10/11/21

MA 633 – Theory of partitions – Lec. 38

$$\lim_{z\to 1}\frac{f(z)}{1-z^{-1}} = \lim_{z\to 1}\frac{f'(z)}{z^{-2}} = f'(1) = \varepsilon(f(z)),$$

$\Rightarrow \sigma(q)$

$$= \varepsilon\left(2\sum_{n=0}^{\infty}\frac{z^n q^{\frac{n(n+1)}{2}}}{(zq)_n} - \frac{(-zq)_\infty(-\frac{1}{z})_\infty(q)_\infty}{(zq)_\infty(-q)_\infty}\right)$$

$$= 2\varepsilon\left(\sum_{n=0}^{\infty}\frac{z^n q^{n(n+1)/2}}{(zq)_n}\right)$$

$$- \lim_{z\to 1}\left[\frac{\frac{d}{dz}\left((-zq)_\infty(-\frac{1}{z})_\infty(q)_\infty\right)}{(zq)_\infty(-q)_\infty} + \frac{d}{dz}\left(\frac{1}{(zq)_\infty(-q)_\infty}\right)(-zq)_\infty(\tfrac{-1}{z})_\infty(q)_\infty\right]$$

$$= 2\varepsilon\left(\sum_{n=0}^{\infty}\frac{z^n q^{n(n+1)/2}}{(zq)_n}\right)$$

$$- \lim_{z\to 1}\left\{\frac{\frac{d}{dz}\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2}}{(q)_\infty(-q)_\infty} + 2(-q)_\infty^2(q)_\infty \times \lim_{z\to 1}\frac{d}{dz}\left(\frac{1}{(zq)_\infty(-q)_\infty}\right)\right\}$$

This is because of the following:

Note that

$$\sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

In this, let $b=\frac{1}{z}$, $a=\frac{q}{(1/z)} = zq$ so that

$$\sum_{n=-\infty}^{\infty} (zq)^{\frac{n(n+1)}{2}} (z^{-1})^{\frac{n(n-1)}{2}} = (-zq;q)_\infty \left(-\frac{1}{z};q\right)_\infty (q;q)_\infty$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} z^n q^{\frac{n(n+1)}{2}} = (-zq;q)_\infty (-z^{-1})_\infty (q)_\infty .$$

Hence, $\sigma(q)$

$$= 2\,\varepsilon\left(\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(zq)_n}\right)$$

$$- \frac{\lim_{z\to 1} \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n q^{n(n+1)/2} + \sum_{n=0}^{\infty} z^{-n-1} q^{n(n+1)/2}\right)}{(q)_\infty (-q)_\infty}$$

$$- \frac{2(-q)_\infty^2 (q)_\infty}{(q)_\infty (-q)_\infty}\, \varepsilon\left(\frac{(q)_\infty}{(zq)_\infty}\right)$$

$$= 2\,\varepsilon\left(\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(zq)_n}\right)$$

$$- \frac{\lim_{z\to 1}\left(\sum_{n=0}^{\infty} n z^{n-1} q^{n(n+1)/2} - \sum_{n=0}^{\infty} (n+1) z^{-n-2} q^{\frac{n(n+1)}{2}}\right)}{(q)_\infty \cdot (-q)_\infty}$$

$$- 2(-q)_\infty \left(\frac{1}{2} + D(q)\right)$$

$$= 2\,\varepsilon\left(\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(zq)_n}\right)$$

$$+ \frac{\sum_{n=0}^{\infty} q^{n(n+1)/2}}{(q)_\infty(-q)_\infty} - S(q) - 2S(q)D(q)$$

(Cor. 7)

$$= 2\,\varepsilon\left(\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(zq)_n}\right)$$

$$+ \frac{\cancel{(q^2;q^2)_\infty}}{(q;q^2)_\infty \cancel{(q)_\infty}\cancel{(-q)_\infty}} - S(q) - 2S(q)D(q)$$

$\Rightarrow \sigma(q)$

$$= 2\,\varepsilon\left(\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(zq)_n}\right) - 2S(q)D(q) \quad \text{II}$$

$\left(\because \dfrac{1}{(q;q^2)_\infty} = S(q), \text{ by Euler's theorem}\right).$

