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MA 633 – Partition theory Tut. 6

① (i) q-analogue of Euler's transformation.

For $|z|<1$, $\left|\frac{abz}{c}\right|<1$, $a,b\neq 0$

$$ {}_2\varphi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};z\right) = \frac{(abz/c)_\infty}{(z)_\infty}\, {}_2\varphi_1\left(\begin{matrix}c/a,\ c/b\\ c\end{matrix};\frac{abz}{c}\right). $$

Proof: By Heine's transformation,

$$ {}_2\varphi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};z\right) = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty}\, {}_2\varphi_1\left(\begin{matrix}c/b,\ z\\ az\end{matrix};b\right) $$

(Lect. 37)

$$ = \frac{(c/b)_\infty (zb)_\infty}{(c)_\infty (z)_\infty}\, {}_2\varphi_1\left(\begin{matrix}\overset{B}{\frac{abz}{c}},\ \overset{A}{b}\\ bz\end{matrix};\overset{Z}{c/b}\right) $$

$$ = \frac{(c/b)_\infty (bz)_\infty}{(c)_\infty (z)_\infty}\, {}_2\varphi_1\left(\begin{matrix}b,\ \frac{abz}{c}\\ \underset{}{\overset{C}{bz}}\end{matrix};\frac{c}{b}\right) $$

(Heine)

$$ = \frac{\cancel{(c/b)_\infty}\cancel{(bz)_\infty}}{\cancel{(c)_\infty}(z)_\infty}\cdot \frac{\left(\frac{abz}{c}\right)_\infty \cancel{(c)_\infty}}{\cancel{(bz)_\infty}\cancel{(c/b)_\infty}}\, {}_2\varphi_1\left(\begin{matrix}c/a,\ c/b\\ c\end{matrix};\frac{abz}{c}\right) $$

$$ = \frac{(abz/c)_\infty}{(z)_\infty}\, {}_2\varphi_1\left(\begin{matrix}c/a,\ c/b\\ c\end{matrix};\frac{abz}{c}\right). $$

(ii) $(a)_{n-m} = \dfrac{(a)_n}{(q^{1-n}/a)_m}\left(\dfrac{-q}{a}\right)^m q^{\frac{m(m-1)}{2}-nm}$

(iii) (q-analogue of Pfaff-Saalschütz)

$${}_3\varphi_2\left(\begin{matrix} a, & b, & q^{-n} \\ c, & & \frac{abq^{1-n}}{c} \end{matrix}; q, q\right) = \frac{(c/a)_n (c/b)_n}{(c)_n \left(\frac{c}{ab}\right)_n},$$

Proof: By q-analogue of Euler's transformation,

$${}_2\varphi_1\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right) = \frac{(abz/c)_\infty}{(z)_\infty}\, {}_2\varphi_1\left(\begin{matrix} c/a, & c/b \\ & c \end{matrix}; \frac{abz}{c}\right) \quad (1)$$

RHS of (1)

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{ab}{c}\right)_k z^k}{(q)_k} \sum_{m=0}^{\infty} \frac{(c/a)_m (c/b)_m}{(c)_m (q)_m} \left(\frac{abz}{c}\right)^m$$

$$= \sum_{k,m=0}^{\infty} \frac{\left(\frac{ab}{c}\right)_k}{(q)_k} \frac{(c/a)_m (c/b)_m}{(c)_m (q)_m} \left(\frac{ab}{c}\right)^m z^{m+k}$$

$(m+k=n)$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\left(\frac{ab}{c}\right)_{n-m}}{(q)_{n-m}} \frac{\left(\frac{c}{a}\right)_m \left(\frac{c}{b}\right)_m}{(c)_m (q)_m} \left(\frac{ab}{c}\right)^m z^n$$

By (iii),

$$(a)_{n-m} = \frac{(a)_n}{(q^{1-n}/a)_m}\left(-\frac{q}{a}\right)^m q^{\frac{m(m-1)}{2}-nm},$$

Hence

$$\frac{\left(\frac{ab}{c}\right)_{n-m}}{(q)_{n-m}} = \frac{\dfrac{\left(\frac{ab}{c}\right)_n}{\left(\frac{cq^{1-n}}{ab}\right)_m}\left(\cancel{-}\frac{qc}{ab}\right)^m q^{\cancel{\frac{m(m-1)}{2}}-nm}}{\dfrac{(q)_n}{(q^{-n})_m}\cancel{(-1)^m}\, q^{\cancel{\frac{m(m-1)}{2}}-nm}}$$

$$= \frac{\left(\frac{ab}{c}\right)_n (q^{-n})_m}{\left(\frac{cq^{1-n}}{ab}\right)_m (q)_n}\left(\frac{cq}{ab}\right)^m$$

