3/11/2021

MA 633 – Partition theory Lec. 36

Göllnitz – Gordon identities

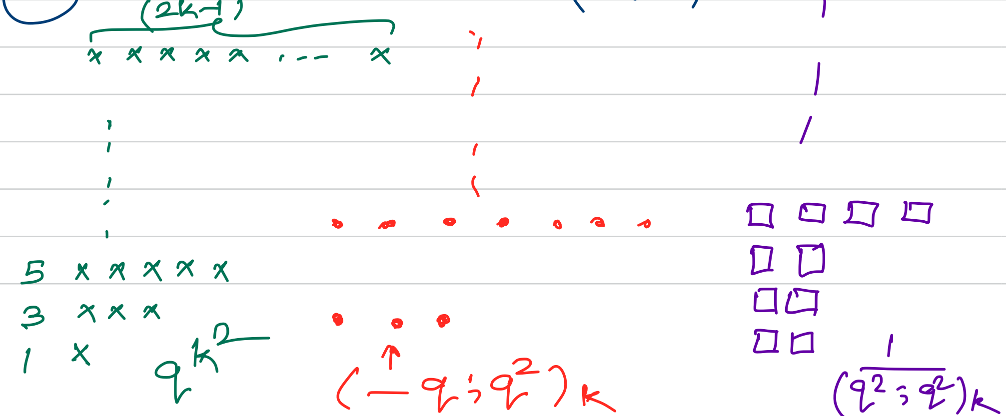
(A) $$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k\, q^{k^2}}{(q^2;q^2)_k} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}$$

(B) $$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k\, q^{k^2+2k}}{(q^2;q^2)_k} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}$$

– Combinatorial interpretations of these identities by H. Göllnitz (1956) – unpublished PhD thesis.
– Rediscovered by Basil Gordon (1965)
– Analytic versions by L. J. Slater
– Equivalent forms of the above identities are in Ramanujan's Lost Notebook.

Combinatorial interpretations:

(A) $k^2 = 1+3+5+\cdots+(2k-1)$



Thus LHS of A generates partitions of a positive integer whose parts differ by at least 2 and, in addition, the even parts differ by at least 4.

(for the addnal condn, we can also say no consecutive multiples of 2)

B = A + added restriction that parts should be ≥ 3.

PROOF OF THEOREM 46 (Sum-of-tails identities)

(i) $$\sum_{n=0}^{\infty}\left((-q;q)_\infty-(-q;q)_n\right)=(-q;q)_\infty D(q)+\frac{1}{2}\sigma(q)$$

Ingredients:

① We will assume a beautiful reciprocity theorem of Ramanujan:

Let $$\rho(a,b):=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}\, a^n b^{-n}}{(-aq)_n}$$

Then
$$\rho(a,b)-\rho(b,a)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{\left(\frac{aq}{b}\right)_\infty\left(\frac{bq}{a}\right)_\infty(q)_\infty}{(-aq)_\infty(-bq)_\infty}.$$

② Define a differential operator

$$\mathcal{E}(f(z)) = f'(1).$$

③ $$\mathcal{E}\left(\frac{(zq)_\infty}{(q)_\infty}\right) = \frac{1}{(q)_\infty}\frac{d}{dz}(zq)_\infty \Big|_{z=1}$$

$$= \frac{1}{(q)_\infty}(zq)_\infty \frac{d}{dz}\log(zq)_\infty \Big|_{z=1}$$

$$= \frac{(zq)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{d}{dz}\log(1-zq^n)\Big|_{z=1}$$

$$= \frac{(zq)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{-q^n}{1-zq^n}\Big|_{z=1}$$

$$= -\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} = -\frac{1}{2} - D(q). \quad \text{—} \; *_1$$

$$\left(\text{Since } D(q) = -\frac{1}{2} + \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right)$$

Also, verify that

$$\mathcal{E}\left(\frac{(q)_\infty}{(zq)_\infty}\right) = \frac{1}{2} + D(q). \quad \text{—} \; *_2$$

Proof of (i): Let $S(q) = (-q;q)_\infty$ & $S_n(q) = (-q;q)_n$. Then we have to show that

$$2\sum_{n=0}^{\infty}\left(S(q) - S_n(q)\right) - 2S(q)D(q) = \sigma(q) \qquad \text{—— (a)}$$

To that end,

LHS of (a)

$$= 2\sum_{n=1}^{\infty} nq^n(-q;q)_{n-1} - 2S(q)D(q)$$

(by Thm. 47)

$$= 2\sum_{n=0}^{\infty}(n+1)q^{n+1}(-q)_n - 2S(q)D(q)$$

$$= 2\,\mathcal{E}\left(1 + \sum_{n=0}^{\infty} z^{n+1}q^{n+1}(-q)_n\right) + S(q)$$

$$+ 2S(q)\,\mathcal{E}\left(\frac{(zq)_\infty}{(q)_\infty}\right)$$

$$= 2\mathcal{E}\left(\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(zq)_n}\right) + S(q)$$

$$+ 2S(q)\,\mathcal{E}\left(\frac{(zq)_\infty}{(q)_\infty}\right)$$