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MA 633 — Partition theory Lec. 37

Proof of (i): Let $S(q) = (-q;q)_\infty$ & $S_n(q) = (-q;q)_n$. Then we have to show that

$$2\sum_{n=0}^{\infty}\left(S(q) - S_n(q)\right) - 2S(q)D(q) = \sigma(q) \quad \text{--- (a)}.$$

To that end,
LHS of (a)

$$= 2\sum_{n=1}^{\infty} n q^n (-q;q)_{n-1} - 2S(q)D(q)$$

(by Thm. 47)

$$= 2\sum_{n=0}^{\infty}(n+1) q^{n+1}(-q)_n - 2S(q)D(q)$$

$$= 2\varepsilon\left(1 + \sum_{n=0}^{\infty} z^{n+1} q^{n+1}(-q)_n\right) + S(q)$$

$$+ 2S(q)\,\varepsilon\left(\frac{(zq)_\infty}{(q)_\infty}\right)$$

$\Rightarrow$ LHS of (a)

$$= 2\varepsilon\left(\sum_{n=0}^{\infty}\frac{z^n q^{n(n+1)/2}}{(zq)_n}\right) + S(q) \quad \text{--- I}$$

$$+ 2S(q)\,\varepsilon\left(\frac{(zq)_\infty}{(q)_\infty}\right)$$

because of the following identity which we will now prove.

$$\sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n+1)}{2}}}{(zq)_n} = 1 + \sum_{n=0}^{\infty} z^{n+1} q^{n+1} (-q)_n \quad \text{---} (\star)$$

Heine's transformation:

$${}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \, {}_2\varphi_1\left(\begin{matrix} c/b, z \\ az \end{matrix}; b\right)$$

$$= \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \, {}_2\varphi_1\left(\begin{matrix} z, c/b \\ az \end{matrix}; b\right)$$

$$= \frac{\cancel{(b)_\infty} \cancel{(az)_\infty}}{(c)_\infty (z)_\infty} \cdot \frac{(c/b)_\infty (bz)_\infty}{\cancel{(az)_\infty} \cancel{(b)_\infty}} \, {}_2\varphi_1\left(\begin{matrix} \frac{abz}{c}, b \\ bz \end{matrix}; \frac{c}{b}\right)$$

2nd iterate of Heine's transformation

$$= \frac{(c/b)_\infty (bz)_\infty}{(c)_\infty (z)_\infty} \, {}_2\varphi_1\left(\begin{matrix} \frac{abz}{c}, b \\ bz \end{matrix}; \frac{c}{b}\right) \quad \text{---} (\star\star)$$

Proof of $(\star)$:

Note that

$$\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(zq)_n} = \lim_{\tau \to 0} {}_2\varphi_1\left(\begin{matrix} -q/\tau, q \\ zq \end{matrix}; \tau z\right)$$

(because $\lim_{\tau \to 0} \left(-\frac{q}{\tau}\right)_n \tau^n = \lim_{\tau \to 0} \left(1 + \frac{q}{\tau}\right)\left(1 + \frac{q^2}{\tau}\right) \cdots \left(1 + \frac{q^n}{\tau}\right) \tau^n$

$= q^{n(n+1)/2}$.)

$$= \lim_{\tau \to 0} \frac{(z)_\infty (q\tau z)_\infty}{(zq)_\infty (\tau z)_\infty} \, {}_2\varphi_1\left(\begin{matrix} -q, & q & ; & z \\ & \tau z q & ; & \end{matrix}\right)$$

(from ☆☆)

$$= \frac{(z)_\infty}{(zq)_\infty} \sum_{n=0}^{\infty} (-q)_n z^n$$

$$= (1-z) \sum_{n=0}^{\infty} (-q)_n z^n$$

$$= \sum_{n=0}^{\infty} (-q)_n z^n - \sum_{n=0}^{\infty} (-q)_n z^{n+1}$$

$$= 1 + \sum_{n=1}^{\infty} (-q)_n z^n - \sum_{n=1}^{\infty} (-q)_{n-1} z^n$$

$$= 1 + \sum_{n=1}^{\infty} (-q)_{n-1} z^n (\cancel{1} + q^n - \cancel{1})$$

$$= 1 + \sum_{n=1}^{\infty} (-q)_{n-1} z^n q^n$$

$$= 1 + \sum_{n=0}^{\infty} (-q)_n z^{n+1} q^{n+1}.$$

We now use the reciprocity theorem of Ramanujan:

$$\rho(a,b) := \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n}$$

Then

$$\rho(a,b)-\rho(b,a) = \left(\frac{1}{b}-\frac{1}{a}\right) \frac{\left(\frac{aq}{b}\right)_\infty \left(\frac{bq}{a}\right)_\infty (q)_\infty}{(-aq)_\infty (-bq)_\infty}.$$

