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MA 633 – Partition Theory – Tut. 5

(2) $j, k, n, m \in \mathbb{Z}$

Consider the gen. fn.

$$\sum_{n=-\infty}^{\infty} \sum_{j+k=n} \frac{(a)_j (a)_k}{(b)_j (b)_k} (-1)^k z^n$$

$$= \left(\sum_{j=-\infty}^{\infty} \frac{(a)_j}{(b)_j} z^j \right) \left(\sum_{k=-\infty}^{\infty} \frac{(a)_k}{(b)_k} (-z)^k \right)$$

$({}_1\psi_1)$

$$= \frac{\left(az, \frac{q}{az}, q, \frac{b}{a}; q\right)_\infty}{\left(z, \frac{b}{az}, b, \frac{q}{a}; q\right)_\infty} \cdot \frac{\left(-az, -\frac{q}{az}, q, \frac{b}{a}; q\right)_\infty}{\left(-z, -\frac{b}{az}, b, \frac{q}{a}; q\right)_\infty}$$

$$= \frac{(az, -az; q)_\infty \left(\frac{q}{az}, -\frac{q}{az}; q\right)_\infty (q)_\infty^2 \left(\frac{b}{a}\right)_\infty^2}{(z, -z; q)_\infty \left(\frac{b}{az}, -\frac{b}{az}; q\right)_\infty (b)_\infty^2 \left(\frac{q}{a}\right)_\infty^2}$$

$$= \frac{(a^2 z^2; q^2)_\infty \left(\frac{q^2}{a z^2}; q^2\right)_\infty (q)_\infty^2 \left(\frac{b}{a}\right)_\infty^2}{(z^2; q^2)_\infty \left(\frac{b^2}{a z^2}; q^2\right)_\infty (b)_\infty^2 \left(\frac{q}{a}\right)_\infty^2} \quad \text{— I}$$

${}_1\psi_1$ summation formula is

$$\sum_{l=-\infty}^{\infty} \frac{(a)_l}{(b)_l} z^l = \frac{\left(az, \frac{q}{az}, q, \frac{b}{a}; q\right)_\infty}{\left(z, \frac{b}{az}, b, \frac{q}{a}; q\right)_\infty}.$$

$q \to q^2$, $a \to a^2$, $b \to b^2$, $z \to z^2$. Then

$$\sum_{l=-\infty}^{\infty} \frac{(a^2;q^2)_l}{(b^2;q^2)_l} z^{2l} = \frac{\left(a^2z^2, \frac{q^2}{a^2z^2}, q^2, \frac{b^2}{a^2}; q^2\right)_\infty}{\left(z^2, \frac{b^2}{a^2z^2}, b^2, \frac{q^2}{a^2}; q^2\right)_\infty}$$

$$= \frac{\left(a^2z^2, \frac{q^2}{a^2z^2}; q^2\right)_\infty}{\left(z^2, \frac{b^2}{a^2z^2}; q^2\right)_\infty} \; \frac{(q)_\infty(-q)_\infty\left(\frac{b}{a}\right)_\infty\left(-\frac{b}{a}\right)_\infty}{(b)_\infty(-b)_\infty\left(\frac{q}{a}\right)_\infty\left(-\frac{q}{a}\right)_\infty} \quad \text{——— } \mathrm{II}$$

From I & II,

$$\sum_{n=-\infty}^{\infty} \sum_{j+k=n} \frac{(a)_j (a)_k}{(b)_j (b)_k} (-1)^k z^n$$

$$= \frac{(a^2z^2;q^2)_\infty \left(\frac{q^2}{az^2};q^2\right)_\infty (q)_\infty^2 \left(\frac{b}{a}\right)_\infty^2}{(z^2;q^2)_\infty \left(\frac{b^2}{az^2};q^2\right) (b)_\infty^2 \left(\frac{q}{a}\right)^2}$$

$$= \frac{\left(a^2z^2, \frac{q^2}{a^2z^2}; q^2\right)_\infty}{\left(z^2, \frac{b^2}{a^2z^2}; q^2\right)_\infty} \frac{(q)_\infty(-q)_\infty\left(\frac{b}{a}\right)_\infty\left(-\frac{b}{a}\right)_\infty}{(b)_\infty(-b)_\infty\left(\frac{q}{a}\right)_\infty\left(-\frac{q}{a}\right)_\infty}$$

$$\times \frac{(q)_\infty(b/a)_\infty(-b)\,(-q/a)_\infty}{(-q)_\infty\left(-\frac{b}{a}\right)_\infty(b)_\infty(q/a)_\infty}$$

$$= \frac{(q)_\infty(b/a)_\infty(-b)\,(-q/a)_\infty}{(-q)_\infty\left(-\frac{b}{a}\right)_\infty(b)_\infty(q/a)_\infty} \sum_{m=-\infty}^{\infty} \frac{(a^2;q^2)_m}{(b^2;q^2)_m} z^{2m}$$

Now equate the coefficients of z^n on both sides to get the required result

(3) $j, k, l, m, n \in \mathbb{Z}^+ \cup \{0\}$.

