ANALOGUES OF A TRANSFORMATION FORMULA OF RAMANUJAN

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Dedicated to Professor Bruce C. Berndt on the occasion of his 70th birthday

Abstract. We derive two new analogues of a transformation formula of Ramanujan involving the Gamma function and Riemann zeta function $\zeta(s)$ present in the Lost Notebook. Both involve infinite series consisting of Hurwitz zeta functions and yield modular-type relations. As a special case of the first formula, we obtain an identity involving polygamma functions given by A.P. Guinand and as a limiting case of the second formula, we derive the transformation formula of Ramanujan.

1. Introduction

In the volume [11] containing Ramanujan's Lost Notebook are present some manuscripts of Ramanujan in the handwriting of G.N. Watson. The first of these manuscripts contains the following beautiful claim (see [11, p. 220]).

Theorem 1.1. *Define*

$$\phi(x) := \psi(x) + \frac{1}{2x} - \log x, \tag{1.1}$$

where

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{m=0}^{\infty}\left(\frac{1}{m+x} - \frac{1}{m+1}\right) \tag{1.2}$$

is the logarithmic derivative of the Gamma function. Let the Riemann ξ-function be defined by

$$\xi(s) := (s-1)\pi^{-\frac{1}{2}s}\Gamma(1+\tfrac{1}{2}s)\zeta(s),$$

and let

$$\Xi(t) := \xi(\tfrac{1}{2}+it) \tag{1.3}$$

be the Riemann Ξ-function. If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{k=1}^{\infty}\phi(k\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{k=1}^{\infty}\phi(k\beta)\right\}$$
$$= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt, \tag{1.4}$$

where γ denotes Euler's constant.

2010 *Mathematics Subject Classification.* Primary 11M06, Secondary 11M35.
Keywords and phrases. Riemann Ξ-function, Riemann zeta function, Hurwitz zeta function, Gamma function.

A.P. Guinand [4, 5] rediscovered the first equality in (1.4) in a slightly different form. Recently, B.C. Berndt and A. Dixit [1] proved both parts of (1.4). A key element in their proof was the identity [9, p. 260, eqn. (22)] or [10, p. 77, eqn. (22)], true for n real,

$$\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\left(\Xi\left(\frac{1}{2}t\right)\right)^2\frac{\cos nt}{1+t^2}\,dt$$
$$=\pi^{3/2}\int_0^\infty\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)dx. \tag{1.5}$$

Ramanujan's paper [9] contains other identities similar to (1.5). Motivated by the use of (1.5) in deriving (1.4), we work with two other identities in [9, eqns. (19), (20)] to derive two new analogues of (1.4). This shows that Ramanujan's result (1.4) is not isolated and that there may be many more results of this type. Some such results, apart from some of the ones discussed here, are obtained in [2] and [3]. But the techniques employed in this paper are different than the ones used in those papers in that here we use only the theory of special functions and do not use contour integration.

At this stage, it must be pointed out that identity (19) in [9], which will be used in proving our first analogue, as it stands, is incorrect. The second term on its right-hand side, namely, $-\frac{1}{4}(4\pi)^{\frac{(s-3)}{2}}\Gamma(s)\zeta(s)\cosh n(1-s)$ should not be present. Furthermore, there is another identity in [9], namely identity (21), which is incorrect, since the second term on the right-hand side, namely, $-\frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\Gamma(1+s)\zeta(1+s)\cosh n(1+s)$, should not be present. Identities (19) (corrected), (20) and (21) (corrected) are respectively as follows:

Theorem 1.2. *For* Re $s>1$ *and n real,*

$$\int_0^\infty \Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\frac{\cos nt}{(s+1)^2+t^2}\,dt$$
$$=\frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\int_0^\infty\frac{x^s}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}\,dx, \tag{1.6}$$

where $\Xi(t)$ *is as defined in* (1.3).

Theorem 1.3. *For* $-1<$ Re $s<1$ *and n real,*

$$\int_0^\infty \Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\frac{\cos nt}{(s+1)^2+t^2}\,dt$$
$$=\frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\int_0^\infty x^s\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)dx. \tag{1.7}$$

The identity (1.5) is the special case $s=0$ of (1.7).

