13/8/21

MA 633 – Partition Theory – Lec. 7

• **q-binomial thm.:** For $|z|<1$ & $|q|<1$, then

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty}.$$

$\boxed{\text{Thm. 11}}$

• **Ramanujan's $_1\psi_1$ summation formula**

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty \left(\frac{q}{az}\right)_\infty (q)_\infty (b/a)_\infty}{(z)_\infty \left(\frac{b}{az}\right)_\infty (b)_\infty (q/a)_\infty}$$

when $\left|\frac{b}{a}\right| < |z| < 1$ & $|q|<1$,

Proof: Let $f(z) = \dfrac{(az)_\infty \left(\frac{q}{az}\right)_\infty}{(z)_\infty \left(\frac{b}{az}\right)_\infty}$. ——— (1)

[Note $|z|<1$ & $\left|\frac{b}{az}\right|<1$ ensure that the denominator of $f(z)$ is never zero.]

Hence $f(z)$ is analytic in $\left|\frac{b}{a}\right| < |z| < 1$, which is an annulus and hence, can be expanded as the Laurent series, i.e.;

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

Replace z by qz in (1) to get

$$f(qz) = \frac{(aqz)_\infty \left(\frac{1}{az}\right)_\infty}{(qz)_\infty \left(\frac{b}{aqz}\right)_\infty}.$$

$$= \frac{(1-z)\left(1-\frac{1}{az}\right)}{(1-az)\left(1-\frac{b}{aqz}\right)} f(z)$$

$$= (1-z)\frac{\frac{-1}{az}}{\left(\frac{aqz-b}{aqz}\right)} f(z)$$

$$\Rightarrow \quad q(1-z)f(z) = (b-aqz)f(qz) \quad \text{— (2)}$$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad \text{on} \quad \left|\frac{b}{a}\right| < |z| < 1.$$

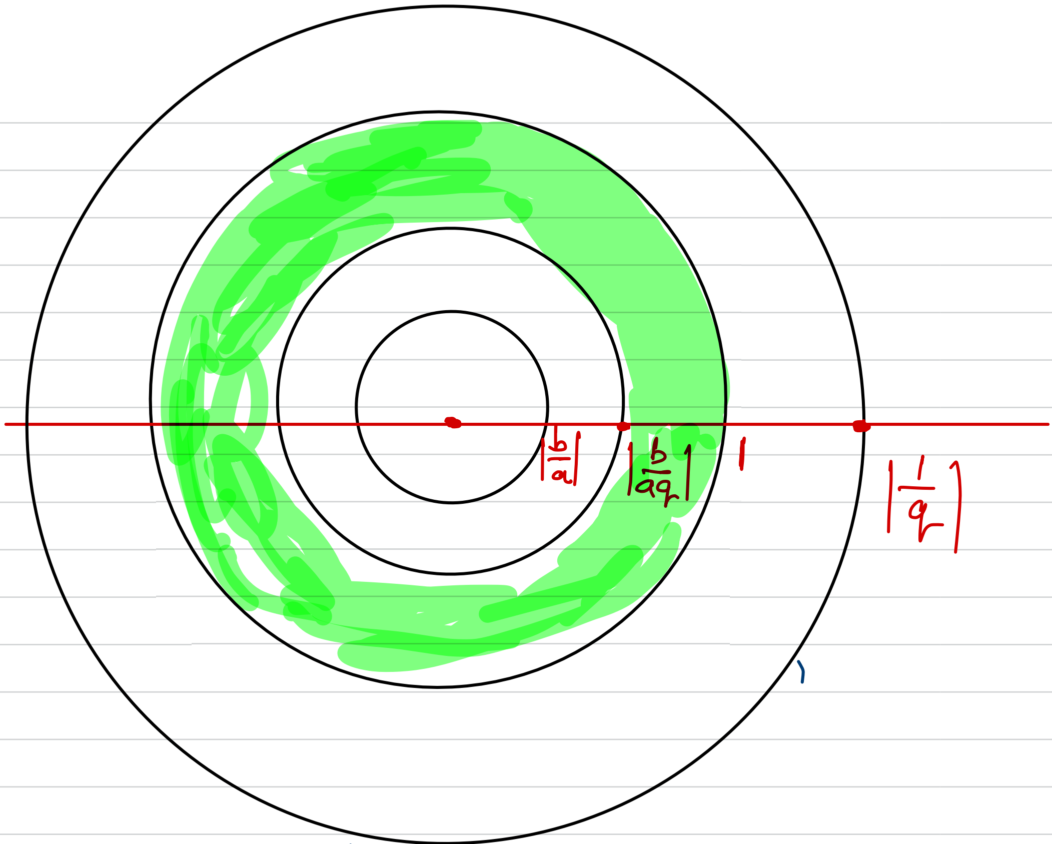
Similarly,

$$f(qz) = \sum_{n=-\infty}^{\infty} c_n q^n z^n \quad \text{on} \quad \left|\frac{b}{aq}\right| < |z| < \frac{1}{|q|}$$

