16/9/21

MA 633 – Partition Theory – Tut. 4

(1) $|q|<|x|<1$, $|q|<|y|<1$.

$$\sum_{k=-\infty}^{\infty} \frac{x^k}{1-yq^k} = \sum_{k=-\infty}^{\infty} \frac{y^k}{1-xq^k}.$$

Proof: We show the following:

$$\sum_{k=0}^{\infty} \frac{x^k}{1-yq^k} = \sum_{k=0}^{\infty} \frac{y^k}{1-xq^k}. \longrightarrow (*)$$

$$\sum_{k=-\infty}^{-1} \frac{x^k}{1-yq^k} = \sum_{k=-\infty}^{-1} \frac{y^k}{1-xq^k}. \longrightarrow (**)$$

Proof of $(*)$: $|yq^k|<1$ for $k\geq 0$ since $|y|<1$, $|q^k|<|q|<1$ for $k\geq 1$.

$$\sum_{k=0}^{\infty} \frac{x^k}{1-yq^k} = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{\infty} (yq^k)^m$$

$$= \sum_{m=0}^{\infty} y^m \sum_{k=0}^{\infty} (xq^m)^k$$

$$= \sum_{m=0}^{\infty} \frac{y^m}{1-xq^m} \qquad (\because |xq^m|<1).$$

This proves $(*)$.

Proof of (**) :

$$\sum_{k=-\infty}^{-1} \frac{x^k}{1-yq^k} = \sum_{k=1}^{\infty} \frac{x^{-k}}{1-yq^{-k}}$$

$$= -\sum_{k=1}^{\infty} \frac{x^{-k}}{yq^{-k}} \cdot \frac{1}{1-\frac{q^k}{y}}$$

$$= -\frac{1}{y}\sum_{k=1}^{\infty} \left(\frac{q}{x}\right)^k \sum_{m=0}^{\infty} \left(\frac{q^k}{y}\right)^m$$

$|q^k|\,|q| < |y| < 1$
for $k \geq 1$
$\Rightarrow \left|\frac{q^k}{y}\right| < 1$

$$= -\sum_{m=0}^{\infty} \frac{1}{y^{m+1}} \sum_{k=1}^{\infty} \left(\frac{q^{m+1}}{x}\right)^k$$

$$= -\sum_{m=0}^{\infty} \frac{1}{y^{m+1}} \frac{q^{m+1}/x}{1-q^{m+1}/x} \qquad \left(\because \left|\frac{q^{m+1}}{x}\right| < 1\right)$$

$$= -\sum_{m=1}^{\infty} \frac{\frac{q^m}{x}}{y^m\left(1-\frac{q^m}{x}\right)} = \sum_{m=1}^{\infty} \frac{y^{-m}}{(1-xq^{-m})}$$

$$= \sum_{k=-\infty}^{-1} \frac{y^k}{1-xq^k} .$$

This proves (**) & hence the result.

② $$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n(n+1)}.$$

Proof: From Cor. 37,

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} q^{n(n+1)/2} = (aq;q^2)_\infty(-q;q)_\infty$$

Replace q by q^2, then let $a=-q$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n(n+1)} = (-q\cdot q^2;q^4)_\infty(-q^2;q^2)_\infty$$

$$= (-q^3;q^4)_\infty(-q^2;q^2)_\infty$$

$$= (-q^3;q^4)_\infty(-q^2;q^4)_\infty(-q^4;q^4)_\infty$$

$$= \frac{(-q;q^4)_\infty(-q^2;q^4)_\infty(-q^3;q^4)_\infty(-q^4;q^4)_\infty}{(-q;q^4)_\infty}$$

$$= \frac{(-q;q)_\infty}{(-q;q^4)_\infty}.$$

③ $i,j \in \mathbb{N}$, let $y_i \neq y_j$ whenever $i \neq j$.
Then,

ⓐ $$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y_1}{\lambda}\right)\cdots\left(1-\frac{y_j}{\lambda}\right)}$$

$$= \frac{1}{(1-x)(1-xy_1)\cdots(1-xy_j)}.$$

Proof: Use induction on j.

(i) $j=1$: We have already proved

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)(1-y_1/\lambda)} = \frac{1}{(1-x)(1-xy_1)}.$$

(ii) Assume that

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y_1}{\lambda}\right)\cdots\left(1-\frac{y_{j-1}}{\lambda}\right)}$$

$$= \frac{1}{(1-x)(1-xy_1)\cdots(1-xy_{j-1})}.$$

(iii) Show that ⓐ holds.

By partial fraction expansion,

$$\frac{1}{\left(1-\frac{y_{j-1}}{\lambda}\right)\left(1-\frac{y_j}{\lambda}\right)} = \frac{1}{y_{j-1}-y_j}\left(\frac{y_{j-1}}{1-\frac{y_{j-1}}{\lambda}} - \frac{y_j}{1-\frac{y_j}{\lambda}}\right)$$

