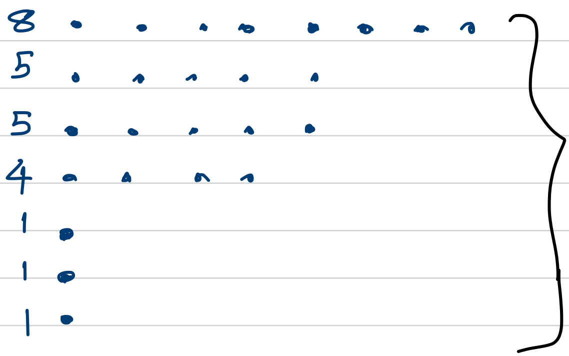
18/8/21

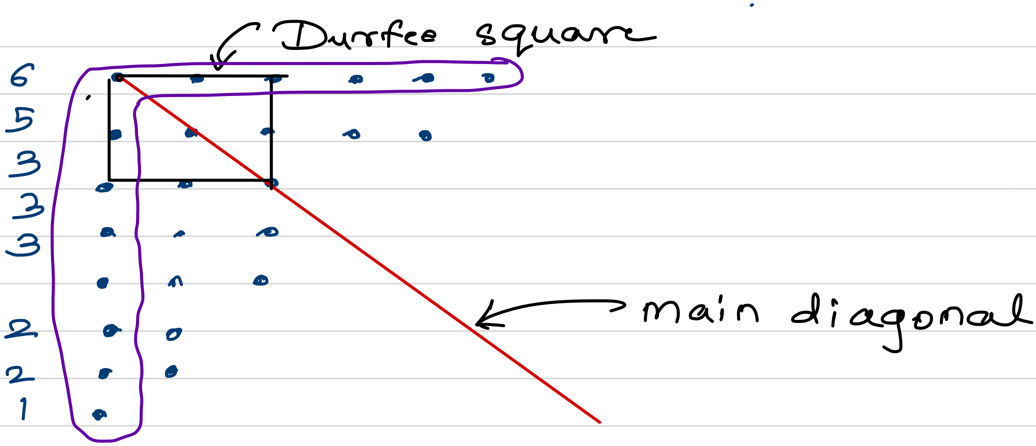
MA 633 – Partition Theory – Lec. 9

- Partitions represented through Ferrers diagrams

Consider the partition: $8+5+5+4+1+1+1$



Ferrers diagram of the partition,



Conjugate of a partition:

6 • • • • • •
5 • • • • •
3 • • •
3 • • •
2 • •
2 • •
1 •

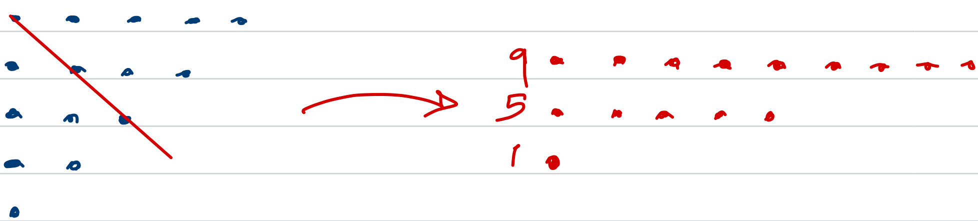
reflecting along the main diagonal gives

7 • • • • • • •
6 • • • • • •
4 • • • •
2 • •
2 • •
1 •

Conjugate of $(6,5,3,3,2,2,1)$ is $(7,6,4,2,2,1)$

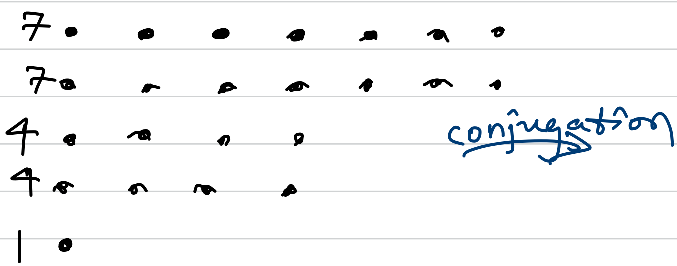
Defn. If $\lambda = (\lambda_1, \lambda_2, ---, \lambda_n)$ is a partition then the partition $\lambda' = (\lambda_1', \lambda_2', ----, \lambda_m')$ by choosing λ_i' to be the number of parts of λ greater than or equal to i is called the conjugate of λ.

• **Thm.** The number of self-conjugate partitions of an integer n equals the number of partitions of n into distinct odd parts.



Thm. 13 The number of partitions of n with at most m parts equals the number of partitions of n in which no part exceeds m.