Hence $q(1-z)\sum_{n=-\infty}^{\infty} c_n z^n = (b-aqz)\sum_{n=-\infty}^{\infty} c_n q^n z^n$

holds on the intersection of the 2 annuli,

i.e. $\left|\frac{b}{aq}\right| < |z| < 1$



Thus we require an extra condition at this point of time, i.e. $\left|\frac{b}{a}\right| < |q|$, but we will later remove it by analytic continuation.

We have

$$q(1-z)\sum_{n=-\infty}^{\infty} c_n z^n = (b - aqz)\sum_{n=-\infty}^{\infty} c_n q^n z^n .$$

$$\Rightarrow \quad q c_n - q c_{n-1} = b c_n q^n - a q^n c_{n-1}$$

$$\Rightarrow q c_n (1 - bq^{n-1}) = q c_{n-1} (1 - aq^{n-1})$$

$$\Rightarrow c_n = \left(\frac{1-aq^{n-1}}{1-bq^{n-1}}\right) c_{n-1} \longleftarrow (3)$$

Definition: For any $n \in \mathbb{Z}$,

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty} . \longrightarrow (*)$$

If $n = 0$ ✓

If $n > 0$, $\text{RHS} = \dfrac{(1-a)(1-aq)\cdots(1-aq^{n-1})(aq^n)_\infty}{(aq^n)_\infty}$

$$= (a)_n$$

If $n < 0$, say, $n = -m$, $m > 0$,

$$\frac{(a)_\infty}{(aq^n)_\infty} = \frac{(a)_\infty}{(aq^{-m})_\infty}$$

$$= \frac{(a)_\infty}{\left(1-\frac{a}{q^m}\right)\left(1-\frac{a}{q^{m-1}}\right)\cdots\left(1-\frac{a}{q}\right)(a)_\infty}$$

$$= \frac{1}{\left(1-\frac{a}{q^{-n}}\right)\left(1-\frac{a}{q^{-n-1}}\right)\cdots\left(1-\frac{a}{q}\right)}$$

$$= (a)_n .$$

From ③, for $n>0$,

$$c_n = \left(\frac{1-aq^{n-1}}{1-bq^{n-1}}\right)\left(\frac{1-aq^{n-2}}{1-bq^{n-2}}\right)\cdots\left(\frac{1-a}{1-b}\right)\cdot c_0$$

$$\Rightarrow c_n = \frac{(a)_n}{(b)_n} c_0 \quad \text{for } n>0.$$

From ③, for $n<0$,

$$c_{n-1} = \left(\frac{1-bq^{n-1}}{1-aq^{n-1}}\right) c_n$$

& iterate to get

$$c_{n-1} = \frac{(1-bq^{n-1})(1-bq^{n})\cdots\cdots(1-bq^{-1})}{(1-aq^{n-1})(1-aq^{n})\cdots(1-aq^{-1})} c_0$$

$$= \frac{\dfrac{1}{\left(1-\frac{a}{q^{-n+1}}\right)\left(1-\frac{a}{q^{-n}}\right)\cdots\left(1-\frac{a}{q}\right)}}{\dfrac{1}{\left(1-\frac{b}{q^{-n+1}}\right)\left(1-\frac{b}{q^{-n}}\right)\cdots\left(1-\frac{b}{q}\right)}}\cdot c_0$$

Replacing n by $n+1$, we have

$$c_n = \frac{\dfrac{1}{\left(1-\frac{a}{q^{-n}}\right)\left(1-\frac{a}{q^{-n+1}}\right)\cdots\left(1-\frac{a}{q}\right)}}{\dfrac{1}{\left(1-\frac{b}{q^{-n+1}}\right)\left(1-\frac{b}{q^{-n}}\right)\cdots\left(1-\frac{b}{q}\right)}}\cdot c_0$$

$$\Rightarrow c_n = \frac{(a)_n}{(b)_n} c_0 \qquad (\text{from } \circledast)$$

Hence from ①, we have

$$c_0 \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty}{(z)_\infty} \frac{\left(\frac{q}{az}\right)_\infty}{\left(\frac{b}{az}\right)_\infty} \quad \text{for } \left|\frac{b}{aq}\right| < |z| < 1,$$

By analytic continuation, the above result holds for $\left|\frac{b}{a}\right| < |z| < 1$.

Goal: To prove $c_0 = \dfrac{(b)_\infty (q/a)_\infty}{(q)_\infty (b/a)_\infty}$.