Theorem 1.4. *For* $-3 < \text{Re } s < -1$ *and* n *real,*

$$\int_0^\infty \Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\frac{\cos nt}{(s+1)^2+t^2}\,dt$$
$$= \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\int_0^\infty x^s\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}+\frac{1}{2}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}+\frac{1}{2}\right)dx. \tag{1.8}$$

Now we state the two key theorems in this paper which give two new analogues of (1.4).

Theorem 1.5. *Let* $\zeta(z, a)$ *denote the Hurwitz zeta function defined for* $\text{Re } z > 1$ *by*

$$\zeta(z,a) = \sum_{k=0}^\infty \frac{1}{(k+a)^z}. \tag{1.9}$$

If α *and* β *are positive numbers such that* $\alpha\beta = 1$*, then for* $\text{Re } z > 2$ *and* $1 < c < \text{Re } z - 1$*,*

$$\alpha^{-\frac{z}{2}}\sum_{k=1}^\infty \zeta\left(z, 1+\frac{k}{\alpha}\right) = \beta^{-\frac{z}{2}}\sum_{k=1}^\infty \zeta\left(z, 1+\frac{k}{\beta}\right)$$
$$= \frac{\alpha^{\frac{z}{2}}}{2\pi i\Gamma(z)}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\,ds$$
$$= \frac{8(4\pi)^{\frac{z-4}{2}}}{\Gamma(z)}\int_0^\infty \Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt, \tag{1.10}$$

where Ξ *is defined as in* (1.3).

Theorem 1.6. *Let* $0 < \text{Re } z < 2$*. Define* $\varphi(z, x)$ *as*

$$\varphi(z,x) = \zeta(z,x) - \frac{1}{2}x^{-z} + \frac{x^{1-z}}{1-z}, \tag{1.11}$$

where $\zeta(z, x)$ *denotes the Hurwitz zeta function. Then if* α *and* β *are any positive numbers such that* $\alpha\beta = 1$*,*

$$\alpha^{\frac{z}{2}}\left(\sum_{k=1}^\infty \varphi(z,k\alpha) - \frac{\zeta(z)}{2\alpha^z} - \frac{\zeta(z-1)}{\alpha(z-1)}\right) = \beta^{\frac{z}{2}}\left(\sum_{k=1}^\infty \varphi(z,k\beta) - \frac{\zeta(z)}{2\beta^z} - \frac{\zeta(z-1)}{\beta(z-1)}\right)$$
$$= \frac{8(4\pi)^{\frac{z-4}{2}}}{\Gamma(z)}\int_0^\infty \Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt, \tag{1.12}$$

where $\Xi(t)$ *is defined in* (1.3).

This paper is organized as follows. In Section 2, some basic properties of Mellin transforms are reviewed. Then in Section 3, we derive an analogue of (1.4), namely Theorem 1.5, using two different methods, both of which make use of (1.6). In Section 4, we derive a second analogue of (1.4), namely Theorem 1.6, which makes use of (1.7), and gives (1.4) as a limiting case. Finally in Sections 5 and 6, identities (1.6) and (1.8) are respectively proved.

2. Basic Properties of Mellin Transforms

Let $F(z)$ denote the Mellin transform of $f(x)$, i.e.,

$$F(z) = \int_0^\infty x^{z-1} f(x)\, dx. \tag{2.1}$$

Then the inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(z) x^{-z}\, dz, \tag{2.2}$$

where c lies in the fundamental strip (or the strip of analyticity) for which $F(z)$ is defined. We also note the Mellin convolution theorem [8, p. 83] which states that if $F(z)$ and $G(z)$ are Mellin transforms of $f(x)$ and $g(x)$ respectively, then

$$\int_0^\infty x^{z-1} f(x) g(x)\, dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) G(z-s)\, ds, \tag{2.3}$$

c again being in the associated fundamental strip.

