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MA 633 – Partition theory Lec. 42

- $$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}$$

- $n^2 = 1+3+5+\cdots+(2n-1)$
- $$\frac{1}{(q)_n} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$

↳ generates partitions into parts whose number is $\le n$, i.e. say,

$$(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m), \qquad m \le n.$$

Suppose we pad zeros at the end of the above partition to make exactly n parts

$$(\lambda_1, \lambda_2, \ldots, \lambda_m, \underbrace{0, 0, 0 \ldots 0}_{n-m \text{ parts}})$$

$$\begin{array}{lll}
\lambda_1 & + & 2n-1 \\
\lambda_2 & + & 2n-3 \\
\vdots & & \vdots \\
\lambda_m & + & \vdots \\
0 & & \vdots \\
0 & & \\
0 & + & 5 \\
0 & + & 3 \\
 & + & 1
\end{array}$$

Hence $\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}$ generates partitions in which parts differ by at least 2.

• Göllnitz – Gordon :

$$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k}{(q^2;q^2)_k}\, q^{k^2}$$

$(-q;q^2)_k$	$\frac{1}{(q^2;q^2)_k}$	q^{k^2}
odd + distinct $\to \lambda_1$	even $\to \mu_1$	$2k-1$
$\to \lambda_2$	$\to \mu_2$	$2k-3$
$\to \lambda_3$	$\to \mu_3$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$\to \lambda_m$	$\vdots$	$\vdots$
$k-m$ $\left\{ 0 \right.$	$\vdots$	7
0		5
0	$\vdots$	3
0	μ_k	1
$\vdots$		

Thm. 54

$$\Phi_2(q) = \frac{1}{(q;q)_\infty (q^2;q^{12})_\infty (q^3;q^{12})_\infty (q^9;q^{12})_\infty (q^{10};q^{12})_\infty}$$

Proof: We know from the proof of Thm. 53 that

$\Phi_k(q)$ is the constant term in the Laurent series expansion of

$$\frac{1}{(q;q)_\infty^k} \prod_{j=1}^{k} \sum_{m_j=-\infty}^{\infty} (-\zeta^j z)^{m_j} q^{\frac{m_j(m_j+1)}{2}}.$$

Let $k=2$. Then $m_2 = -m_1$. Hence,

$$\Phi_2(q) = \frac{1}{(q;q)_\infty^2} \sum_{m_1,m_2=-\infty}^{\infty} (-1)^{m_1+m_2} \zeta^{m_1+2m_2} q^{\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}}$$

$$= \frac{1}{(q;q)_\infty^2} \sum_{m_1=-\infty}^{\infty} \zeta^{m_1-2m_1} q^{\frac{m_1(m_1+1)+(-m_1(-m_1+1))}{2}}$$

$$= \frac{1}{(q;q)_\infty^2} \sum_{m_1=-\infty}^{\infty} \zeta^{-m_1} q^{m_1^2}$$

$$= \frac{1}{(q;q)_\infty^2} \sum_{m_1=-\infty}^{\infty} \zeta^{m_1} q^{m_1^2} \qquad (m_1 \to -m_1)$$

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$$= \frac{1}{(q;q)_\infty^2} (-\zeta q; q^2)_\infty (-\zeta^{-1} q; q^2)_\infty (q^2; q^2)_\infty$$

(Note that $f(\zeta q, \zeta^{-1} q)$

$$= \sum_{n=-\infty}^{\infty} (\zeta q)^{\frac{n(n+1)}{2}} (\zeta^{-1} q)^{\frac{n(n-1)}{2}}$$

$$= \sum_{n=-\infty}^{\infty} \zeta^n q^{n^2} \Big)$$

$$= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \prod_{n=1}^{\infty} (1+\zeta q^{2n-1})(1+\zeta^{-1} q^{2n-1})$$

$$= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \prod_{n=1}^{\infty} (1+(\zeta+\zeta^{-1}) q^{2n-1} + q^{4n-2})$$

$$= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \prod_{n=1}^{\infty} (1-q^{2n-1}+q^{4n-2})$$

$$\Big(\because \zeta+\zeta^{-1} = e^{2\pi i/3} + e^{-2\pi i/3} = 2\cos(2\pi/3)$$

$$= -2\cos(\pi/3) = -2 \cdot \frac{1}{2} = -1 \Big)$$

$$= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \prod_{n=1}^{\infty} \frac{(1+q^{6n-3})}{(1+q^{2n-1})}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \frac{(-q^3;q^6)_\infty}{(-q;q^2)_\infty} \frac{(q^3;q^6)_\infty}{(q^3;q^6)_\infty}$$

$$= \frac{\cancel{(q^2;q^2)_\infty}(q^6;q^{12})_\infty}{(q;q)_\infty \underbrace{(q;q^2)_\infty \cancel{(q^2;q^2)_\infty}}_{=(q;q)_\infty} (-q;q^2)_\infty (q^3;q^6)_\infty}$$

$$= \frac{(q^6;q^{12})_\infty}{(q;q)_\infty (q^2;q^4)_\infty (q^3;q^6)_\infty}$$

$$= \frac{\cancel{(q^6;q^{12})_\infty}}{\left\{ \begin{array}{l} (q;q)_\infty (q^2;q^{12})_\infty \cancel{(q^6;q^{12})_\infty} (q^{10};q^{12})_\infty \\ (q^3;q^{12})_\infty (q^9;q^{12})_\infty \end{array} \right\}}$$

$$= \frac{1}{(q;q)_\infty (q^2;q^{12})_\infty (q^3;q^{12})_\infty (q^9;q^{12})_\infty (q^{10};q^{12})_\infty}$$

$\blacksquare$