}, & -\sqrt{a}, & \frac{aq}{b}, & \frac{aq}{c}, & \frac{aq}{d}, & \frac{aq}{e}, & aq^{N+1} & \end{matrix} ; \frac{a^2q^{N+2}}{bcde}\right]
$$

$$
= \frac{(aq)_N\left(\frac{aq}{de}\right)_N}{\left(\frac{aq}{d}\right)_N\left(\frac{aq}{e}\right)_N}\, {}_4\phi_3\left[\begin{matrix} \frac{aq}{bc}, & d, & e, & q^{-N} \\ \frac{deq^{-N}}{a}, & \frac{aq}{b}, & \frac{aq}{c} & \end{matrix} ; q\right]
$$

We will use this to prove the following result:

Thm. 50 (Entry 7, Ch. 16, Ramanujan's 2nd notebook).

$$
\sum_{n=0}^{\infty} \frac{(a)_n\left(\frac{d}{b}\right)_n\left(\frac{d}{c}\right)_n\left(\frac{d}{q}\right)_n\left(1-dq^{2n-1}\right)\left(\frac{bc}{a}\right)^n q^{n(n-1)}}{(b)_n(c)_n\left(\frac{d}{a}\right)_n(q)_n\left(1-\frac{d}{q}\right)}
$$

$$
= \frac{(a)_\infty(d)_\infty}{(b)_\infty(c)_\infty}\sum_{n=0}^{\infty}\frac{(b/a)_n(c/a)_n}{(d/a)_n(q)_n}a^n .
$$

Proof: We want to let $N \to \infty$ & then $c \to \infty$ in the ${}_8\phi_7$ transf. of Watson.

Observe that

$$\lim_{N \to \infty} \frac{(q^{-N})_n \, q^{Nn}}{(aq^{N+1})_n} = \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{1}$$

$$= (-1)^n q^{n(n-1)/2},$$

using

$$(wq^{-N})_n = (-w)^n q^{\frac{n(n-1)}{2} - nN} \frac{(q/w)_N}{(q/w)_{N-n}} \quad \text{— } (*)$$

[Proof of $(*)$:

$$(wq^{-N})_n = \left(1 - \frac{w}{q^N}\right)\left(1 - \frac{w}{q^{N-1}}\right) \cdots \left(1 - \frac{w}{q^{N-(n-1)}}\right)$$

$$= \left(\frac{-w}{q^N}\right)\left(\frac{-w}{q^{N-1}}\right) \cdots \left(\frac{-w}{q^{N-(n-1)}}\right)$$

$$\times \left(1 - \frac{q^N}{w}\right)\left(1 - \frac{q^{N-1}}{w}\right) \cdots \left(1 - \frac{q^{N-(n-1)}}{w}\right)$$

$$= \frac{(-w)^n}{q^{Nn - \frac{n(n-1)}{2}}} \frac{\left(\frac{q^{N-(n-1)}}{w}\right)_n \left(\frac{q}{w}\right)_{N-n}}{\left(\frac{q}{w}\right)_{N-n}}$$

which proves $(*)$.]