Now let $F(z)$ be related to $f(x)$ by (2.1) and (2.2), where $f(x)$ is locally integrable on $(0, \infty)$, is $O(x^{-a})$ as $x \to 0^+$ and $O(x^{-b})$, where $b > 1$, as $x \to \infty$ and $a < c < b$. Then for $\max\{1, a\} < c < b$, we have

$$\sum_{k=1}^\infty f(kx) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \zeta(s) x^{-s}\, ds, \tag{2.4}$$

where $\zeta(s)$ denotes the Riemann zeta function (see [8, p. 117]).

3. The first modular relation involving Hurwitz zeta functions

In this section, we prove the first of the two "modular relations", i.e., Theorem 1.5, mentioned in the introduction. Generally a "modular relation" is a relation between two expressions governed by the transformation $z \mapsto -1/z$, where z is a complex variable. Any such relation can be cast into the form $\alpha\beta =$ constant which is the formulation Ramanujan employed and which we also adopt in this paper.

Theorem 3.1. *Let $\zeta(z, a)$ denote the Hurwitz zeta function defined for* Re $z > 1$ *by* (1.9). *If α and β are positive numbers such that $\alpha\beta = 1$, then for* Re $z > 2$ *and* $1 < c <$ Re $z - 1$,

$$\begin{aligned}
&\alpha^{-\frac{z}{2}} \sum_{k=1}^\infty \zeta\left(z, 1 + \frac{k}{\alpha}\right) = \beta^{-\frac{z}{2}} \sum_{k=1}^\infty \zeta\left(z, 1 + \frac{k}{\beta}\right) \\
&= \frac{\alpha^{\frac{z}{2}}}{2\pi i \Gamma(z)} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\, ds \\
&= \frac{8(4\pi)^{\frac{z-4}{2}}}{\Gamma(z)} \int_0^\infty \Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\, dt,
\end{aligned} \tag{3.1}$$

where Ξ *is defined as in* (1.3).

First proof: Replace s by $z-1$ in (1.6) and then multiply the resulting two sides by $8(4\pi)^{\frac{z-4}{2}}e^{-nz}$, so that for Re $z>2$ and n real, we have

$$8(4\pi)^{\frac{z-4}{2}}e^{-nz}\int_0^\infty \Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos nt}{z^2+t^2}\,dt$$
$$=e^{-nz}\int_0^\infty \frac{x^{z-1}}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}\,dx. \tag{3.2}$$

The integral on the right-hand side of (3.2) can be viewed as the Mellin transform of the product of $\dfrac{1}{e^{xe^n}-1}$ and $\dfrac{1}{e^{xe^{-n}}-1}$.

But for Re $z>1$, the integral representation of $\zeta(z)$ [12, p. 18, eqn. (2.4.1)] gives

$$\Gamma(z)\zeta(z)=\int_0^\infty \frac{t^{z-1}}{e^t-1}\,dt. \tag{3.3}$$

Employing a change of variable $t=xe^n$ in (3.3), we deduce that

$$e^{-nz}\Gamma(z)\zeta(z)=\int_0^\infty \frac{x^{z-1}}{e^{xe^n}-1}\,dx. \tag{3.4}$$

Similarly letting $t=xe^{-n}$ in (3.3), we find that

$$e^{nz}\Gamma(z)\zeta(z)=\int_0^\infty \frac{x^{z-1}}{e^{xe^{-n}}-1}\,dx. \tag{3.5}$$

Thus from (2.3), (3.4) and (3.5), it can be seen that for $1<c<\text{Re } z-1$,

$$\int_0^\infty \frac{x^{z-1}}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}\,dx=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-ns}\Gamma(s)\zeta(s)e^{n(z-s)}\Gamma(z-s)\zeta(z-s)\,ds, \tag{3.6}$$

which can be written as

$$e^{-nz}\int_0^\infty \frac{x^{z-1}}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}\,dx=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-2ns}\Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\,ds. \tag{3.7}$$

Letting $n=\frac{1}{2}\log\alpha$ in (3.2) and (3.7) and combining them together, we arrive at

$$8(4\pi)^{\frac{z-4}{2}}\alpha^{-\frac{z}{2}}\int_0^\infty \Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt$$
$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\,ds. \tag{3.8}$$

