5/8/21

MA 633 - Partition Theory - Lec. 3

Defn. $(a)_0 = (a;q)_0 = 1$

q ← base or "nome"

$$(a)_n = (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$
$$= (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

If $|q|<1$,

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k)$$
$$= (1-a)(1-aq)(1-aq^2)\cdots\cdots$$

Replace a by q^a, where $a>0$, and then L'Hôpital's rule so that

$$\lim_{q\to1}\frac{(q^a;q)_n}{(1-q)^n} = \lim_{q\to1}\frac{(1-q^a)}{1-q}\frac{(1-q^{a+1})}{1-q}\frac{(1-q^{a+2})}{1-q}\cdots\frac{(1-q^{a+n-1})}{1-q}$$
$$= \lim_{q\to1}\left(\frac{-aq^{a-1}}{-1}\right)\left(\lim_{q\to1}\frac{-(a+1)q^a}{-1}\right)$$
$$\cdots\left(\lim_{q\to1}\frac{-(a+n-1)q^{a+n-2}}{(-1)}\right) \quad \text{———} (*)$$
$$= a(a+1)(a+2)\cdots\cdots(a+n-1).$$ ← Rising or shifted factorial

When $a=1$, $a(a+1)\cdots(a+n-1) = 1\cdot2\cdot3\cdots n$
$$= n!$$

Defn. Gaussian polynomials : If $m, n \in \mathbb{N}$, then the Gaussian polynomial is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \dfrac{(q)_n}{(q)_m (q)_{n-m}} & , \text{ if } 0 \le m \le n \\ 0 & , \text{ otherwise}, \end{cases}$$

q-binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(1-q)(1-q^2)(1-q^3)\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^m)\,(1-q)(1-q^2)\cdots(1-q^{n-m})}$$

- $$\lim_{q \to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m}$$

From (*), $\displaystyle\lim_{q \to 1} \frac{(q^a;q)_n}{(1-q)^n} = a(a+1)\cdots(a+n-1)$.

Hence

$$\lim_{q \to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \lim_{q \to 1} \frac{\dfrac{(q;q)_n}{(1-q)^n}}{\dfrac{(q;q)_m}{(1-q)^m} \cdot \dfrac{(q;q)_{n-m}}{(1-q)^{n-m}}}$$

$$= \frac{n!}{m!\,(n-m)!} = \binom{n}{m}.$$

* For $n \geq 1$,

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}$$

(q-analogue of 1st Pascal formula)

$$RHS = \frac{(q)_{n-1}}{(q)_{m-1}(q)_{n-m}} + q^m \frac{(q)_{n-1}}{(q)_m (q)_{n-1-m}}$$

$$= \frac{(q)_{n-1}}{(q)_{m-1}(q)_{n-1-m}} \left\{ \frac{1}{1-q^{n-m}} + \frac{q^m}{1-q^m} \right\}$$

$$= \frac{(q)_{n-1}}{(q)_{m-1}(q)_{n-1-m}} \; \frac{1-\cancel{q^m} + q^m(\cancel{1}-q^{n-m})}{(1-q^{n-m})(1-q^m)}$$

$$= \frac{(q)_n}{(q)_m (q)_{n-m}} = \begin{bmatrix} n \\ m \end{bmatrix}$$

Thm. 2 $\begin{bmatrix} n \\ m \end{bmatrix}$ is a polynomial in q of degree $m(n-m)$.

Proof: (Induction on n)

1. $n=1$: $\begin{bmatrix} 1 \\ m \end{bmatrix} = \frac{(q)_1}{(q)_m (q)_{1-m}}$

$$\begin{cases} = \dfrac{(q)_1}{(q)_0 (q)_1} = 1 & , \quad m = 0 \\ \dfrac{(q)_1}{(q)_1 (q)_0} = 1 & , \quad m = 1 \end{cases}$$

Hence $\begin{bmatrix} 1 \\ m \end{bmatrix}$ is trivially a polynomial of degree 0 $\left(= 1(1-1) \text{ or } 0(1-0) \right)$.

