12/8/21

MA 633 – Partition Theory – Lec. 6

Cor. 7 $\varphi(q) = (-q;q^2)_\infty^2 (q^2;q^2)_\infty$

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

Proof: $f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty$

$$\varphi(q) = f(q,q) = (-q;q^2)_\infty^2 (q^2;q^2)_\infty .$$

$$\psi(q) = f(q,q^3) = (-q;q^4)_\infty (-q^3;q^4)_\infty (q^4;q^4)_\infty$$

$$= (-q;q^2)_\infty (-q^2;q^2)_\infty (q^2;q^2)_\infty$$

$$= (-q;q)_\infty (q^2;q^2)_\infty$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} .$$

$$f(-q) = f(-q, -q^2)$$
$$= (q;q^3)_\infty (q^2;q^3)_\infty (q^3;q^3)_\infty$$
$$= (q;q)_\infty$$

Now $f(-q) = \sum_{n=-\infty}^{\infty} (-q)^{\frac{n(n+1)}{2}} (-q^2)^{\frac{n(n-1)}{2}}$

$$= \sum_{n=-\infty}^{\infty} (-1)^{n^2} q^{\frac{3n^2-n}{2}} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}$$

Hence we get

Cor. 8 (Euler's pentagonal number theorem)

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}}$$

Cor. 9 of (Euler's PNT)

Let $w_j = \frac{j(3j-1)}{2}$, $-\infty < j < \infty$.

Then
$$p(n) = \sum_{0 < w_j \le n} (-1)^{j+1} p(n - w_j).$$

Proof: $1 = (q;q)_\infty \cdot \frac{1}{(q;q)_\infty}$

$$= \left(\sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(3j-1)}{2}}\right) \cdot \left(\sum_{m=0}^{\infty} p(m) q^m\right)$$

$$= \sum_{m=0}^{\infty} p(m) q^m + \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \sum_{m=0}^{\infty} (-1)^j p(m) q^{m+\omega_j}$$

$$= \sum_{m=0}^{\infty} p(m) q^m - \sum_{n=1}^{\infty} \sum_{0<\omega_j\leq n} (-1)^{j+1} p(n-\omega_j) q^n$$

Since $\left\{ \left(\sum_{m=0}^{\infty} a(m) z^m\right)\left(\sum_{k=0}^{\infty} b(k) \cdot z^k\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} b(k) a(n-k) z^n \right\}$

$$\Rightarrow 1 = 1 + \sum_{n=1}^{\infty} p(n) q^n - \sum_{n=1}^{\infty} q^n \left(\sum_{0<\omega_j\leq n} (-1)^{j+1} p(n-\omega_j)\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} p(n) q^n = \sum_{n=1}^{\infty} \left(\sum_{0<\omega_j\leq n} (-1)^{j+1} p(n-\omega_j)\right) q^n$$

Now the result follows by comparing the coeff. of q^n, $n \geq 1$, on both sides. $\blacksquare$

Cor. 10 (Legendre's combinatorial version of Euler's PNT).

Let $D_e(n)$ (resp. $D_o(n)$) denotes the number of partitions of n into distinct parts where the number of parts is even (resp. odd).

Then
$$D_e(n) - D_o(n) = \begin{cases} (-1)^j, & \text{if } n = \frac{j(3j \pm 1)}{2} \\ 0, & \text{else}, \end{cases}$$

<u>Example:</u> $n = 5$

5 ✓
4+1 ✓
3+2 ✓
3+1+1
2+2+1
2+1+1+1
1+1+1+1+1

$D_e(5) = 2$
$D_o(5) = 1$

$D_e(5) - D_o(5) = 1$,

since $5 = \frac{2(3(2)-1)}{2}$

<u>$n = 6$</u> 6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1

$D_e(6) = 2 = D_o(6)$; note that 6 is $\boxed{\text{NOT}}$ a gen. pentagonal number

Proof: Let us first examine

$$(q;q)_\infty = (1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)\cdots \quad (*)$$

$$\Rightarrow (q;q)_\infty = 1 + \sum_{n=1}^{\infty} (D_e(n) - D_o(n))\, q^n, \quad (A)$$

since we get a plus sign when we multiply even number of powers of q in $(*)$ & a minus sign when we multiply odd numbers of q.

From Euler's PNT & eqn. (A),

$$1 + \sum_{n=1}^{\infty} (D_e(n) - D_o(n))\, q^n = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(3j-1)}{2}},$$

$$= 1 + \sum_{m=1}^{\infty} a(m)\, q^m,$$

corresponds to the $j=0$ term

where $a(m) = \begin{cases} (-1)^j, & \text{if } m = \frac{j(3j\pm 1)}{2} \\ 0, & \text{else} \end{cases}$

$\Rightarrow$ For all $n \geq 1$,

$$D_e(n) - D_o(n) = \begin{cases} (-1)^j, & n = \frac{j(3j\pm 1)}{2} \\ 0, & \text{else}, \end{cases}$$

$\boxtimes$

Replace q by q^2 & then z by $-q/z$ in the above formula so that for $\left|-\frac{q}{z}\right|<1$, we have

$$\sum_{m=0}^{\infty} \frac{(-q/z)^m}{(q^2;q^2)_m} = \frac{1}{(-q/z;q^2)_\infty}.$$

Hence the theorem is proved for $|q|<|z|$

By analytic continuation, the result follows for all $z\neq 0$.