Upon simplification, this gives the last equality in (3.1). Now since $1<c<\text{Re } z-1$, on the vertical line Re $s=c$, we have Re $(z-s)>1$. Therefore we can use the representation

$$\zeta(z-s)=\sum_{k=1}^\infty \frac{1}{k^{z-s}}. \tag{3.9}$$

Using (3.9) on the right-hand side of (3.8), by absolute convergence, it can be seen that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\,ds=\sum_{k=1}^{\infty}k^{-z}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\zeta(s)\Gamma(z-s)\left(\frac{\alpha}{k}\right)^{-s}\,ds$$

$$=\Gamma(z)\sum_{k=1}^{\infty}k^{-z}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\Gamma(s)\Gamma(z-s)}{\Gamma(z)}\right)\zeta(s)\left(\frac{\alpha}{k}\right)^{-s}\,ds. \tag{3.10}$$

Now we know that for $0<\text{Re }s<\text{Re }z$, Euler's beta integral $B(s,z-s)$ is given by

$$B(s,z-s)=\int_0^\infty\frac{x^{s-1}}{(1+x)^z}\,dx=\frac{\Gamma(s)\Gamma(z-s)}{\Gamma(z)}. \tag{3.11}$$

In other words, $B(s,z-s)$ is the Mellin transform of $\frac{1}{(1+x)^z}$. For $\text{Re }z>2$, $f(x):=\frac{1}{(1+x)^z}$ is locally integrable on $(0,\infty)$. Also, as $x\to0^+$, $f(x)=O(1)$ and as $x\to\infty$, $f(x)\sim\frac{1}{x^z}=O\left(\frac{1}{x^{\text{Re}(z)}}\right)$. In particular, for $1<c<\text{Re }z-1$, using (2.4) and (3.11), we find that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}B(s,z-s)\zeta(s)x^{-s}\,ds=\sum_{m=1}^{\infty}(1+xm)^{-z}. \tag{3.12}$$

The integral in the second expression in (3.10) can also be directly evaluated using formula 5.78 in [7, p. 202].

From (3.10), (3.11) and (3.12), we arrive at

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\,ds=\Gamma(z)\sum_{k=1}^{\infty}k^{-z}\sum_{m=1}^{\infty}\left(1+\frac{\alpha m}{k}\right)^{-z}$$

$$=\Gamma(z)\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}(k+\alpha m)^{-z}=\alpha^{-z}\Gamma(z)\sum_{k=1}^{\infty}\zeta\left(z,1+\frac{k}{\alpha}\right). \tag{3.13}$$

Invoking (3.13) in (3.8), simplifying and rearranging, it is observed that

$$\alpha^{-\frac{z}{2}}\sum_{k=1}^{\infty}\zeta\left(z,1+\frac{k}{\alpha}\right)$$

$$=\frac{8(4\pi)^{\frac{(z-4)}{2}}}{\Gamma(z)}\int_0^\infty\Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt. \tag{3.14}$$

This shows that the extreme left and right-hand sides in Theorem 3.1 are equal. Now replacing α by β in (3.14), and making use of the fact that $\alpha\beta=1$ and that $\cos\theta$ is an even function of θ, we obtain the equality of second and fourth expressions in Theorem 3.1 as well. This completes the proof of Theorem 3.1. □

Remark: An alternative way to proceed from (3.8) is to use the series definition of $\zeta(s)$, interchange the order of summation and integration, and then use the following formula from [7, p. 202, formula 5.78] valid for $0 < c = \text{Re } s < \text{Re } z - 1$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(z-s)\zeta(z-s)a^{-s}x^{-s}\,ds = \Gamma(z)\zeta(z, 1+ax), \tag{3.15}$$

with $x = 1$. But since the evaluation of (3.8) given previously is short and self-contained, we chose to do it that way.