② Assume that $\begin{bmatrix} n \\ m \end{bmatrix}$ is a poly. in q of degree $m(n-m)$ for every $m \ni 0 \le m \le n$,

③ Consider

$$\begin{bmatrix} n+1 \\ m \end{bmatrix} = \underbrace{\begin{bmatrix} n \\ m-1 \end{bmatrix}}_{\text{poly.}} + q^m \underbrace{\begin{bmatrix} n \\ m \end{bmatrix}}_{\text{poly.}}$$

Since $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}$

Hence $\begin{bmatrix} n+1 \\ m \end{bmatrix}$ is a poly. in q of deg. $m(n+1-m)$

because $\begin{bmatrix} n \\ m-1 \end{bmatrix}$ is of deg. $(m-1)(n-m+1)$

$$= m(n-m) + m - n + m - 1$$
$$= m(n-m) + 2m - n - 1$$

& $\begin{bmatrix} n \\ m \end{bmatrix}$ is of degree $m(n-m)$

$\Rightarrow q^m \begin{bmatrix} n \\ m \end{bmatrix}$ is of degree $m + m(n-m)$
$= m(n+1-m)$

$$m(n+1-m) - (m-1)(n+1-m)$$
$$= (n+1-m) > 0 \qquad (0 \leq m \leq n)$$

Hence the result follows by induction. $\blacksquare$

Ex. 1 Prove that $n \geq 1$,

$$\begin{bmatrix} n \\ m \end{bmatrix} = q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ m \end{bmatrix}.$$

Ex. 2
$$\sum_{j=0}^{n} \begin{bmatrix} m+j \\ j \end{bmatrix} q^j = \begin{bmatrix} m+n+1 \\ m+1 \end{bmatrix}$$
for any $m, n \geq 0$.

Yesterday: $H \subseteq \mathbb{N}$.

$$\sum_{n=0}^{\infty} p(\text{"}H\text{"}, n)\, q^n = \prod_{n \in H} \frac{1}{1-q^n}.$$

When $H = \mathbb{N}$, then

$$\sum_{n=0}^{\infty} p(n)\, q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \text{ (Euler)}$$

$|q| < 1$.

Thm. 3 Euler's theorem

The number of partitions of n into odd parts equals the number of partitions into distinct parts.

Proof: Let $p_o(n) =$ —— n —— into odd parts.

$P_d(n) =$ —— n —— into distinct parts.

$$\sum_{n=0}^{\infty} p_o(n) q^n = \prod_{\substack{n \in \mathbb{N} \\ n \text{ odd}}} \frac{1}{1-q^n} = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}}$$

$$= \frac{1}{(q;q^2)_\infty}$$

$$= \frac{1}{(1-q)(1-q.q^2)(1-q^3.q^2)(1-q^5.q^2)\ldots}$$

$$= \frac{1}{(1-q)(1-q^3)(1-q^5)\ldots}$$

$$= \frac{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\ldots}{(1-q)(1-q^2)(1-q^3)(1-q^4)\ldots}$$

$$= (1+q)(1+q^2)(1+q^3)(1+q^4)\ldots$$

$$= \sum_{n=1}^{\infty} P_d(n) q^n. \qquad = (-q;q)_\infty$$

$$\downarrow$$

$$(1-(-q))(1-(-q).q)(1-(-q)q^2)\ldots$$

$$\Longrightarrow \boxed{\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty},$$

Ex. Prove that the number of partitions of n ($a(n)$) into parts which are not divisible by 3 is equal to the number of partitions of n in which no part appears more than twice, ($b(n)$).

Proof:

$$\sum_{n=0}^{\infty} a(n) q^n = \frac{1}{(1-q)(1-q^2)(1-q^4)(1-q^5)\cdots}$$

$$= \frac{1}{(q;q^3)_\infty (q^2;q^3)_\infty} = \frac{(q^3;q^3)_\infty}{\underbrace{(q;q^3)_\infty (q^2;q^3)_\infty (q^3;q^3)_\infty}},$$

$$= (1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)\cdots$$

$$= \frac{(q^3;q^3)_\infty}{(q;q)_\infty} = \prod_{n=1}^{\infty} \frac{1-q^{3n}}{1-q^n} = \prod_{n=1}^{\infty}(1+q^n+q^{2n})$$

$$(a^3-b^3 = (a-b)(a^2+ab+b^2)) \qquad = \sum_{n=0}^{\infty} b(n) q^n$$

$$(q;q)_\infty = (q;q^4)_\infty (q^2;q^4)_\infty (q^3;q^4)_\infty (q^4;q^4)_\infty$$