Proofs of the Rogers–Ramanujan identities due to G.N. Watson

Consider the following $_8\phi_7$ to $_4\phi_3$ transf. of Watson,

$$
{}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-N} \\ \sqrt{a}, & -\sqrt{a}, & \frac{aq}{b}, & \frac{aq}{c}, & \frac{aq}{d}, & \frac{aq}{e}, & aq^{N+1} & \end{matrix}; \frac{a^2q^{N+2}}{bcde}\right]
$$

$$
= \frac{(aq)_N\left(\frac{aq}{de}\right)_N}{\left(\frac{aq}{d}\right)_N\left(\frac{aq}{e}\right)_N}\ {}_4\phi_3\left[\begin{matrix} \frac{aq}{bc}, & d, & e, & q^{-N} \\ \frac{deq^{-N}}{a}, & \frac{aq}{b}, & \frac{aq}{c} & \end{matrix}; q\right]
$$

We will use this to prove the following result:

Thm. 50 (Entry 7, Ch. 16, Ramanujan's 2nd notebook).

$$
\sum_{n=0}^{\infty} \frac{(a)_n\left(\frac{d}{b}\right)_n\left(\frac{d}{c}\right)_n\left(\frac{d}{q}\right)_n\left(1-dq^{2n-1}\right)\left(\frac{bc}{a}\right)^n q^{n(n-1)}}{(b)_n(c)_n\left(\frac{d}{a}\right)_n(q)_n\left(1-\frac{d}{q}\right)}
$$

$$
= \frac{(a)_\infty(d)_\infty}{(b)_\infty(c)_\infty}\sum_{n=0}^{\infty}\frac{(b/a)_n(c/a)_n}{(d/a)_n(q)_n}a^n .
$$

Proof: We want to let $N \to \infty$ & then $c \to \infty$ in the ${}_8\phi_7$ transf. of Watson.

Observe that

$$\lim_{N\to\infty} \frac{(q^{-N})_n\, q^{Nn}}{(aq^{N+1})_n} = \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{1} \quad (\because |q|<1)$$

$$= (-1)^n q^{n(n-1)/2},$$

using

$$(wq^{-N})_n = (-w)^n q^{\frac{n(n-1)}{2} - nN} \frac{(q/w)_N}{(q/w)_{N-n}}. \quad — (*)$$

[Proof of (*):

$$(wq^{-N})_n = \left(1 - \frac{w}{q^N}\right)\left(1 - \frac{w}{q^{N-1}}\right) \cdots \left(1 - \frac{w}{q^{N-(n-1)}}\right)$$

$$= \left(-\frac{w}{q^N}\right)\left(\frac{-w}{q^{N-1}}\right) \cdots \left(\frac{-w}{q^{N-(n-1)}}\right)$$

$$\times \left(1 - \frac{q^N}{w}\right)\left(1 - \frac{q^{N-1}}{w}\right) \cdots \left(1 - \frac{q^{N-(n-1)}}{w}\right)$$

$$= \frac{(-w)^n}{q^{Nn - \frac{n(n-1)}{2}}} \frac{\left(\frac{q^{N-(n-1)}}{w}\right)_n \left(\frac{q}{w}\right)_{N-n}}{\left(\frac{q}{w}\right)_{N-n}}$$

which proves (*).]

Term on RHS :

$$\lim_{N\to\infty} \frac{(q^{-N})_n}{\left(\frac{deq^{-N}}{a}\right)_n} = \lim_{N\to\infty} \frac{\dfrac{(-1)^n (q)_N}{(q)_{N-n}}}{\left(-\frac{de}{a}\right)^n \dfrac{\left(\frac{qa}{de}\right)_N}{\left(\frac{qa}{de}\right)_{N-n}}}$$

$$= \left(\frac{a}{de}\right)^n.$$

Also let $c \to \infty$. Then,

$$\lim_{c\to\infty} \frac{(c)_n c^{-n}}{\left(aq/c\right)_n} = (-1)^n q^{n(n-1)/2}$$

Hence

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n (d)_n (e)_n (1-aq^{2n})}{\left(\frac{aq}{b}\right)_n \left(\frac{aq}{d}\right)_n \left(\frac{aq}{e}\right)_n (1-a)(q)_k} \left(\frac{a^2}{bde}\right)^n q^{n(n+1)}$$

$$= \frac{(aq)_\infty \left(\frac{aq}{de}\right)_\infty}{\left(\frac{aq}{d}\right)_\infty \left(\frac{aq}{e}\right)_\infty} \sum_{n=0}^{\infty} \frac{(d)_n (e)_n}{\left(\frac{aq}{b}\right)_n (q)_n} \left(\frac{aq}{de}\right)^n$$

Now replace d by $\frac{d}{b}$, e by $\frac{d}{c}$, a by $\frac{d}{q}$ and then b by a. Then LHS becomes what we want.

The RHS becomes

$$\frac{(d)_\infty \left(\frac{bc}{d}\right)_\infty}{(b)_\infty (c)_\infty} \sum_{n=0}^{\infty} \frac{(d/b)_n (d/c)_n}{(d/a)_n (q)_n} \left(\frac{bc}{q}\right)^n$$

q-analogue of Euler's transf.

$$= \frac{(d)_\infty (a)_\infty}{(b)_\infty (c)_\infty} \sum_{n=0}^{\infty} \frac{(b/a)_n (c/a)_n \, a^n}{(d/a)_n (q)_n}.$$

This proves the result.