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# MA 633 — Partition Theory — Lec. 5

**Theorem 4** (q-binomial theorem)
For $|q|<1$ & $|z|<1$,

$$\sum_{n=0}^{\infty} \frac{(a)_n z^n}{(q)_n} = \frac{(az)_\infty}{(z)_\infty}.$$

**Recall:** $(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$

**Proof:** We note that $\frac{(az)_\infty}{(z)_\infty}$ converges uniformly on compact subsets of $|z|<1$ and hence represents an analytic function on $|z|<1$.
Hence

$$\frac{(az)_\infty}{(z)_\infty} = \sum_{n=0}^{\infty} A_n z^n, \quad |z|<1 \qquad (1).$$

Let us call both sides of (1) by $F(z)$.

$$F(z) = \frac{(az)_\infty}{(z)_\infty} = \frac{(1-az)}{(1-z)} \frac{(aqz)_\infty}{(qz)_\infty}$$

$$= \frac{1-az}{1-z} \cdot F(qz)$$

$$\Rightarrow (1-z)F(z) = (1-az)F(qz)$$

$$\Rightarrow (1-z)\sum_{n=0}^{\infty} A_n z^n = (1-az)\sum_{n=0}^{\infty} A_n q^n z^n$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n z^n - \sum_{n=0}^{\infty} A_n z^{n+1} = \sum_{n=0}^{\infty} A_n q^n z^n - a\sum_{n=0}^{\infty} A_n q^n z^{n+1}$$

$$\Rightarrow A_0 + \sum_{n=1}^{\infty} (A_n - A_{n-1}) z^n$$

$$= A_0 + \sum_{n=1}^{\infty} \left(A_n q^n - a A_{n-1} q^{n-1}\right) z^n$$

Compare the coefficients of $z^n$, $n \geq 1$ on both sides to get

$$A_n - A_{n-1} = A_n q^n - a q^{n-1} A_{n-1}$$

$$\Rightarrow A_n (1-q^n) = A_{n-1}(1-aq^{n-1})$$

$$\Rightarrow A_n = \frac{1-aq^{n-1}}{1-q^n} \cdot A_{n-1}, \quad n \geq 1 \qquad \text{—— (2)}$$

Note also that the series expansion of $\frac{(az)_\infty}{(z)_\infty}$ begins with 1, i.e., $A_0 = 1$. —— (3)

From (2) & (3),

$$A_n = \frac{(1-aq^{n-1}) \cdots (1-a)}{(1-q^n) \cdots (1-q)} \cdot A_0 = \frac{(a)_n}{(q)_n}$$

—— (4)

The result now follows from (1) & (4). $\blacksquare$

Reason for calling Thm. 4 a q-analogue of binomial thm.

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty} , \quad |z|<1.$$

Replace a by $q^a$, where 'a' is a positive integer.

$$\lim_{q\to 1} \sum_{n=0}^{\infty} \frac{(q^a)_n}{(q)_n} z^n = \lim_{q\to 1} \frac{(q^a z)_\infty}{(z)_\infty}.$$

$$\text{LHS} = \sum_{n=0}^{\infty} \lim_{q\to 1} \frac{(q^a)_n}{(q)_n} z^n$$

$$\lim_{q\to 1} \frac{(q^a)_n}{(q)_n} = \lim_{q\to 1} \frac{1-q^a}{1-q} \cdot \frac{1-q^{a+1}}{1-q^2} \cdots \frac{1-q^{a+n-1}}{1-q^n}$$

$$= \frac{a(a+1)\cdots(a+n-1)}{n!}.$$

$$\text{RHS} = \lim_{q\to 1} \frac{(1-q^a z)(1-q^{a+1}z)\cdots}{(1-z)(1-qz)(1-q^2z)\cdots}$$

$$= \lim_{q \to 1} \frac{1}{(1-z)(1-qz)\cdots(1-q^{a-1}z)} = \frac{1}{(1-z)^a}.$$

Hence $$\sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)}{n!} z^n = (1-z)^{-a}; \quad |z|<1$$

which is the binomial theorem.