Let $a=-z$ and $b=1$ so that

$$\rho(-z,1)-\rho(1,-z) = \left(1+\frac{1}{z}\right) \frac{(-zq)_\infty \left(-\frac{q}{z}\right)_\infty (q)_\infty}{(zq)_\infty (-q)_\infty}$$

$$= \frac{(-zq)_\infty \left(-\frac{1}{z}\right)_\infty (q)_\infty}{(zq)_\infty (-q)_\infty}$$

$$\Rightarrow \rho(1,-z) = \rho(-z,1) - \frac{(-zq)_\infty \left(-\frac{1}{z}\right)_\infty (q)_\infty}{(zq)_\infty (-q)_\infty}.$$

Now divide both sides by $1-z^{-1}$ & let $z \to 1$.

Note that

$$\lim_{z\to 1} \frac{\rho(1,-z)}{1-z^{-1}} = \lim_{z\to 1} \frac{\cancel{\left(1-\frac{1}{z}\right)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} (-z)^{-n}}{(-q)_n}}{\cancel{\left(1-\frac{1}{z}\right)}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n} = \sigma(q).$$

Let

$$f(z) := \rho(-z,1) - \frac{(-zq)_\infty(-\frac{1}{z})_\infty(q)_\infty}{(zq)_\infty(-q)_\infty}.$$

Then we claim that

$$\lim_{z\to 1} f(z) = 0.$$

(Proof of the claim:

$$\lim_{z\to 1} f(z) = \lim_{z\to 1}\left\{2\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}(-z)^n}{(zq)_n} - \frac{(-zq)_\infty(-\frac{1}{z})_\infty(q)_\infty}{(zq)_\infty(-q)_\infty}\right\}$$

$$= 2\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q)_n} - \frac{2(-q)_\infty^2(q)_\infty}{(q)_\infty(-q)_\infty}$$

$$= 2\left(\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q)_n} - (-q)_\infty\right)$$

$= 0$ (by a cor. of q-binomial thm; also it can be proved by noting that both expns. are the g.f's of the number of ptns. ~~of~~ a positive integer into distinct parts)

Hence $\frac{f(z)}{1-z^{-1}}$ is of the form $\frac{0}{0}$ (when $z=1$)!

So by L'Hôpital's rule,

$$\lim_{z\to 1}\frac{f(z)}{1-z^{-1}} = \lim_{z\to 1}\frac{f'(z)}{z^{-2}} = f'(1) = \varepsilon(f(z)),$$

$$\Rightarrow \sigma(q)$$

$$= \varepsilon\left(2\sum_{n=0}^{\infty}\frac{z^n q^{\frac{n(n+1)}{2}}}{(zq)_n} - \frac{(-zq)_\infty\left(-\frac{1}{z}\right)_\infty(q)_\infty}{(zq)_\infty(-q)_\infty}\right)$$

$$= 2\varepsilon\left(\sum_{n=0}^{\infty}\frac{z^n q^{n(n+1)/2}}{(zq)_n}\right)$$

$$- \lim_{z\to 1}\left[\frac{\frac{d}{dz}\left((-zq)_\infty\left(-\frac{1}{z}\right)_\infty(q)_\infty\right)}{(zq)_\infty(-q)_\infty} + \frac{d}{dz}\left(\frac{1}{(zq)_\infty(-q)_\infty}\right)(-zq)_\infty\left(-\frac{1}{z}\right)_\infty(q)_\infty\right]$$

$$= 2\varepsilon\left(\sum_{n=0}^{\infty}\frac{z^n q^{n(n+1)/2}}{(zq)_n}\right)$$

$$- \lim_{z\to 1}\left\{\frac{\frac{d}{dz}\sum_{n=0}^{\infty} z^n q^{n(n+1)/2}}{(q)_\infty(-q)_\infty} + 2(-q)_\infty^2(q)_\infty \times \lim_{z\to 1}\frac{d}{dz}\left(\frac{1}{(zq)_\infty(-q)_\infty}\right)\right\}$$