11/8/2021

MA 633 – Tutorial 1

(Glaisher)

(2) Let $a(n)$ = # of ptns. of n in which no part is divisible by k.

$b(n)$ = # of ptns. of n in which there are strictly $< k$ copies of each part.

To show: $a(n) = b(n)$.

Proof:
$$\sum_{n=0}^{\infty} a(n) q^n = \prod_{n=0}^{\infty} \frac{1}{(1-q^{nk+1})(1-q^{nk+2})\cdots(1-q^{nk+k-1})}$$

$$= \frac{1}{(q;q^k)_\infty (q^2;q^k)_\infty \cdots (q^{k-1};q^k)_\infty}$$

$$= \frac{(q^k;q^k)_\infty}{(q;q)_\infty}$$

$$\sum b(n) q^n = \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + q^{3n} + \cdots + q^{(k-1)n})$$

$$= \prod_{n=1}^{\infty} \left(\frac{1-q^{nk}}{1-q^n} \right) = \frac{(q^k;q^k)_\infty}{(q;q)_\infty}.$$

$\blacksquare$

③ Let $a(n)$ = # ptns. of n in which each part appears exactly 2, 3 or 5 times

$b(n)$ = # of ptns. of n in which parts $\equiv \pm 2, \pm 3$ or $6 \pmod{12}$

Prove: $a(n) = b(n)$.

Proof: $$\sum_{n=0}^{\infty} a(n) q^n = \prod_{n=1}^{\infty} (1 + q^{2n} + q^{3n} + q^{5n}) \quad \text{— (1)}$$

$$\sum_{n=0}^{\infty} b(n) q^n = \prod_{n=0}^{\infty} \frac{1}{\{(1-q^{12n+2})(1-q^{12n+3})(1-q^{12n+6})(1-q^{12n+9})(1-q^{12n+10})\}}$$

$$= \frac{1}{(q^2, q^3, q^6, q^9, q^{10}; q^{12})_\infty} \quad \text{— (2)}$$

From (1),

$$\sum a(n) q^n = \prod_{n=1}^{\infty} (1+q^{2n})(1+q^{3n})$$

$$= \prod_{n=1}^{\infty} \left(\frac{1-q^{4n}}{1-q^{2n}}\right)\left(\frac{1-q^{6n}}{1-q^{3n}}\right)$$

$$= \frac{(q^4; q^4)_\infty (q^6; q^6)_\infty}{(q^2; q^2)_\infty (q^3; q^3)_\infty}$$

$$= \frac{(\cancel{q^4};q^{12})_\infty(\cancel{q^8};q^{12})_\infty(\cancel{q^{12}};q^{12})_\infty(\cancel{q^6};q^{12})_\infty(\cancel{q^{12}};q^{12})_\infty}{(q^2,\cancel{q^4},\cancel{q^6},\cancel{q^8},q^{10},\cancel{q^{12}};q^{12})_\infty(q^3,q^6,q^9,\cancel{q^{12}};q^{12})_\infty}$$

$$= \frac{1}{(q^2,q^3,q^6,q^9,q^{10};q^{12})_\infty}$$

$\boxtimes$

④ $a(n)$ = # ptns. of n in which no parts appear exactly once

$b(n)$ = # of ptns. of n into parts $\not\equiv \pm 1 \pmod 6$.

Show $a(n) = b(n)$.

<u>Proof</u>: $\displaystyle\sum_{n=0}^{\infty} a(n)q^n = \prod_{n=1}^{\infty}(1+q^{2n}+q^{3n}+q^{4n}+\dots)$

$$= \prod_{n=1}^{\infty}\left(\frac{1}{1-q^n} - q^n\right) = \prod_{n=1}^{\infty}\left(\frac{1-q^n+q^{2n}}{1-q^n}\right)$$

$$= \prod_{n=1}^{\infty}\left(\frac{1+q^{3n}}{1-q^{2n}}\right) = \frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}$$

$$\boxed{(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}}$$

$$= \frac{1}{(q^2,q^4,q^6;q^6)_\infty(q^3;q^6)_\infty}$$

$$= \sum_{n=1}^{\infty} b(n)q^n.$$

$\boxtimes$

⑤ $f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n-1)}{2}} b^{\frac{n(n+1)}{2}}$, $|ab|<1$

① $f(a,b) = f(b,a)$ trivial

② $f(1,a) = 2f(a,a^3)$.

Proof: 1st method:

$$f(1,a) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}$$

$$= \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} a^{\frac{n(n+1)}{2}} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} a^{\frac{n(n+1)}{2}}$$

$$= \sum_{n=-\infty}^{\infty} a^{\frac{2n(2n+1)}{2}} + \sum_{n=-\infty}^{\infty} a^{\frac{(2n-1)2n}{2}}$$

$$= \sum_{n=-\infty}^{\infty} a^{n(2n+1)} + \sum_{n=-\infty}^{\infty} a^{n(2n-1)}$$

replace n by $-n$

$$= 2\sum_{n=-\infty}^{\infty} a^{n(2n+1)}$$

$$f(a,a^3) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n-1)}{2} + \frac{3n(n+1)}{2}} = \sum_{n=-\infty}^{\infty} a^{n(2n+1)}.$$

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

<u>2nd method</u> <u>(JTPI)</u>:

$$f(1,a) = (-1;a)_\infty(-a;a)_\infty(a;a)_\infty \quad \text{--- (1)}$$

$$2f(a,a^3) = 2(-a;a^4)_\infty(-a^3;a^4)_\infty(a^4;a^4)_\infty \quad \text{--- (2)}$$

From (1),

$$f(1,a) = 2(-a;a)_\infty(-a;a)_\infty(a;a)_\infty$$

$$= 2(-a;a^2)_\infty(-a^2;a^2)_\infty(a^2;a^2)_\infty$$

$$= 2(-a;a^4)_\infty(-a^3;a^4)_\infty(a^4;a^4)_\infty$$

(3) $f(-1,a) = 0$.

JTPI implies

$$f(-1,a) = (1;-a)_\infty(-a;-a)_\infty(-a;-a)_\infty$$

$$= 0.$$

(4) For $n \in \mathbb{Z}$,

$$f(a,b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^{-n}).$$

Proof:

$$a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^{-n})$$

$$= a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \sum_{m=-\infty}^{\infty} \left(a(ab)^n\right)^{\frac{m(m-1)}{2}} \left(b(ab)^{-n}\right)^{\frac{m(m+1)}{2}}$$

$$= a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \sum_{m=-\infty}^{\infty} a^{\frac{m(m-1)}{2} + \frac{mn(m-1)}{2} - \frac{mn(m+1)}{2}} \times b^{\frac{mn(m-1)}{2} + \frac{m(m+1)}{2} - \frac{mn(m+1)}{2}}$$

Now exponent of a

$$= \frac{n(n+1) + m(m-1) + mn(m-1) - mn(m+1)}{2}$$

$$= \frac{n^2+n+m^2-m-2mn}{2} = \frac{(m-n)^2-(m-n)}{2}$$

$$= \frac{(m-n)(m-n-1)}{2}$$

Similarly, exponent of $b = \frac{(m-n)(m-n+1)}{2}$

$$\Rightarrow \text{RHS} = \sum_{m=-\infty}^{\infty} a^{\frac{(m-n)(m-n-1)}{2}} b^{\frac{(m-n)(m-n+1)}{2}}$$

$$= \sum_{N=-\infty}^{\infty} a^{\frac{N(N-1)}{2}} b^{\frac{N(N+1)}{2}}$$

($N = m-n$ as m ranges from $-\infty$ to ∞, so does n)

$$= f(a, b).$$