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# MA 633 – Partition Theory – Tut. 2

① $$1+\sum_{n=1}^{\infty}\frac{(1/\alpha;q^2)_n(-\alpha q z)^n}{(\beta q^2;q^2)_n}+\sum_{n=1}^{\infty}\frac{(1/\beta;q^2)_n(-\beta q)^n z^{-n}}{(\alpha q^2;q^2)_n}$$
$$=\frac{(q^2,\alpha\beta q^2,-qz,-q/z;q^2)_\infty}{(\alpha q^2,\beta q^2,-\alpha q z,-\frac{\beta q}{z};q^2)_\infty}.$$
for $|\beta q|<|z|<\frac{1}{|\alpha q|}$

Proof: $q\to q^2$, $z\to -\alpha q z$, $a=\frac{1}{\alpha}$, $b=\beta q^2$ in 1Ψ1 summation:

$$\sum_{n=-\infty}^{\infty}\frac{(a)_n}{(b)_n}z^n=\frac{(az)_\infty\left(\frac{q}{az}\right)_\infty(q)_\infty(b/a)_\infty}{(z)_\infty\left(\frac{b}{az}\right)_\infty(b)_\infty(q/a)_\infty}$$
$\left(\left|\frac{b}{a}\right|<|z|<1,\ |q|<1\right)$

$$\Rightarrow 1+\sum_{n=1}^{\infty}\frac{(1/\alpha;q^2)_n}{(\beta q^2;q^2)_n}(-\alpha q z)^n+\sum_{n=-\infty}^{-1}\frac{(1/\alpha;q^2)_n}{(\beta q^2;q^2)_n}(-\alpha q z)^n$$
$$=\frac{(q^2,\alpha\beta q^2,-qz,q/z,q)_\infty}{(\alpha q^2,\beta q^2,-\alpha q z,-\frac{\beta q}{z};q^2)_\infty}$$

Now $\displaystyle\sum_{n=-\infty}^{-1} \frac{(1/\alpha;q^2)_n}{(\beta q^2;q^2)_n}(\alpha q z)^n = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{\alpha};q^2\right)_{-n}}{\left(\beta q^2;q^2\right)_{-n}}(-\alpha q z)^{-n}$ ——— (1)

$$\left(\frac{1}{\alpha};q^2\right)_{-n} = \frac{\left(\frac{1}{\alpha};q^2\right)_\infty}{\left(\frac{q^{-2n}}{\alpha};q^2\right)_\infty}$$

$$= \frac{1}{\left(1-\frac{1}{\alpha q^{2n}}\right)\left(1-\frac{1}{\alpha q^{2n-2}}\right)\cdots\cdots\left(1-\frac{1}{\alpha q^2}\right)}$$

$$= \frac{(-1)^n \alpha^n q^{\frac{2n(n+1)}{2}}}{(1-\alpha q)(1-\alpha q^2)\cdots\cdots(1-\alpha q^{2n})}$$

$$= \frac{(-\alpha)^n q^{n(n+1)}}{(\alpha q^2;q^2)_n}$$

$$\Rightarrow (-\alpha q)^{-n}\left(\frac{1}{\alpha};q^2\right)_n = \frac{q^{n^2}}{(\alpha q^2;q^2)_n}.$$ ——— (2)

Similarly, $(\beta q^2; q^2)_{-n} = \dfrac{(\beta q^2; q^2)_\infty}{(\beta q^{2-2n}; q^2)_\infty}$

$$= \frac{1}{\left(1-\frac{\beta}{q^{2n-2}}\right)\left(1-\frac{\beta}{q^{2n-4}}\right)\cdots\cdots(1-\beta)}$$

$$= \frac{q^{2+4+6+\cdots+2n-2}}{(q^{2n-2}-\beta)(q^{2n-4}-\beta)\cdots\cdots(q^2-\beta)(1-\beta)}$$

$$= \frac{(-\beta)^{-n}\, q^{2(1+2+\cdots+n-1)}}{\left(1-\frac{q^{2n-2}}{\beta}\right)\left(1-\frac{q^{2n-4}}{\beta}\right)\cdots\cdots\left(1-\frac{q^2}{\beta}\right)\left(1-\frac{1}{\beta}\right)}$$

$$= \frac{(-\beta)^{-n}\, q^{\frac{2n(n-1)}{2}}}{(1/\beta; q^2)_n} \qquad \text{--- (3)}$$

From (1), (2) & (3), we have

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(\frac{1}{\alpha}; q^2)_{-n}}{(\beta q^2; q^2)_{-n}} (-\alpha q z)^{-n}$$

$$= \sum_{n=1}^{\infty} \frac{\dfrac{q^{n^2}}{(\alpha q^2;q^2)_n} \cdot z^{-n}}{\dfrac{(-\beta)^{-n} q^{\frac{2n(n-1)}{2}}}{(\frac{1}{\beta};q^2)_n}}$$

$$= \sum_{n=1}^{\infty} \frac{(\frac{1}{\beta};q^2)_n (-\beta q)^n z^{-n}}{(\alpha q^2;q^2)_n}.$$

This completes the proof. ∎

② Prove JTPI $f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$ from ${}_1\psi_1$.

Proof: ${}_1\psi_1$ is given by:

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty (\frac{q}{az})_\infty (q)_\infty (b/a)_\infty}{(z)_\infty (\frac{b}{az})_\infty (b)_\infty (q/a)_\infty}$$

$$\left( \left|\frac{b}{a}\right| < |z| < 1, \quad |q| < 1 \right)$$

① Let $b=0$.

