3/8/2021

MA 633 - Partition Theory - Lecture 1

Defn: A partition of a number n is a non-incr--easing sequence of positive integers which sum to n.

Eg: $n=4$

$$4$$
$$3+1$$
$$2+2$$
$$2+1+1$$
$$1+1+1+1$$

• **Defn:** Partition function $p(n)$ counts the number of partitions of n.

$\Rightarrow$ $p(4)=5$.

$p(5)=7$

$$5$$
$$4+1$$
$$3+2$$
$$3+1+1$$
$$2+2+1$$
$$2+1+1+1$$
$$1+1+1+1+1$$

$p(6)=11$

$p(20)=627$

$p(100)=190569292$

$p(200)=3972999029388$.

- Ramanujan's congruences for $p(n)$ (1919)

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$
$$p(11n+6) \equiv 0 \pmod{11}$$

"It appears that there are no equally simple properties for any moduli involving primes other than these 3".

— proved by Scott Ahlgren & Matt Boylan (2003)

- Ramanujan's more general conjecture:
Let $\delta = 5^a 7^b 11^c$ & λ is any integer $\ni$ $24\lambda \equiv 1 \pmod \delta$. Then

$$p(n\delta + \lambda) \equiv 0 \pmod \delta.$$

Chowla: $(24)(243) \equiv 1 \pmod{7^3}$
but $7^3 \nmid p(243)$

Correct conjecture:

$$\delta' = 5^a 7^{b'} 11^c, \quad b' = \begin{cases} b, & \text{if } b = 0,1,2 \\ \left\lfloor \frac{b+2}{2} \right\rfloor, & \text{if } b > 2. \end{cases}$$

$$p(n\delta' + \lambda) \equiv 0 \pmod{\delta'}.$$

* Sometimes we are not interested in all partitions of n but only those belonging to a particular subset of n.

Eg.

- $P_o(n)$ = the number of partitions of n into odd parts.

$n=5$

5	✓	
$4+1$	✗	$P_o(5) = 3$
$3+2$	✗	
$3+1+1$	✓	
$2+2+1$	✗	
$2+1+1+1$	✗	
$1+1+1+1+1$	✓	

- $P_d(n)$ = number of partitions of n into distinct parts

5	✓	
$4+1$	✓	$P_d(5) = 3$
$3+2$	✓	
$3+1+1$	✗	
$2+2+1$	✗	
$2+1+1+1$	✗	
$1+1+1+1+1$	✗	

$$\Rightarrow P_o(5) = P_d(5)$$

$n = 6$

6	✗	$p(6) = 11$
$5+1$	✓	
$4+2$	✗	$p_o(6) = 4$
$4+1+1$	✗	
$3+3$	✓	
$3+2+1$	✗	
$3+1+1+1$	✓	
$2+2+2$	✗	
$2+2+1+1$	✗	
$2+1+1+1+1$	✗	
$1+1+1+1+1+1$	✓	

6	✓	
$5+1$	✓	
$4+2$	✓	$p_d(6) = 4.$
$4+1+1$	✗	
$3+3$	✗	
$3+2+1$	✓	
$3+1+1+1$	✗	
$2+2+2$	✗	
$2+2+1+1$	✗	
$2+1+1+1+1$	✗	
$1+1+1+1+1+1$	✗	

$$\Longrightarrow \quad p_o(6) = p_d(6).$$

Euler: $p_o(n) = p_d(n) \quad \forall\, n \in \mathbb{N}.$

Defn.

Generating function: The generating function $f(q)$ for a sequence $a_0, a_1, a_2, \ldots$ is the power series $\sum_{n=0}^{\infty} a_n q^n$.

$H \subseteq \mathbb{N}$

Defn. Let H be the set of positive integers, "H" denotes the set of all partitions whose parts lie in H.

$p(\text{"}H\text{"}, n)$ = number of partitions of n that have their parts in H.

H_o = set of all odd positive integers.

$p(\text{"}H_o\text{"}, n) = p_o(n)$.

Defn. Let "H"$(\leq d)$ denote the set of all partitions of n whose parts lie in H & do not appear more than 'd' times.

$p(\text{"}\mathbb{N}\text{"}(\leq 1), n) = p_d(n)$.

Thm. 1 Let $H \subseteq \mathbb{N}$ & let

$$f(q) = \sum_{n=0}^{\infty} p(\text{"}H\text{"}, n)\, q^n$$

$$f_d(q) = \sum_{n=0}^{\infty} p(\text{"}H\text{"}(\leq d), n)\, q^n.$$

Then for $|q|<1$,

$$f(q) = \prod_{n\in H} \frac{1}{1-q^n} \, ;$$

$$f_d(q) = \prod_{n\in H} \left(1 + q^n + q^{2n} + \cdots + q^{dn}\right) = \prod_{n\in H} \frac{1-q^{(d+1)n}}{1-q^n} .$$