$\omega = e^{2\pi i/3}$.

$$\sum_{j+k+l=n} \frac{(a)_j(a)_k(a)_l}{(q)_j(q)_k(q)_l} \omega^{k+2l} = \begin{cases} 0, & \text{if } 3 \nmid n \\ \frac{(a^3;q^3)_m}{(q^3;q^3)_m} & \text{if } n = 3m. \end{cases}$$

Hint: $(1-z)(1-z\omega)(1-z\omega^2)$

$$\left.\begin{aligned} &= (1-z)(1-z(\omega+\omega^2)+z^2\omega^3) \\ &= (1-z)(1+z+z^2) \\ &= 1-z^3. \end{aligned}\right\} (*)$$

Note that

$j+k+l$

$$\sum_{n=0}^{\infty}\left(\sum_{j+k+l=n}\frac{(a)_j(a)_k(a)_l}{(q)_j(q)_k(q)_l}\omega^{k+2l}\right)z^{n}$$

$$=\left(\sum_{j=0}^{\infty}\frac{(a)_j}{(q)_j}z^j\right)\left(\sum_{k=0}^{\infty}\frac{(a)_k}{(q)_k}(\omega z)^k\right)\left(\sum_{l=0}^{\infty}\frac{(a)_l}{(q)_l}(\omega^2 z)^l\right)$$

(q-binomial)

$$=\frac{(az)_\infty}{(z)_\infty}\cdot\frac{(a\omega z)_\infty}{(\omega z)_\infty}\frac{(a\omega^2 z)_\infty}{(\omega^2 z)_\infty}$$

$$=\frac{(a^3z^3;q^3)_\infty}{(z^3;q^3)_\infty}\Bigg\}\text{(using (*))}$$

(q-binomial)

$$=\sum_{m=0}^{\infty}\frac{(a^3;q^3)_m}{(q^3;q^3)_m}z^{3m}$$

Now equate the coeff. of z^n on extreme sides of the above string of equalities.

① Logarithmically differentiate both sides of

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty} \text{ gives}$$

$$\frac{\sum_{n=0}^{\infty} n p(n) q^{n-1}}{\sum_{n=0}^{\infty} p(n) q^n} = \frac{d}{dq} \log\left(\frac{1}{(q;q)_\infty}\right)$$

$$= -\frac{d}{dq} \log \prod_{n=1}^{\infty} (1-q^n)$$

$$= -\frac{d}{dq} \sum_{n=1}^{\infty} \log(1-q^n)$$

$$= -\sum_{n=1}^{\infty} \frac{-nq^{n-1}}{1-q^n} = \sum_{n=1}^{\infty} \frac{nq^{n-1}}{1-q^n}.$$

$$\Rightarrow \sum_{n=0}^{\infty} n p(n) q^n = \left(\sum_{n=1}^{\infty} \frac{n q^n}{1-q^n}\right)\left(\sum_{n=0}^{\infty} p(n) q^n\right)$$

But $\displaystyle\sum_{n=1}^{\infty} \frac{n q^n}{1-q^n} = \sum_{n=1}^{\infty} n \sum_{m=1}^{\infty} q^{mn}$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n q^{mn}$$

Cauchy product

- $$\left(\sum_{k=0}^{\infty} a(k)\, z^{k}\right)\left(\sum_{m=0}^{\infty} b(m)\, z^{m}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a(k)\, b(n-k)\right) z^{n}$$

$n = m+k$

- $$\left.\begin{aligned}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a(k)\, b(m)\, z^{km} &= \sum_{n=1}^{\infty}\left(\sum_{km=n} a(k)\, b(m)\right) z^{n} \\ &= \sum_{n=1}^{\infty}\left(\sum_{k\mid n} a(k)\, b\!\left(\frac{n}{k}\right)\right) z^{n},\end{aligned}\right\} \circledast$$

$km = n$

Hence using $\circledast$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n\, q^{mn} = \sum_{\ell=1}^{\infty}\left(\sum_{n\mid \ell} n\right) q^{\ell} = \sum_{\ell=1}^{\infty} \sigma(\ell)\, q^{\ell},$$

Thus

$$\sum_{n=1}^{\infty} n\,p(n)\,q^n = \left(\sum_{\ell=1}^{\infty} \sigma(\ell)\,q^{\ell}\right)\left(\sum_{n=0}^{\infty} p(n)\,q^n\right)$$

$$= \sum_{k=1}^{\infty}\left(\sum_{n=0}^{k-1} p(n)\,\sigma(k-n)\right)q^k$$

$\ell + n = k$

$$= \sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1} p(j)\,\sigma(n-j)\right)q^n .$$

$$\Rightarrow \quad n\,p(n) = \sum_{j=0}^{n-1} p(j)\,\sigma(n-j) \quad \text{for } n \geq 1.$$

$n=1$,

LHS $= 1$ RHS $= p(0)\,\sigma(1-0) = 1 \cdot 1 = 1$