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MA 633 – Theory of partitions – Lec. 39

By iteration,

$$A_n(q) = q^{\frac{n(n+1)}{2}} A_0(q)$$

Hence, $\varphi(z) = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} A_0(q) z^n$

<u>Claim</u>: $A_0(q) = \dfrac{1}{\prod_{n=1}^{\infty}(1-q^n)}$.

Proof of the claim:

Note that $\varphi(z) := \prod_{n=1}^{\infty} (1+zq^n)(1+z^{-1}q^{n-1})$.

We first understand how the constant term $z^0 q^N$ arise. It arises exactly as many times as one gets the pairs of terms $q^{a_1+a_2+\cdots+a_m} z^m$ with $a_1 > a_2 > \cdots > a_m \geq 1$ & $q^{b_1+b_2+\cdots+b_m} z^{-m}$, with $b_1 > b_2 > \cdots > b_m \geq 0$, and with added condition that

$$a_1 + a_2 + \cdots + a_m + b_1 + b_2 + \cdots + b_m = N.$$

In other words, each contribution to q^N in the constant term comes from each F-partition of N of the form

$$\begin{pmatrix} a_1-1 & a_2-1 & \cdots\cdots & a_m-1 \\ b_1 & b_2 & \cdots\cdots & b_m \end{pmatrix},$$

where $a_1-1 > a_2-1 > \cdots\cdots > a_m-1 \geq 0$
& $b_1 > b_2 > \cdots > b_m \geq 0$, &

$$N = m + \sum_{i=1}^{m}(a_i-1) + \sum_{i=1}^{m} b_i$$

Now observe ~~that~~ there is a 1-1 correspondence between the number of F-partitions of N & the number of ordinary partitions of N

Hence the constant term in $\varphi(z)$ is nothing but the gen. fn. for partitions, i.e;
$\frac{1}{(q)_\infty}$.

$$\Rightarrow A_0(q) = \frac{1}{(q;q)_\infty}.$$

The general principle

If $f_A(z,q) = f_A(z) = \sum p_A(m,n) z^m q^n$ denotes the generating function for $p_A(m,n)$, which is the number of ptns. of n into m parts subject to the restriction A, then the function $f_A(zq) f_B(z^{-1})$ has, as its constant term, the generating function

$$\Phi_{A,B}(q) = \sum_{n\geq 0} \phi_{A,B}(n) q^n,$$

where $\phi_{A,B}(n) =$ the number of F-partitions $\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$ in which the top row is subject to the set of restrictions A & the bottom row is subject to the set of restrictions B.

Example: In the proof of JTPI, $A = B = D$ where D is the set of restriction that the parts are distinct & non-negative.

$$\underbrace{\prod_{n=0}^{\infty}(1+zq^n)} = \sum_{m,n=0}^{\infty} P_d(m,n)\, z^m q^n .$$

Definitions and examples

① A_k denotes the condition that "each part is repeated at most k times".

$$\Phi_K(q) = \Phi_{A_k,A_k}(q) := \sum_{n=0}^{\infty} \phi_{A_k,A_k}(n)\, q^n = \sum_{n=0}^{\infty} \phi_k(n) q^n .$$

Note: $\phi_1(n) = p(n)$.

Now let us take an example of $\phi_2(3) =$ the number of F-partitions of 3 where the parts are allowed to repeat atmost twice, are given by