4/8/2021

MA 633 – Partition Theory – Lecture 2

Defn. Generating function: The generating function $f(q)$ for a sequence $a_0, a_1, a_2, \ldots$ is the power series $\sum_{n=0}^{\infty} a_n q^n$.

Defn. Let H be the set of positive integers, $H \subseteq \mathbb{N}$. "H" denotes the set of all partitions whose parts lie in H.

$p(\text{"}H\text{"}, n)$ = number of partitions of n that have their parts in H.

H_o = set of all odd positive integers.

$p(\text{"}H_o\text{"}, n) = p_o(n)$.

Defn. Let "H"$(\leq d)$ denote the set of all partitions of n whose parts lie in H & do not appear more than 'd' times.

$p(\text{"}\mathbb{N}\text{"}(\leq 1), n) = p_d(n)$.

Thm. 1 Let $H \subseteq \mathbb{N}$ & let

$$f(q) = \sum_{n=0}^{\infty} p(\text{"}H\text{"}, n)\, q^n$$

$$f_d(q) = \sum_{n=0}^{\infty} p(\text{"}H\text{"}(\leq d), n)\, q^n.$$

Then for $|q|<1$,

$$f(q) = \prod_{n \in H} \frac{1}{1-q^n};$$

$$f_d(q) = \prod_{n \in H} \left(1+q^n+q^{2n}+\cdots+q^{dn}\right) = \prod_{n \in H} \frac{1-q^{(d+1)n}}{1-q^n}.$$

Proof: Since $|q|<1$,

$$\prod_{n \in H} \frac{1}{1-q^n} = \prod_{n \in H} \left(1+q^n+q^{2n}+q^{3n}+\cdots\right)$$

$$= \left(1+q^{h_1}+q^{2h_1}+q^{3h_1}+\cdots\right)\left(1+q^{h_2}+q^{2h_2}+q^{3h_2}+\cdots\right)$$
$$\left(1+q^{h_3}+q^{2h_3}+q^{3h_3}+\cdots\right)\cdots$$

$$= \left(\sum_{m_1=0}^{\infty} q^{m_1 h_1}\right)\left(\sum_{m_2=0}^{\infty} q^{m_2 h_2}\right)\left(\sum_{m_3=0}^{\infty} q^{m_3 h_3}\right)\cdots$$

$$= \sum_{m_1, m_2, m_3, \ldots = 0}^{\infty} q^{m_1 h_1 + m_2 h_2 + m_3 h_3 + \cdots}.$$

Suppose $N = m_1 h_1 + m_2 h_2 + m_3 h_3 + \cdots$.

We see that the exponent of q is the partition $\left(h_1^{m_1}\, h_2^{m_2}\, h_3^{m_3} \cdots\right)$

frequencies of $h_1, h_2, h_3, \ldots$

Thus, q^N will occur in the above multi-sum (or, equivalently, in the infinite product) once for each partition of N with parts from H,

$$\Rightarrow \prod_{n \in H} \frac{1}{1-q^n} = \sum_{N=0}^{\infty} p(\text{"}H\text{"}, N)\, q^N .$$

Remark :

If $H = \mathbb{N}$, then

$$\sum_{n=0}^{\infty} p(n)\, q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$$

If $H = 2\mathbb{N}+1$,

$$\sum_{n=0}^{\infty} p_o(n)\, q^n = \frac{1}{(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})} .$$

2^{nd} part :

$$\prod_{n \in H} (1+q^n+q^{2n}+\cdots+q^{dn})$$

$$= \left(\sum_{m_1=0}^{d} q^{m_1 h_1}\right)\left(\sum_{m_2=0}^{d} q^{m_2 h_2}\right) \cdots\cdots$$

$$= \sum_{m_1, m_2, \ldots = 0}^{d} q^{m_1 h_1 + m_2 h_2 + \cdots\cdots}$$

$$= \sum_{N=0}^{\infty} p(\text{"}H(\leq d)\text{"}, N)\, q^N .$$

Multiplication of finitely many absolutely convergent series is justified, but what about infinitely many absolutely conv. series?

