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MA 633 – Theory of partitions – Lec. 41

$$= \prod_{j=1}^{k} \prod_{n=1}^{\infty} (1-\zeta^j z q^n)(1-\zeta^{-j} z^{-1} q^{n-1})$$

$$= \prod_{j=1}^{k} (\zeta^j z q)_\infty (\zeta^{-j} z^{-1})_\infty$$

$$= \frac{1}{(q;q)_\infty^k} \prod_{j=1}^{k} \left[(\zeta^j z q)_\infty (\zeta^{-j} z^{-1})_\infty (q)_\infty \right]$$

$$= \frac{1}{(q;q)_\infty^k} \prod_{j=1}^{k} \sum_{m_j=-\infty}^{\infty} (-\zeta^j z)^{m_j} q^{\frac{m_j(m_j+1)}{2}} .$$

(From JTPI :

$$\sum_{n=-\infty}^{\infty} x^n q^{\frac{n(n+1)}{2}} = (-xq;q)_\infty (-x^{-1})_\infty (q)_\infty .)$$

To extract the constant term from the above product, one has to let the exponent of z to be 0, i.e. $m_1 + m_2 + \cdots + m_k = 0$

$\iff m_k = -m_1 - m_2 - \cdots - m_{k-1}$.

Hence

$$\Phi_k(q) = \frac{1}{(q)_\infty^k} \sum_{m_1,\dots,m_{k-1}=-\infty}^{\infty} \zeta^{m_1+2m_2+\cdots+(k-1)m_{k-1}+k(-m_1-m_2-\cdots-m_{k-1})}$$

$$\times q^{\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+\cdots+\frac{m_{k-1}(m_{k-1}+1)}{2}+\frac{(-m_1-\cdots-m_{k-1})(-m_1-\cdots-m_{k-1}+1)}{2}}$$

Note that

$$m_1+2m_2+\cdots+(k-1)m_{k-1}+k(-m_1-m_2-\cdots-m_{k-1})$$
$$= -(k-1)m_1-(k-2)m_2-\cdots-m_{k-1} \quad \&$$

$$\& \quad \sum_{\ell=1}^{k-1}\frac{m_\ell(m_\ell+1)}{2}+\frac{(-m_1-\cdots-m_{k-1})(-m_1-\cdots-m_{k-1}+1)}{2}$$

$$= m_1^2+m_2^2+\cdots+m_{k-1}^2+\sum_{1\le i<j\le k-1} m_i m_j.$$

We prove the latter by induction.

Base case: • $k=1$ trivial $(0=0)$

• $k=2$: $\text{LHS} = \frac{m_1(m_1+1)}{2}+\frac{(-m_1)(-m_1+1)}{2}$

$$= m_1^2 = \text{RHS}.$$

Assume the given statement for $k-1$ & prove it for k, i.e. to show

$$\sum_{\ell=1}^{k}\frac{m_\ell(m_\ell+1)}{2}+\frac{(-m_1-\cdots-m_{k-1}-m_k)(-m_1-\cdots-m_{k-1}-m_k+1)}{2}$$

$$= m_1^2+m_2^2+\cdots+m_{k-1}^2+m_k^2+\sum_{1\le i<j\le k} m_i m_j.$$

$$\text{LHS} = \sum_{l=1}^{k-1} \frac{m_l(m_l+1)}{2} + \frac{m_k(m_k+1)}{2}$$

$$+ \frac{(-m_1 - \cdots - m_{k-1})(-m_1 - \cdots - m_{k-1}+1)}{2}$$

$$+ (-m_1 - \cdots - m_{k-1})(-m_k) + \frac{m_k^2 - m_k}{2}$$

Induction hyp.

$$m_1^2 + \cdots + m_{k-1}^2 + m_k^2 + \sum_{1 \le i < j \le k} m_i m_j .$$

Thus the result is proved by induction.

To prove the theorem, the last step is to replace m_i's by $-m_i$'s for all $1 \le i \le k$. $\boxtimes$

Special cases of the above theorem.

① $$\Phi_1(q) = \frac{1}{(q;q)_\infty}$$

② $$\Phi_2(q) = \frac{1}{(q;q)_\infty (q^2;q^{12})_\infty (q^3;q^{12})_\infty (q^9;q^{12})_\infty (q^{10};q^{12})_\infty},$$

The Rogers-Ramanujan continued fraction (RRCF)

High school: Find the value of $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$.

Let $x = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$.

$$= \frac{1}{x} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cdots}} = 1+x$$

$$\Rightarrow \frac{1}{x} = 1+x$$

$$\Rightarrow x^2+x-1=0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1+4}}{2}$$

But $x>0 \Rightarrow x = \dfrac{-1+\sqrt{5}}{2}$,

Def.
RRCF: $R(q) := \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cdots}}}$ $\quad |q|<1$,

Notation:

$$R(q) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cdots}}}$$

Thm. 54 (Rogers–Ramanujan)

Let $G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}$ & $H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n}$

be the Rogers–Ramanujan functions.

Then $R(q) = q^{1/5} \frac{H(q)}{G(q)}$

$$= \frac{q^{1/5} (q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

(from Rogers–Ramanujan identities)