MA 633 – Partition theory Lec. 35

Cor. 51
$$1+\sum_{n=1}^{\infty}\frac{(-1)^n(1-aq^{2n})a^{2n}q^{\frac{n(5n-1)}{2}}(aq)_{n-1}}{(q)_n}$$
$$=(aq)_\infty\sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{(q)_n}.$$

Proof: From (*),

$$1+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n(d)_n(e)_n(1-aq^{2n})}{\left(\frac{aq}{b}\right)_n\left(\frac{aq}{d}\right)_n\left(\frac{aq}{e}\right)_n(1-a)(q)_k}\left(\frac{a^2}{bde}\right)^n q^{n(n+1)}$$
$$=\frac{(aq)_\infty\left(\frac{aq}{de}\right)_\infty}{\left(\frac{aq}{d}\right)_\infty\left(\frac{aq}{e}\right)_\infty}\sum_{n=0}^{\infty}\frac{(d)_n(e)_n}{\left(\frac{aq}{b}\right)_n(q)_n}\left(\frac{aq}{de}\right)^n$$

Now let b, d & $e\longrightarrow\infty$. Then.

Note that $\frac{(x)_n}{x^n}=\frac{(1-x)}{x}\frac{(1-xq)}{x}\cdots\frac{(1-xq^{n-1})}{x}$
$$=\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-q\right)\cdots\left(\frac{1}{x}-q^{n-1}\right)$$
$$\longrightarrow(-1)^n q^{n(n-1)/2}\quad\text{as } x\to\infty,$$

$$\underbrace{\frac{2n^2+2n+3n^2-3n}{2}} = \frac{5n^2-n}{2}$$

Hence

$$1+\sum_{n=1}^{\infty} \frac{(a)_n (1-aq^{2n})\, a^{2n}\, q^{n(n+1)+\frac{3n(n-1)}{2}} (-1)^n}{(1-a)\,(q)_n}$$

$$= (aq)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n+n(n-1)}}{(q)_n}$$

$$\Rightarrow 1+\sum_{n=1}^{\infty} \frac{(-1)^n a^{2n} (aq)_{n-1} (1-aq^{2n})\, q^{\frac{n(5n-1)}{2}}}{(q)_n}$$

$$= (aq)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_n}.$$

Thm. 52 (Rogers–Ramanujan identities)

(i) $$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

(ii) $$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

Proof: (i) In Cor. 51, we let $a=1$.

$$\Rightarrow 1+\sum_{n=1}^{\infty} \frac{(-1)^n (q)_{n-1} (1-q^{2n})\, q^{\frac{n(5n-1)}{2}}}{(q)_n}$$

$$= (q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}$$

$$\text{LHS} = 1+\sum_{n=1}^{\infty} (-1)^n (1+q^n)\, q^{\frac{n(5n-1)}{2}}$$

$$= 1+\sum_{n=1}^{\infty} (-1)^n q^{\frac{n(5n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(5n+1)}{2}}$$

$n \longrightarrow -n$

$$= 1+\sum_{n=1}^{\infty} (-1)^n q^{\frac{n(5n-1)}{2}} + \sum_{n=-\infty}^{-1} (-1)^n q^{\frac{n(5n-1)}{2}}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(5n-1)}{2}}$$

$$= f(-q^2, -q^3) \quad \left(\because \sum_{n=-\infty}^{\infty} (-q^2)^{\frac{n(n+1)}{2}} (-q^3)^{\frac{n(n-1)}{2}} \right.$$

$$= \sum_{n=-\infty}^{\infty} (-1)^{n^2} q^{\frac{n(n+1)+3n(n-1)}{2}}$$

$$\left. = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2-n}{2}} \right)$$

From JTPI,

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

$$\Rightarrow f(-q^2,-q^3) = (q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty$$

$$\Rightarrow (q^2,q^3,q^5;q^5)_\infty = (q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}.$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{(\cancel{q^2},\cancel{q^3},\cancel{q^5};q^5)_\infty}{(q,\cancel{q^2},\cancel{q^3},q^4,\cancel{q^5};q^5)_\infty}$$

$$= \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}.$$

(ii) Let $a=q$ in (*), that is,

$$1+\sum_{n=1}^{\infty} \frac{(-1)^n a^{2n}(aq)_{n-1}(1-aq^{2n})\, q^{\frac{n(5n-1)}{2}}}{(q)_n}$$

$$= (aq)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_n},$$

to get

$$1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n}(q^2)_{n-1}(1-q^{2n+1})\,q^{\frac{n(5n-1)}{2}}}{(q)_n}$$

$$=(q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n}$$

Multiplying both sides by $(1-q)$, we get

$$(1-q)+\sum_{n=1}^{\infty}(-1)^n q^{2n}(1-q^{2n+1})\,q^{\frac{n(5n-1)}{2}}$$

$$=(q)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n}$$

$$\text{LHS}=(1-q)+\sum_{n=1}^{\infty}(-1)^n q^{2n+\frac{n(5n-1)}{2}}$$

$$-\sum_{n=1}^{\infty}(-1)^n q^{2n+2n+1+\frac{n(5n-1)}{2}}$$

$$=(1-q)+\sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}}$$

$$\underbrace{-\sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2+7n+2}{2}}}_{n\to -n-1}$$

$$= 1 - q + \sum_{n=1}^{\infty} (-1)^n q^{\frac{5n^2+3n}{2}} + \sum_{n=-\infty}^{-2} (-1)^n q^{\frac{5n^2+3n}{2}}$$

$n=-1$ term of

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(5n+3)}{2}}$$

$$= f(-q,-q^4) \quad \left(\sum_{n=-\infty}^{\infty} (-q)^{\frac{n(n+1)}{2}} (-q^4)^{\frac{n(n-1)}{2}} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n+1)+4n(n-1)}{2}} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2-3n}{2}} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n^2+3n}{2}} \right)$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q)_\infty} f(-q,-q^4)$$

$$= \frac{(\cancel{q}, \cancel{q^4}, \cancel{q^5}; q^5)_\infty}{(\cancel{q}, q^2, q^3, \cancel{q^4}, \cancel{q^5}; q^5)_\infty}$$

$$= \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$$

Göllnitz–Gordon identities

$$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k \, q^{k^2+2k}}{(q^2;q^2)_k} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}.$$

Note that

$$k^2+2k = 3+5+7+ \cdots + (2k+1).$$