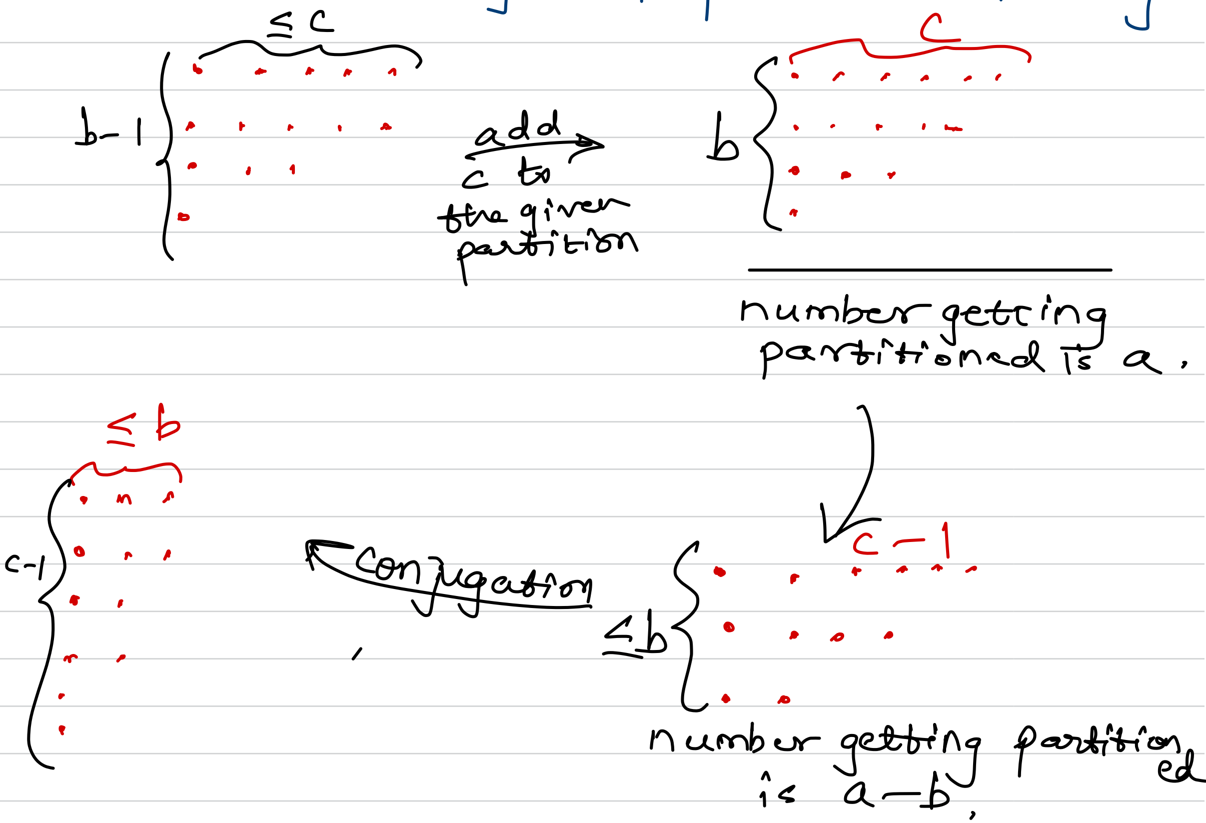
Proof:



Proof follows from the fact conjugation is a 1-1 correspondence ⊠

Thm. 14 The number of partitions of $a-c$ into exactly $b-1$ parts, not exceeding c equals the number of partitions of $a-b$ into exactly $c-1$ parts, not exceeding b.

Proof: Consider the following partition of $a-c$ into exactly $b-1$ parts not exceeding c,



Since each of the operations (maps) above are bijections we take their composition. ∎

Thm. 14 Franklin's proof of Euler's pentagonal number thm.

Let $p_e(\mathcal{D}, n)$ = number of partitions of n into even number of distinct parts.

Similarly $p_o(\mathcal{D}, n)$.

Then $p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n) = \begin{cases} (-1)^r, & \text{if } n = \frac{r(3r\pm1)}{2} \\ 0, & \text{else.} \end{cases}$

Proof:

We get an "almost" bijection between the sets enumerated by $p_e(\mathcal{D}, n)$ & $p_o(\mathcal{D}, n)$ which fails only when n is a generalized pentagonal number.

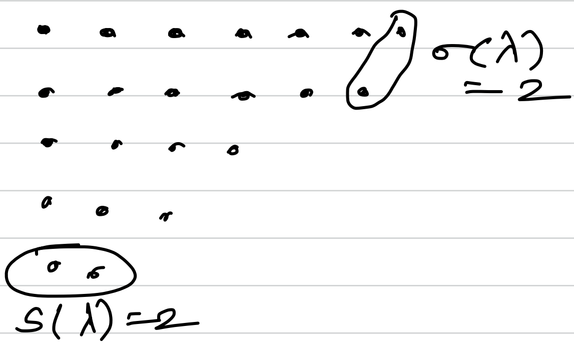
Consider a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$.