From I,

LHS of (a)

$$= 2\varepsilon\left(\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(zq)_n}\right) + S(q)$$

$$+ 2S(q)\,\varepsilon\left(\frac{(zq)_\infty}{(q)_\infty}\right)$$

$$= 2\varepsilon\left(\sum_{n=0}^{\infty}\frac{z^n q^{n(n+1)/2}}{(zq)_n}\right) + S(q)$$

$$+ 2S(q)\left(-\frac{1}{2} - D(q)\right)$$

$$= 2\varepsilon\left(\sum_{n=0}^{\infty}\frac{z^n q^{n(n+1)/2}}{(zq)_n}\right) - 2S(q)D(q)$$

$$= \sigma(q) \qquad (\text{from } \text{II})$$

$\blacksquare$

Generalized Frobenius Partitions

Here, we study two-rowed arrays of non-negative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

— 2 rows are of the same length

— a_i's are arranged in non-incr. order, and so are b_i's. $\begin{pmatrix} a_1 > a_2 > \cdots a_r \geq 0 \\ b_1 > b_2 > \cdots b_r \geq 0 \end{pmatrix}$

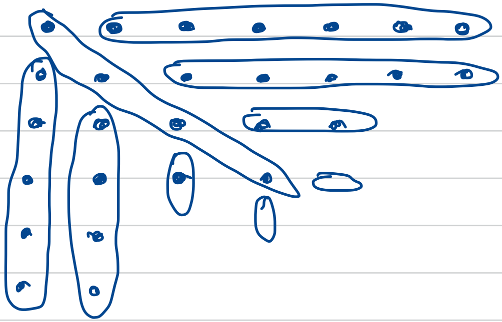
The above array is said to be a generalized Frobenius partition (or simply F-partition) of n if

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i .$$

- Frobenius desired a notation for partitions which would exhibit immediately the conjugate of a partition.

What he did was the following:

Consider a partition
$7+7+5+4+2+2$.



Frobenius symbol of the above partition is $\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix}$.

For conjugate, the Frobenius symbol would be $\begin{pmatrix} 5 & 4 & 1 & 0 \\ 6 & 5 & 2 & 0 \end{pmatrix}$.

The Jacobi triple product identity and the general principle

JTPI: $$\prod_{n=1}^{\infty}(1+zq^n)(1+z^{-1}q^{n-1})$$

$$= \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{m=-\infty}^{\infty} z^m q^{m(m+1)/2}$$

Proof: Let $\varphi(z) := \prod_{n=1}^{\infty}(1+zq^n)(1+z^{-1}q^{n-1})$.

Then

$$\varphi(qz) = \prod_{n=1}^{\infty} (1 + zq^{n+1})(1 + z^{-1} q^{n-2})$$

$$= (1 + z^{-1} q^{-1}) \prod_{n=1}^{\infty} (1 + zq^{n+1})(1 + z^{-1} q^{n-1})$$

$$= z^{-1} q^{-1} (1 + zq) \prod_{n=1}^{\infty} (1 + zq^{n+1})(1 + z^{-1} q^{n-1})$$

$$= z^{-1} q^{-1} \prod_{n=1}^{\infty} (1 + zq^{n})(1 + z^{-1} q^{n-1})$$

$$= z^{-1} q^{-1} \varphi(z). \longrightarrow (1)$$

Note that $\varphi(z)$ can be expanded as a Laurent series in a nbhd. of $z = 0$; i.e;

$$\varphi(z) = \sum_{n=-\infty}^{\infty} A_n(q) z^n \overset{(\text{by } 1)}{=} zq\, \varphi(qz)$$

$$= \sum_{n=-\infty}^{\infty} A_n(q) z^{n+1} q^{n+1}$$

$$= \sum_{n=-\infty}^{\infty} A_{n-1}(q) z^n q^n.$$

$$\Longrightarrow A_n(q) = q^n A_{n-1}(q), \quad n \in \mathbb{Z}.$$

By iteration,

$$A_n(q) = q^{\frac{n(n+1)}{2}} A_0(q)$$

Hence, $$\varphi(z) = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} A_0(q) z^n$$

Claim: $$A_0(q) = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)}.$$