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(0)_n} z^n = \frac{(az, \frac{q}{az}, q; q)_\infty}{(z, \frac{q}{a}; q)_\infty}$$

,

$(0)_0 = 1$, $(0)_n = 1$ $(n>0)$

For $n<0$,

$$(0)_n \overset{?}{=} \frac{(0)_\infty}{(0q^n)_\infty} = \frac{1}{1} = 1.$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} (a)_n z^n = \frac{\left(az, \frac{q}{az}, q; q\right)_\infty}{\left(z, \frac{q}{a}; q\right)_\infty}, \quad \underline{0<|z|<1},$$

Replace $a$ by $c$ so that

$$\sum_{n=-\infty}^{\infty} (c)_n z^n = \frac{\left(cz, \frac{q}{cz}, q; q\right)_\infty}{\left(z, \frac{q}{c}; q\right)_\infty}, \quad 0<|z|<1$$

Let $z = -b/c$. Hence of $0<\left|\frac{b}{c}\right|<1$,

$$\sum_{n=-\infty}^{\infty} (c)_n \left(-\frac{b}{c}\right)^n = \frac{\left(-b, -\frac{q}{b}, q; q\right)_\infty}{\left(-\frac{b}{c}, \frac{q}{c}; q\right)_\infty}$$

For $n \geq 0$,

$$\lim_{c\to\infty} (c)_n \left(-\frac{b}{c}\right)^n = (-b)^n \lim_{c\to\infty} \left(\frac{1-c}{c}\right)\left(\frac{1-cq}{c}\right) \cdots \left(\frac{1-cq^{n-1}}{c}\right)$$

$$= (-b)^n \lim_{c\to\infty} \left(\frac{1}{c}-1\right)\left(\frac{1}{c}-q\right)\cdots\left(\frac{1}{c}-q^{n-1}\right)$$

$$= (-b)^n (-1)(-q)\cdots(-q^{n-1})$$

$$= b^n q^{n(n-1)/2}.$$

For $n<0$, say, $n=-m$, $m>0$

$$\lim_{c\to\infty} (c)_{-m} \left(\frac{-b}{c}\right)^{-m} = (-b)^{-m} \lim_{c\to\infty} (c)_{-m}\, c^m$$

$$= (-b)^{-m} \lim_{c\to\infty} \frac{(c)_\infty}{(cq^{-m})_\infty}\, c^m$$

$$= (-b)^{-m} \lim_{c\to\infty} \frac{c^m}{\left(1-\frac{c}{q^m}\right)\left(1-\frac{c}{q^{m-1}}\right)\cdots\left(1-\frac{c}{q}\right)}$$

$$= (-b)^{-m} \lim_{c\to\infty} \frac{c^m\, q^{1+2+\cdots+m}}{(-c)^m\left(1-\frac{q^m}{c}\right)\left(1-\frac{q^{m-1}}{c}\right)\cdots\left(1-\frac{q}{c}\right)}$$

$$= b^{-m} q^{\frac{m(m+1)}{2}}$$

$$= b^n q^{\frac{-n(-n+1)}{2}} = b^n q^{\frac{n(n-1)}{2}}.$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} b^n q^{\frac{n(n-1)}{2}} = \left(-b, -\frac{q}{b}, q; q\right)_\infty$$

for $b \neq 0$.

Now let $q = ab$ so that for $|ab|<1$,

$$\sum_{n=-\infty}^{\infty} b^n (ab)^{\frac{n(n-1)}{2}} = (-b; ab)_\infty \left(-\frac{ab}{b}; ab\right)_\infty (ab; ab)_\infty$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} a^{\frac{n(n-1)}{2}} b^{\frac{n(n+1)}{2}} = (-b; ab)_\infty (-a; ab)_\infty (ab; ab)_\infty$$

for $|ab|<1$. ∎

(3) ${}_1\psi_1 \rightarrow q$-binomi

$$1 + \sum_{n=1}^{\infty} \frac{(1/\alpha; q^2)_n (-\alpha q z)^n}{(\beta q^2; q^2)_n} + \sum_{n=1}^{\infty} \frac{(1/\beta; q^2)_n (-\beta q)^n z^{-n}}{(\alpha q^2; q^2)_n}$$

$$= \frac{(q^2, \alpha\beta q^2, -qz, -q/z; q^2)_\infty}{(\alpha q^2, \beta q^2, -\alpha q z, -\frac{\beta q}{z}; q^2)_\infty}$$

for $|\beta q| < |z| < \frac{1}{|\alpha q|}$

Let $\beta = 1$, and replace $z$ by $-\frac{z}{\alpha q}$

to obtain, for $|q| < \left|-\frac{z}{\alpha q}\right| < \frac{1}{|\alpha q|}$, $\equiv |\alpha q^2| < |z| < 1$

$$1 + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{\alpha}; q^2\right)_n}{(q^2; q^2)_n} z^n$$

$$= \frac{\left(q^2, \alpha q^2, \frac{z}{\alpha}, \frac{\alpha q^2}{z}; q^2\right)_\infty}{\left(\alpha q^2, q^2, z, \frac{\alpha q^2}{z}; q^2\right)_\infty}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{\left(\frac{1}{\alpha}; q^2\right)_n}{(q^2; q^2)_n} z^n = \frac{\left(\frac{z}{\alpha}; q^2\right)_\infty}{(z; q^2)_\infty}.$$

Now let $\alpha = \frac{1}{a}$ & $q^2 \longrightarrow q$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \quad \text{for } \left|\frac{q}{a}\right| < |z| < 1.$$

But both sides are analytic in $|z| < 1$,

Hence by analytic continuation, the formula holds for $|z| < 1$.

∎