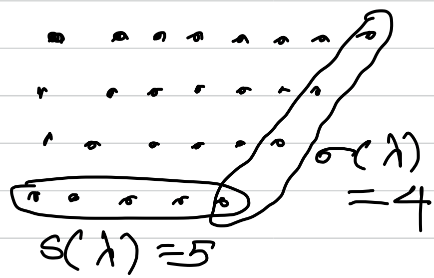
Let $s(\lambda) = \lambda_r$ (the smallest part of λ),

Largest part $= \lambda_1$.

$\sigma(\lambda) =$ the number consecutive integers in the partition beginning with λ_1.

$\lambda = (7, 6, 4, 3, 2)$ $\qquad\qquad$ $\lambda = (8\ 7\ 6\ 5)$



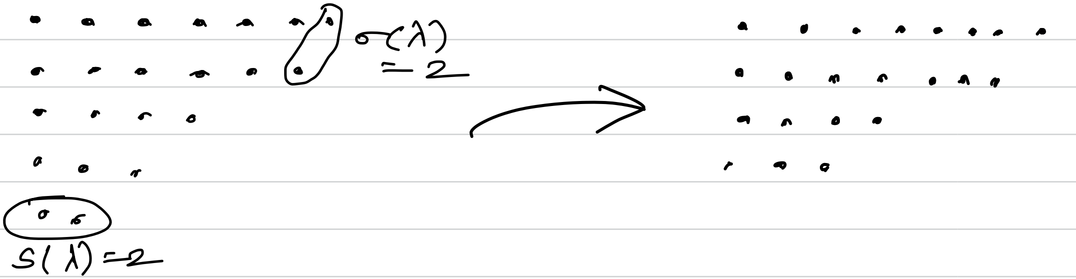


We now transform the partitions as follows:

Case 1:

(i) $s(\lambda) \leq \sigma(\lambda)$: We add one node to each of $s(\lambda)$ largest parts of λ and delete the smallest part.

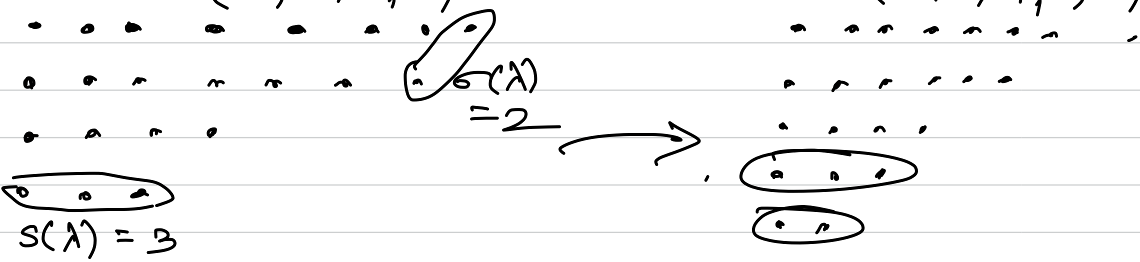
$\lambda = (7,6,4,3,2)$ $\qquad\qquad$ $\lambda = (8,7,4,3)$



Case 2

(ii) $s(\lambda) > \sigma(\lambda)$: In this case, we subtract one node from each of $\sigma(\lambda)$ largest parts of λ and insert a new smallest part of size $\sigma(\lambda)$.

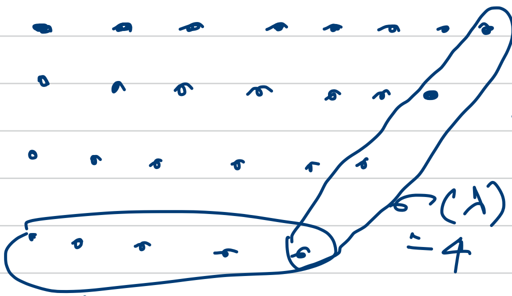
$\lambda = (8,7,4,3)$ $\qquad\qquad$ $\lambda = (7,6,4,3,2)$



These 2 rules change parity of the number of parts of the partition, and note that exactly one case is applicable to any partition λ, it seems as if we have obtained 1-1 corresp.

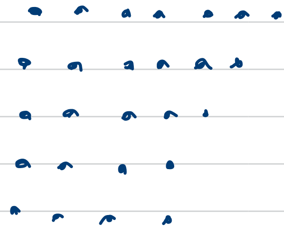
But it fails for certain partitions.

$\lambda = (8, 7, 6, 5) \longrightarrow (7, 6, 5, 4, 4)$



$\sigma(\lambda) = 4$

$s(\lambda) = 5$



not a partition into distinct parts!

This fails precisely when $s(\lambda) = r + 1$, $\sigma(\lambda) = r$.
The number we are partitioning here is

$$(r+1) + (r+2) + (r+3) + \cdots + (r+r)$$
$$= r^2 + \frac{r(r+1)}{2} = \frac{3r^2 + r}{2} = \frac{r(3r+1)}{2}.$$