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y_1}{\lambda}\right)\cdots\left(1-\frac{y_{j-1}}{\lambda}\right)\left(1-\frac{y_j}{\lambda}\right)}$$

$$= \frac{1}{y_{j-1}-y_j}\left[\underset{\geq}{\Omega} \frac{y_{j-1}}{(1-\lambda x)\left(1-\frac{y_1}{\lambda}\right)\cdots\left(1-\frac{y_{j-1}}{\lambda}\right)} - \underset{\geq}{\Omega} \frac{y_j}{(1-\lambda x)\left(1-\frac{y_1}{\lambda}\right)\cdots\left(1-\frac{y_j}{\lambda}\right)}\right]$$

$$= \frac{1}{(y_{j-1}-y_j)}\left\{\frac{y_{j-1}}{(1-x)(1-xy_1)\cdots(1-xy_{j-1})} - \frac{y_j}{(1-x)(1-xy_1)\cdots(1-xy_{j-2})(1-xy_j)}\right\}$$

$$= \frac{1}{(1-x)(1-xy_1)\cdots(1-xy_{j-2})}$$

$$\times\left\{\frac{1}{y_{j-1}-y_j}\left(\frac{y_{j-1}}{(1-xy_{j-1})} - \frac{y_j}{1-xy_j}\right)\right\}$$

$$= \frac{1}{(1-x)(1-xy_1)\cdots(1-xy_{j-2})(1-xy_{j-1})(1-xy_j)}$$

$\square$

(b) Write $\dfrac{1}{(1-\lambda x)(1-\lambda y)}$

as $\dfrac{1}{x-y}\left(\dfrac{x}{1-\lambda x} - \dfrac{y}{1-\lambda y}\right)$

& proceed as before.

(c) $$\frac{1}{\left(1-\frac{z}{\lambda^2}\right)} = \frac{1}{\left(1+\frac{\sqrt{z}}{\lambda}\right)\left(1-\frac{\sqrt{z}}{\lambda}\right)}$$

$$= \frac{1}{2}\,\frac{\left(1+\frac{\sqrt{z}}{\lambda}\right)+\left(1-\frac{\sqrt{z}}{\lambda}\right)}{\left(1+\frac{\sqrt{z}}{\lambda}\right)\left(1-\frac{\sqrt{4}}{\lambda}\right)}$$

$$= \frac{1}{2}\left\{\frac{1}{1-\frac{\sqrt{z}}{\lambda}} + \frac{1}{1+\frac{\sqrt{z}}{\lambda}}\right\}$$

& then use part (b).

(4) $Q_m^{(k,\ell)}(n)$ = number of partitions of n into m parts where each part differs from the next part by at least k and the smallest part $\geq \ell$.

Then $$\sum_{n=1}^{\infty} Q_m^{(k,\ell)}(n)\, q^n = \frac{q^{\ell m + \frac{km(m-1)}{2}}}{(q;q)_m}.$$

Proof: If $n_1, n_2, \dots, n_m$ are parts of a partition enumerated by $Q_m^{(k,l)}(n)$, then $n_1 - n_2 \geq k$, $n_2 - n_3 \geq k$, $\dots$, $n_{m-1} - n_m \geq k$, $n_m \geq l$.

$$\sum_{n=1}^{\infty} Q_m^{(k,l)}(n)\, q^n$$

$$= \underset{\geq}{\Omega} \sum_{n_1, n_2, \dots, n_m = 0}^{\infty} q^{n_1+n_2+\dots+n_m} \lambda_1^{n_1-n_2-k} \lambda_2^{n_2-n_3-k} \cdots \lambda_{m-1}^{n_{m-1}-n_m-k} \lambda_m^{n_m-l}$$

$$= \underset{\geq}{\Omega} \sum_{n_1, n_2, \dots, n_m = 0}^{\infty} (q\lambda_1)^{n_1} \left(\frac{q\lambda_2}{\lambda_1}\right)^{n_2} \left(\frac{q\lambda_3}{\lambda_2}\right)^{n_3} \cdots \left(\frac{q\lambda_{m-1}}{\lambda_{m-2}}\right)^{n_{m-1}} \left(\frac{q\lambda_m}{\lambda_{m-1}}\right)^{n_m} \times \lambda_1^{-k} \lambda_2^{-k} \cdots \lambda_{m-1}^{-k} \lambda_m^{-l}$$

$$= \underset{\geq}{\Omega} \frac{\lambda_1^{-k} \lambda_2^{-k} \cdots \lambda_{m-1}^{-k} \lambda_m^{-l}}{(1-q\lambda_1)\left(1-\frac{q\lambda_2}{\lambda_1}\right)\left(1-\frac{q\lambda_3}{\lambda_2}\right)\cdots\left(1-\frac{q\lambda_{m-1}}{\lambda_{m-2}}\right)\left(1-\frac{q\lambda_m}{\lambda_{m-1}}\right)}$$

From Lemma 28,

$$\underset{\geq}{\Omega} \frac{\lambda^{-\alpha}}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)} = \frac{x^{\alpha}}{(1-x)(1-xy)}$$

So
$$\underset{\geq}{\Omega} \frac{\lambda_1^{-k}}{(1-q\lambda_1)\left(1-\frac{q\lambda_2}{\lambda_1}\right)} = \frac{q^k}{(1-q)(1-q^2\lambda_2)}$$

Similarly, $\underset{\geq}{\Omega} \dfrac{\lambda_2^{-k}}{(1-q^2\lambda_2)(1-q\frac{\lambda_3}{\lambda_2})}$

$$= \frac{q^{2k}}{(1-q^2)(1-q^3\lambda_3)}$$

& so on.

Hence

$$\sum_{n=0}^{\infty} \mathcal{Q}_m^{(k,l)}(n)\, q^n$$

$$= \frac{q^{k+2k+3k+\cdots+(m-1)k}}{(1-q)(1-q^2)\cdots(1-q^{m-1})} .$$

$$\times \underset{\geq}{\Omega} \frac{\lambda_m^{-l}}{(1-q^m\lambda_m)}$$

$$= \frac{q^{\frac{km(m-1)}{2}}}{(q;q)_{m-1}} \cdot \frac{(q^m)^l}{(1-q^m)} \quad \text{(Lemma 28 with } y=0)$$

$$= \frac{q^{lm+\frac{km(m-1)}{2}}}{(q;q)_m} .$$