$\Rightarrow$ RHS of 1

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\left(\frac{ab}{c}\right)_n (q^{-n})_m}{\left(\frac{cq^{1-n}}{ab}\right)_m (q)_n}\left(\frac{cq}{ab}\right)^m \frac{\left(\frac{c}{a}\right)_m\left(\frac{c}{b}\right)_m}{(c)_m (q)_m}\left(\frac{ab}{c}\right)^m z^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{(c/a)_m (c/b)_m (q^{-n})_m\, q^m}{(c)_m\left(\frac{cq^{1-n}}{ab}\right)_m (q)_m}\right)\frac{\left(\frac{ab}{c}\right)_n}{(q)_n} z^n$$

$\longrightarrow$ 2

LHS of 1 $= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} z^n$. $\longrightarrow$ 3

Equating the coeff's of z^n in 2 & 3, we get

$$\sum_{m=0}^{n} \frac{(c/a)_m (c/b)_m (q^{-n})_m q^m}{(c)_m \left(\frac{cq^{1-n}}{ab}\right)_m (q)_m} = \frac{(a)_n (b)_n}{(c)_n \left(\frac{ab}{c}\right)_n}.$$

Now replace a & b by c/a & c/b resp.
This gives

$$\sum_{m=0}^{n} \frac{(a)_m (b)_m (q^{-n})_m q^m}{(c)_m \left(\frac{abq^{1-n}}{c}\right)_m (q)_m} = \frac{(c/a)_n (c/b)_n}{(c)_n \left(\frac{c}{ab}\right)_n}$$

This proves

$$ {}_3\varphi_2 \left(\begin{matrix} a, & b, & q^{-n} \\ c, & \frac{abq^{1-n}}{c} & \end{matrix} ; q, q \right) = \frac{(c/a)_n (c/b)_n}{(c)_n \left(\frac{c}{ab}\right)_n}.$$

$\blacksquare$

2 Do by yourself using the hint.
(In the pdf, the $(aq;q^2)_\infty$ on ~~RHS~~ should be $(aq^2;q^2)_\infty$)

3 (i) Let $a=1$ in prob. 2. This gives,

$$1+\sum_{k=1}^{\infty}\frac{(1-q^{4k})(q^2;q^2)_{k-1}\cancel{(-q;q^2)_k}(-1)^k q^{4k^2-k}}{(q^2;q^2)_k\cancel{(-q;q^2)_k}}$$

$$=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}q^{k^2}.$$

$$\text{LHS}=1+\sum_{k=1}^{\infty}(1+q^{2k})(-1)^k q^{4k^2-k}$$

$$=1+\sum_{k=1}^{\infty}(-1)^k q^{4k^2-k}+\underbrace{\sum_{k=1}^{\infty}(-1)^k q^{4k^2+k}}_{\text{replace } k \text{ by } -k}$$

$$=1+\sum_{k=1}^{\infty}(-1)^k q^{4k^2-k}+\sum_{k=-\infty}^{-1}(-1)^k q^{4k^2-k}$$

$$=\sum_{k=-\infty}^{\infty}(-1)^k q^{4k^2-k}$$

$$=f(-q^3,-q^5)$$

$$=(q^3;q^8)_\infty(q^5;q^8)_\infty(q^8;q^8)_\infty \quad (\text{by JTPI})$$

Hence

$$\sum_{k=0}^{\infty} \frac{(-q;q^2)_k}{(q^2;q^2)_k} q^k = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} (q^3,q^5,q^8;q^8)_\infty$$

$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} (q^3,q^5,q^8;q^8)_\infty \frac{(q;q^2)_\infty}{(q;q^2)_\infty}$$

$$= \frac{(q^2;q^4)_\infty (q^3,q^5,q^8;q^8)_\infty}{(q;q)_\infty}$$

$$= \frac{(\cancel{q^2};q^8)_\infty (\cancel{q^6};q^8)_\infty (\cancel{q^3},\cancel{q^5},\cancel{q^8};q^8)}{(q,\cancel{q^2},\cancel{q^3},q^4,\cancel{q^5},\cancel{q^6},q^7,\cancel{q^8};q^8)_\infty}$$

$$= \frac{1}{(q,q^4,q^7;q^8)_\infty}$$

(ii) Can be proved by letting $a=q^2$ in Prob. 2.