To justify this, truncate the infinite product

$$\prod_{n \in H} \frac{1}{1-q^n} \quad \text{to} \quad \prod_{i=1}^{n} \frac{1}{1-q^{h_i}}$$

This truncated product generates partitions of integers whose parts come from $\{h_1, h_2, \ldots, h_n\}$.

$$\prod_{i=1}^{n} \frac{1}{1-q^{h_i}} = \prod_{i=1}^{n} \left(1 + q^{h_i} + q^{h_i} + \cdots\right)$$

Now the multiplication of finitely many absolutely convergent series makes sense.

Now $q \in \mathbb{R} \ni 0 < q < 1$. Let $M = h_n$,

$$\sum_{j=0}^{M} p(\text{"}H\text{"}, j)\, q^j \le \prod_{i=1}^{n} \frac{1}{1-q^{h_i}}$$

$$\big(1+q^{h_1}+q^{2h_1}+\dots\big)\big(1+q^{h_2}+q^{2h_2}+\dots\big) \cdots\cdots \big(1+q^{h_n}+q^{2h_n}+\dots\big)$$

$$\le \prod_{i=1}^{\infty} \frac{1}{1-q^{h_i}} < \infty$$

$$0<q<1 \Rightarrow 0 < q^{h_i} < 1 \Rightarrow -1 < -q^{h_i} < 0$$

$$\Rightarrow 0 < 1-q^{h_i} < 1$$

$$\Rightarrow \frac{1}{1-q^{h_i}} > 1$$

(JUSTIFICATION)

Complex analysis tells us that

$\displaystyle\prod_{i=1}^{\infty} \frac{1}{1-q^{h_i}} < \infty$ if and only if $\displaystyle\sum_{i=1}^{\infty}\left(\frac{1}{1-q^{h_i}} - 1\right) < \infty$,

if $\displaystyle\frac{1}{1-q^{h_i}} - 1 > 0$.

Now

$$\sum_{i=1}^{\infty} \frac{1}{1-q^{h_i}} - 1 = \sum_{i=1}^{\infty} \frac{q^{h_i}}{1-q^{h_i}} \le \sum_{i=1}^{\infty} \frac{q^i}{1-q^i}$$

It can be proved that

$$\sum_{i=1}^{\infty} \frac{q^i}{1-q^i} < \infty \qquad \text{(Root test)}$$

Since $\sum_{j=0}^{M} p(\text{"H"}, j)\, q^j$ is a bounded increasing seq. and hence converges.

$$\Rightarrow \sum_{j=0}^{\infty} p(\text{"H"}, j)\, q^j \leq \prod_{i=1}^{\infty} \frac{1}{1-q^{h_i}} \quad \text{—— (1)}$$

On the other hand,

$$\sum_{j=0}^{\infty} p(\text{"H"}, j)\, q^j \geq \prod_{i=1}^{n} \frac{1}{1-q^{h_i}}$$

$$\Rightarrow \lim_{n\to\infty} \sum_{j=0}^{\infty} p(\text{"H"}, j)\, q^j \geq \prod_{i=1}^{\infty} \frac{1}{1-q^{h_i}}$$

$$\Rightarrow \sum_{j=0}^{\infty} p(\text{"H"}, j)\, q^j \geq \prod_{i=1}^{\infty} \frac{1}{1-q^{h_i}} \quad \text{—— (2)}$$

From (1) & (2) we have

$$\sum_{j=0}^{\infty} p(\text{"H"}, j)\, q^j = \prod_{i=1}^{\infty} \frac{1}{1-q^{h_i}}.$$

$H = \mathbb{N} = \{1, 2, 3, \ldots\}$
$\{h_1, h_2, h_3, \ldots\}$

$4 = m_1 h_1 + m_2 h_2 + m_3 h_3 + m_4 h_4 + m_5 h_5 + \ldots$

1 2 3 4 0 5

$4 = 1 m_1 + 2 m_2 + 3 m_3 + 4 m_4$

N

$m_1 = 4, \quad m_2 = m_3 = m_4 = 0.$

$1 + 1 + 1 + 1$

$m_1 = 0, \quad m_2 = 2, \quad m_3 = m_4 = 0.$

$2 + 2 \qquad 2 + 1 + 1$

$3 + 1, \quad 4$

q^4