$$\Rightarrow c_n = \frac{(a)_n}{(b)_n} c_0 \qquad (\text{from } \circledast)$$

Hence from ①, we have

$$c_0 \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty}{(z)_\infty} \frac{\left(\frac{q}{az}\right)_\infty}{\left(\frac{b}{az}\right)_\infty} \quad \text{for } \left|\frac{b}{aq}\right| < |z| < 1,$$

By analytic continuation, the above result holds for $\left|\frac{b}{a}\right| < |z| < 1$.

Goal: To prove $C_0 = \dfrac{(b)_\infty (q/a)_\infty}{(q)_\infty (b/a)_\infty}$.

Proof:

$$f(z) = c_0 \sum_{n=-\infty}^{-1} \frac{(a)_n}{(b)_n} z^n + c_0 \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^n$$

$$=: c_0 \left(g(z) + h(z) \right)$$

④

$f(z)$ has a simple pole at $z=1$.

Also, $g(z) = \displaystyle\sum_{n=1}^{\infty} \frac{(a)_{-n}}{(b)_{-n}} \cdot \left(\frac{1}{z}\right)^n$

Now $g(z)$ is a power series about $z=\infty$ converging $|z| > \left|\frac{b}{a}\right|$, and since $|z|<1$ | lies in the neighbourhood of infinity.

Hence $g(z)$ is analytic at $z=1$. —— **

Multiplying both sides of (4) by $1-z$ & then letting $z \to 1$, we get

$$\lim_{z\to 1} (1-z) f(z) = \lim_{z\to 1} c_0 (1-z) g(z) + \lim_{z\to 1} c_0 (1-z) h(z)$$

$$= c_0 \lim_{z\to 1} \underbrace{(1-z) h(z)}_{G(z)} \quad (\text{from } **)$$

(5)

The function $G(z)$ is

1. Analytic in $0 \leq |z| < \frac{1}{|q|}$.
2. The singularity at $z=1$ is removable.
3. The next largest pole of both $f(z)$ & $h(z)$ is at $z = \frac{1}{q}$.

$$G(z) = (1-z)h(z)$$

$$= (1-z)\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}z^n$$

$$= \left(\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}z^n - \sum_{n=1}^{\infty}\frac{(a)_{n-1}}{(b)_{n-1}}z^n\right)$$

$$= \left(\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}z^n - \sum_{n=0}^{\infty}\frac{(a)_{n-1}}{(b)_{n-1}}z^n\right),$$

with the understanding that $\frac{(a)_{-1}}{(b)_{-1}} = 0$.

$$= \sum_{n=0}^{\infty}\left(\frac{(a)_n}{(b)_n} - \frac{(a)_{n-1}}{(b)_{n-1}}\right)z^n.$$

$$G(z) = \lim_{N\to\infty}\sum_{n=0}^{N}\left(\frac{(a)_n}{(b)_n} - \frac{(a)_{n-1}}{(b)_{n-1}}\right)z^n$$

Since $\lim_{z\to 1} G(z) = G(1)$,

$$= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{(a)_n}{(b)_n} - \frac{(a)_{n-1}}{(b)_{n-1}}$$

$$= \lim_{N\to\infty}\left(\frac{(a)_N}{(b)_N} - \frac{(a)_{N-1}}{(b)_{N-1}} + \frac{(a)_{N-1}}{(b)_{N-1}} - \frac{(a)_{N-2}}{(b)_{N-2}}\right.$$

$$+ \cdots\cdots + \left(\frac{(a)_0}{(b)_0} - \frac{(a)_{-1}}{(b)_{-1}}\right)$$

$$= \lim_{N\to\infty} \frac{(a)_N}{(b)_N} = \frac{(a)_\infty}{(b)_\infty}$$

But $\lim_{z\to 1} (1-z)f(z) = \lim_{z\to 1} (1-z)\frac{(az)_\infty}{(z)_\infty} \frac{\left(\frac{q}{az}\right)_\infty}{\left(\frac{b}{az}\right)_\infty}$

$$= \lim_{z\to 1} \frac{(az)_\infty}{(zq)_\infty} \frac{\left(\frac{q}{az}\right)_\infty}{\left(\frac{b}{az}\right)_\infty}$$

$$= \frac{(a)_\infty}{(q)_\infty} \frac{(q/a)_\infty}{(b/a)_\infty}.$$

Hence from (5),

$$\frac{(a)_\infty}{(q)_\infty} \frac{(q/a)_\infty}{(b/a)_\infty} = \frac{c_0 (a)_\infty}{(b)_\infty}$$

$$\Rightarrow \boxed{c_0 = \frac{(b)_\infty}{(q)_\infty} \frac{(q/a)_\infty}{(b/a)_\infty},}$$

This proves the result.

Thm 12 (Jacobi's identity for $(q;q)_\infty^3$)

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1)\, q^{n(n+1)/2} \quad \text{for } |q|<1$$

Proof: From JTPI,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty \qquad (*)$$

Replace z by z^2q to get

$$\sum_{n=-\infty}^{\infty} z^{2n} q^{n^2+n} = (-z^2q^2;q^2)_\infty(-z^{-2};q^2)_\infty(q^2;q^2)_\infty$$

$$= \left(1+\frac{1}{z^2}\right)(-z^2q^2;q^2)_\infty(-z^{-2}q^2;q^2)_\infty(q^2;q^2)_\infty$$

$$\Rightarrow \frac{\displaystyle\sum_{n=-\infty}^{\infty} z^{2n} q^{n^2+n}}{\left(1+\frac{1}{z^2}\right)} = (-z^2q^2;q^2)_\infty(-z^{-2}q^2;q^2)_\infty(q^2;q^2)_\infty$$

for $z \neq 0$.

Multiply both Nr. & Dr. of LHS by z so that

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} z^{2n+1} q^{n+n}}{\left(z+\frac{1}{z}\right)} = (-z^2q^2;q^2)_\infty(-z^{-2}q^2;q^2)_\infty(q^2;q^2)_\infty$$

Let $z \to i$ on both sides. Then

$$\text{RHS} = (q^2;q^2)^3_\infty .$$

$$\text{LHS} = \lim_{z \to i} \frac{\sum_{n=-\infty}^{\infty} z^{2n+1} q^{n^2+n}}{z + 1/z} \qquad \left(\frac{0}{0} \text{ form}\right)$$

because by JTPI,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n} = 0 \qquad \left(\begin{array}{l}\text{to see this,} \\ \text{let } z = -q \text{ in } (*)\end{array}\right)$$

Hence by L'Hôpital's rule,

$$\text{LHS} = \lim_{z \to i} \frac{\sum_{n=-\infty}^{\infty} (2n+1) z^{2n} q^{n^2+n}}{1 - \frac{1}{z^2}} = \frac{1}{2} \sum_{n=-\infty}^{\infty} (2n+1)(-1)^n q^{n^2+n}$$

$$= \frac{1}{2} \left\{ \left(\sum_{n=-\infty}^{-1} + \sum_{n=0}^{\infty} \right) (-1)^n (2n+1) q^{n^2+n} \right\}$$

$n \to -n-1$

$$= \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n^2+n}$$

Hence

$$\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n^2+n} = (q^2;q^2)^3_\infty$$

Now q^2 by q to complete the proof. $\boxtimes$