16/11/2021

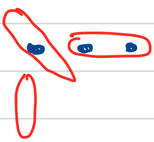
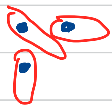
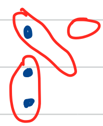
MA 633 – Theory of partitions – Lec. 40

① A_k denotes the condition that 'each part is repeated at most k times'.

$$\Phi_k(q) = \Phi_{A_k, A_k}(q) := \sum_{n=0}^{\infty} \phi_{A_k, A_k}(n)\, q^n = \sum_{n=0}^{\infty} \phi_k(n)\, q^n.$$

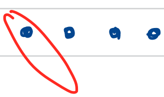
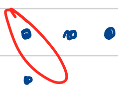
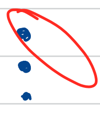
Note: $\phi_1(n) = p(n)$.

Now let us take an example of $\phi_2(3)$ = the number of F-partitions of 3 where the parts are allowed to repeat atmost twice, are given by

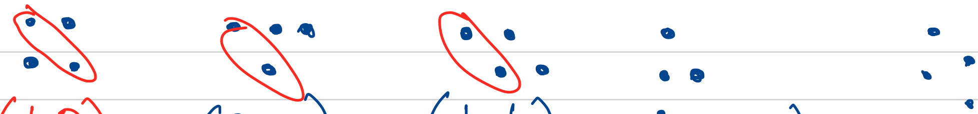
 $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$,  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,  $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

 $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  $\leftarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

② partitions enumerated by $\phi_3(4)$:

$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$



$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \implies \phi_3(4) = 11,$$

<u>Generating function for</u> $\Phi_k(q) := \sum_{n=0}^{\infty} \phi_k(n) q^n.$

<u>Thm. 53</u> For $|q|<1$,

$$\Phi_k(q) = \sum_{m_1, m_2, \ldots, m_{k-1} = -\infty}^{\infty} \frac{\zeta^{(k-1)m_1 + (k-2)m_2 + \cdots + m_{k-1}} \, q^{Q(m_1, m_2, \ldots, m_{k-1})}}{\prod_{n=1}^{\infty} (1-q^n)^k},$$

where $\zeta = \exp\left(\frac{2\pi i}{k+1}\right)$ &

$$Q(m_1, m_2, \ldots, m_{k-1}) = m_1^2 + \cdots + m_{k-1}^2 + \sum_{1 \le i < j \le k-1} m_i m_j.$$

<u>Proof:</u> (Note that $\Phi_1(q)$ is the constant term (w.r.t. z) in $\prod_{n=1}^{\infty} (1+zq^n)(1+z^{-1}q^{n-1})$.

Similarly, $\Phi_k(q)$ is the constant term in

$$\prod_{n=1}^{\infty}\left(1+zq^n+z^2q^{2n}+\dots+z^kq^{kn}\right)\left(1+z^{-1}q^{n-1}+z^{-2}q^{2(n-1)}+\dots+z^{-k}q^{k(n-1)}\right)$$

$$=\prod_{n=1}^{\infty}\frac{1-z^{k+1}q^{(k+1)n}}{1-zq^n}\cdot\frac{1-z^{-(k+1)}q^{(k+1)(n-1)}}{1-z^{-1}q^{n-1}}$$

[If $\zeta=e^{\frac{2\pi i}{k+1}}$, then

$$\prod_{j=0}^{k}(1-\zeta^j x)=1-x^{k+1}$$

(Example: $(1-x)(1-\omega x)(1-\omega^2x)=(1-x^3)$
where $\omega=e^{2\pi i/3}$)]

$$=\prod_{n=1}^{\infty}\frac{\prod_{j=0}^{k}(1-\zeta^j zq^n)(1-\zeta^j z^{-1}q^{n-1})}{(1-zq^n)(1-z^{-1}q^{n-1})}$$

$$=\prod_{n=1}^{\infty}\prod_{j=1}^{k}(1-\zeta^j zq^n)(1-\zeta^j z^{-1}q^{n-1})$$

$$=\prod_{n=1}^{\infty}\prod_{j=1}^{k}(1-\zeta^j zq^n)(1-\bar{\zeta}^j z^{-1}q^{n-1})$$

(replacing j by $k+1-j$ only in the second product)

$$= \prod_{j=1}^{k} \prod_{n=1}^{\infty} (1 - \zeta^{j} z q^{n})(1 - \zeta^{-j} z^{-1} q^{n-1})$$

$$= \prod_{j=1}^{k} (\zeta^{j} z q)_{\infty} (\zeta^{-j} z^{-1})_{\infty}$$

$$= \frac{1}{(q;q)_{\infty}^{k}} \prod_{j=1}^{k} \left[(\zeta^{j} z q)_{\infty} (\zeta^{-j} z^{-1})_{\infty} (q)_{\infty} \right]$$

$$= \frac{1}{(q;q)_{\infty}^{k}} \prod_{j=1}^{k} \sum_{m_j=-\infty}^{\infty} (-\zeta^{j} z)^{m_j} q^{\frac{m_j(m_j+1)}{2}}$$

(From JTPI:

$$\sum_{n=-\infty}^{\infty} x^{n} q^{\frac{n(n+1)}{2}} = (-xq;q)_{\infty} (-x^{-1})_{\infty} (q)_{\infty} \cdot)$$