<u>Cor. 5</u> : (Euler) (i) $$\sum_{n=0}^{\infty} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty}, \quad |z|<1$$

(ii) $$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q)_n} = (z)_\infty, \quad |z|<\infty.$$

<u>Proof:</u> From Thm. 4, for $|z|<1$, $|q|<1$,

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty}. \quad \text{(*)}$$

Simply put $a=0$ to get (i).

(ii) Replace $a$ by $a/b$ & $z$ by $bz$ in (*):

$$\sum_{n=0}^{\infty} \frac{(a/b)_n (bz)^n}{(q)_n} = \frac{\left(\frac{a}{b}\cdot bz\right)_\infty}{(bz)_\infty} = \frac{(az)_\infty}{(bz)_\infty}.$$

Now let $b \to 0$.

Hence

$$\sum_{n=0}^{\infty} \frac{\lim_{b\to 0} \left(\frac{a}{b}\right)_n b^n \cdot z^n}{(q)_n} = (az)_\infty$$

Note that

$$\lim_{b\to 0}\left(\frac{a}{b}\right)_n b^n = \lim_{b\to 0}\left(1-\frac{a}{b}\right)\left(1-\frac{aq}{b}\right)\cdots\left(1-\frac{aq^{n-1}}{b}\right)b^n$$

$$= \lim_{b\to 0}(b-a)(b-aq)\cdots(b-aq^{n-1})$$

$$= (-a)(-aq)\cdots(-aq^{n-1})$$

$$= (-a)^n q^{(n-1)n/2}$$

$$= (-a)^n q^{n(n-1)/2}.$$

Hence $$\sum_{n=0}^{\infty}\frac{(-a)^n q^{n(n-1)/2} z^n}{(q)_n} = (az)_\infty.$$

Now let $a=1$ to get the result for $|z|<\infty$
(Note that both sides are analytic in the whole complex plane.)

<u>Thm. 6</u> (Jacobi triple product identity)
For $|q|<1$ & $z\neq 0$,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty$$

— $**$

Recall: $f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n-1)}{2}} b^{\frac{n(n+1)}{2}}$, $|ab|<1$.

Then (**) can be rephrased as

$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty$.

LHS of (**) is essentially the sol$^n$ to Heat eqn.

Proof: Replace $q$ by $q^2$, and then $z$ by $-qz$ in

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q)_n} = (z)_\infty, \qquad (\S)$$

$$\sum_{n=0}^{\infty} \frac{(qz)^n q^{n^2-n}}{(q^2;q^2)_n} = (-zq;q^2)_\infty$$

$$\Rightarrow (-zq;q^2)_\infty = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} z^n q^{n^2} (q^{2n+2};q^2)_\infty$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} (q^{2n+2};q^2)_\infty, \text{ since}$$

for $(q^{2n+2};q^2)_\infty = 0$ for $n<0$.

Replacing $q$ by $q^2$, and then $z$ by $q^{2n+2}$ in ( ), we have

$$(q^{2n+2};q^2)_\infty = \sum_{m=0}^{\infty} \frac{(-q^{2n+2})^m q^{m(m-1)}}{(q^2;q^2)_m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2-m+2mn+2m}}{(q^2;q^2)_m}$$

Thus,

$$(-zq;q^2)_\infty = \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \sum_{m=0}^{\infty} \frac{(-1)^m q^{(m^2+2mn)+m}}{(q^2;q^2)_m}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^m z^{-m}}{(q^2;q^2)_m} \underbrace{\sum_{n=-\infty}^{\infty} z^{n+m} q^{(n+m)^2}}_{\text{let } m+n=k}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{\left(-\frac{q}{z}\right)^m}{(q^2;q^2)_m} \sum_{k=-\infty}^{\infty} z^k q^{k^2}$$

From Cor. 5 (i), we have

$$\sum_{n=0}^{\infty} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty}, \quad |z|<1$$

Replace $q$ by $q^2$ & then $z$ by $-q/z$ in the above formula so that for $\left|-\frac{q}{z}\right| < 1$, we have

$$\sum_{m=0}^{\infty} \frac{(-q/z)^m}{(q^2;q^2)_m} = \frac{1}{(-q/z;q^2)_\infty}.$$

Hence the theorem is proved for $|q| < |z|$

By analytic continuation, the result follows for all $z \neq 0$,

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