6/8/21

MA 633 – Partition Theory – Lec. 3

Prob. Let $A(n)$ = number of partitions of n into parts $\equiv 2, 5$ or $11 \pmod{12}$

& $B(n)$ = ——— " ——— into distinct parts $\equiv 2, 4, 5 \pmod 6$.

Then prove that $A(n) = B(n)$.

Proof: $$\sum_{n=0}^{\infty} A(n)q^n = \prod_{n=0}^{\infty} \frac{1}{(1-q^{12n+2})(1-q^{12n+5})(1-q^{12n+11})}$$

$$= \frac{1}{(q^2;q^{12})_\infty (q^5;q^{12})_\infty (q^{11};q^{12})_\infty} \longrightarrow (1)$$

&

$$\sum_{n=0}^{\infty} B(n)q^n = \prod_{n=0}^{\infty} (1+q^{6n+2})(1+q^{6n+4})(1+q^{6n+5})$$

$$= (-q^2;q^6)_\infty (-q^4;q^6)_\infty (-q^5;q^6)_\infty \longrightarrow (2)$$

Compact notation:

$$(a_1, a_2, \cdots, a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty$$

We would like to show that the products in (1) & (2) are equal.

$$\text{LHS} = \frac{1}{(q^2;q^{12})_\infty (q^5,q^{11};q^{12})_\infty}$$

$$= \frac{1}{(q;q^6)_\infty(-q;q^6)_\infty(q^5,q^{11};q^{12})_\infty}$$

$$\left[(a^2;q^2)_\infty = (a;q)_\infty(-a;q)_\infty\right]$$

$$(1-q)(1-q^7)(1-q^{13})(1-q^{19})(1-q^{25})\cdots$$

$$= \frac{1}{(-q;q^6)_\infty (q;q^{12})_\infty (q^7;q^{12})_\infty (q^5,q^{11};q^{12})_\infty}$$

$$= \frac{(q^3,q^9;q^{12})_\infty}{(-q;q^6)_\infty (q,q^3,q^5,q^7,q^9,q^{11};q^{12})_\infty}$$

$$= \frac{(q^3,q^9;q^{12})_\infty}{(-q;q^6)_\infty (q;q^2)_\infty}$$

Euler:

$$\left[\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty\right]$$

$$\overset{\text{Euler}}{=} \frac{(q^3;q^6)_\infty(-q;q)_\infty}{(-q;q^6)_\infty}$$

$$\overset{\text{Euler}}{=} \frac{(-q;q)_\infty}{(-q^3;q^3)_\infty(-q;q^6)_\infty}$$

$$= \frac{(\cancel{-q;q^6})_\infty(-q^2;q^6)_\infty(\cancel{-q^3;q^6})_\infty(-q^4;q^6)_\infty(-q^5;q^6)_\infty(\cancel{-q^6;q^6})_\infty}{(\cancel{-q^3;q^6})_\infty(\cancel{-q^6;q^6})_\infty(\cancel{-q;q^6})_\infty}$$

Aside: Ramanujan tau function $\tau(n)$:

$$\sum_{n=0}^{\infty} \tau(n) q^n = q(q;q)_\infty^{24}, \quad |q|<1.$$

Properties: ① $\tau(mn) = \tau(m)\tau(n)$, whenever $(m,n)=1$,

② relation for $\tau(p^{n+1})$ in terms of $\tau(p^n)$ & $\tau(p^{n-1})$

③ $|\tau(p)| \leq 2p^{11/2}$.

p prime

Lehmer's conjecture: $\tau(n) \neq 0 \ \forall n \in \mathbb{N}$.

Theta functions:

Ramanujan's theta function:

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab|<1.$$

Three special cases:

① $$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2} + \frac{n(n-1)}{2}}$$

$$= \sum_{n=-\infty}^{\infty} q^{n^2}, \quad |q|<1.$$

(2) $\psi(q) = f(q, q^3) = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} (q^3)^{\frac{n(n-1)}{2}}$

Exercise $\Rightarrow$ $= \sum_{n=0}^{\infty} q^{n(n+1)/2}$.

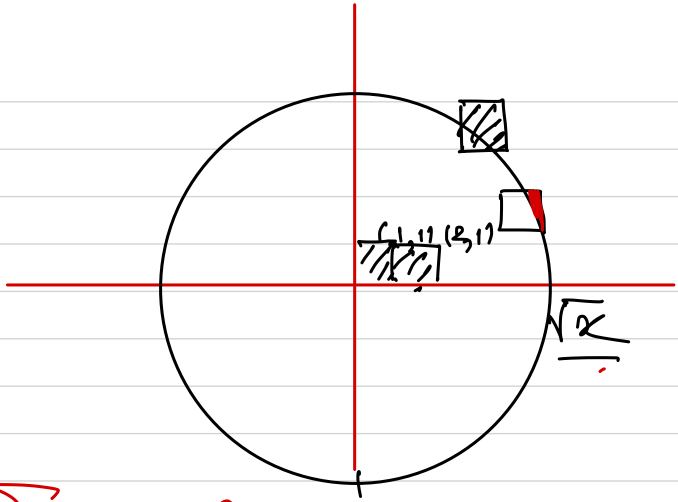
(3) $f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}$,

Remark :

(1) $\varphi^2(q) = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)\left(\sum_{m=-\infty}^{\infty} q^{m^2}\right)$

$$= \sum_{m,n=-\infty}^{\infty} q^{m^2+n^2} = \sum_{l=0}^{\infty} r_2(l) q^l,$$

where $r_2(l)$ = number of representations of l as sum of 2 squares, with order & signs considered distinct.



$x = a^2 + b^2$

$$\sum_{n \le x} r_2(n) = \pi x + \text{error}$$

$$O(\sqrt{x})$$

$$O(x^{1/3} \log x)$$

$$O(x^{1/3})$$

0.31

$$O\left(x^{\frac{131}{416}+\varepsilon}\right)$$

$$O(x^{1/4}) \quad \cancel{O\left(x^{\frac{1}{4}+\varepsilon}\right)}$$