Second proof: Letting $n = \frac{1}{2}\log\alpha$ in (3.2) and multiplying both sides by $\dfrac{1}{\Gamma(z)}$, we see that

$$\begin{aligned}
&\frac{8(4\pi)^{\frac{(z-4)}{2}}}{\Gamma(z)}\int_0^\infty \Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt\\
&=\frac{1}{\Gamma(z)}\int_0^\infty \frac{x^{z-1}}{(e^{x\sqrt{\alpha}}-1)(e^{x/\sqrt{\alpha}}-1)}\,dx\\
&=\frac{\alpha^{-\frac{z}{2}}}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{(e^{t}-1)(e^{t/\alpha}-1)}\,dt.
\end{aligned} \tag{3.16}$$

Now,

$$\begin{aligned}
\frac{\alpha^{-\frac{z}{2}}}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{(e^{t}-1)(e^{t/\alpha}-1)}\,dt &= \frac{\alpha^{-\frac{z}{2}}}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}e^{-t}}{1-e^{-t}}\sum_{k=1}^\infty e^{-kt/\alpha}\,dt\\
&=\frac{\alpha^{-\frac{z}{2}}}{\Gamma(z)}\sum_{k=1}^\infty\int_0^\infty \frac{t^{z-1}e^{-(1+k/\alpha)t}}{1-e^{-t}}\,dt,
\end{aligned} \tag{3.17}$$

where the order of summation and integration can be interchanged because of absolute convergence. But from [12, p. 37, eqn. (2.17.1)], we know that for $\text{Re } z > 1$,

$$\zeta(z,a) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{x^{z-1}e^{-ax}}{1-e^{-x}}\,dx. \tag{3.18}$$

Using (3.18) in (3.17), we deduce that

$$\frac{\alpha^{-\frac{z}{2}}}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{(e^{t}-1)(e^{t/\alpha}-1)}\,dt = \alpha^{-\frac{z}{2}}\sum_{k=1}^\infty \zeta\left(z, 1+\frac{k}{\alpha}\right). \tag{3.19}$$

Thus from (3.16) and (3.19), we derive (3.14). Then following the same argument as in the first proof, we obtain the equality of second and fourth expressions in (3.1) as well. This finishes the second proof.

□

Corollary 3.2. *For* Re $z > 2$, *we have*

$$\sum_{k=1}^\infty \zeta\left(z, 1+k\right) = \zeta(z-1) - \zeta(z)$$

$$= \frac{8(4\pi)^{\frac{z-4}{2}}}{\Gamma(z)} \int_0^\infty \Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{dt}{z^2+t^2}. \tag{3.20}$$

Proof. Set $\alpha = 1$ in (3.1) and note that from [12, p. 35], for $1 < c < \text{Re } z - 1$, we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\, ds = \Gamma(z)\left(\zeta(z-1)-\zeta(z)\right). \tag{3.21}$$

□

3.1. **Guinand's formula as a special case of (3.1).** Let $\psi^{(j)}(x)$ denote the j^{th} derivative of the digamma function $\psi(x)$ defined in (1.2), also known as the polygamma function of order j. In [5], Guinand gave the following formula

$$\sum_{k=1}^\infty \psi^{(j)}(1+kx) = x^{-j-1}\sum_{k=1}^\infty \psi^{(j)}\left(1+\frac{k}{x}\right), \tag{3.22}$$

where $j \geq 2$. We derive this formula as a special case of (3.1). Let $z \in \mathbb{N}, z > 2$. From (1.2), by successive differentiation, it is seen that

$$\psi^{(z-1)}(x) = (-1)^z(z-1)!\sum_{m=1}^\infty \frac{1}{(m-1+x)^z}. \tag{3.23}$$

Thus,

$$\begin{aligned}\alpha^{\frac{z}{2}}\sum_{k=1}^\infty \psi^{(z-1)}(1+k\alpha) &= (-1)^z(z-1)!\alpha^{\frac{z}{2}}\sum_{k=1}^\infty\sum_{m=1}^\infty \frac{1}{(m+k\alpha)^z}\\ &= (-1)^z(z-1)!\alpha^{\frac{z}{2}}\sum_{m=1}^\infty\sum_{k=1}^\infty \frac{1}{(m+k\alpha)^z}\\ &= (-1)^z(z-1)!\alpha^{\frac{-z}{2}}\sum_{m=1}^\infty\sum_{k=1}^\infty \frac{1}{(k+m/\alpha)^z}\\ &= (-1)^z(z-1)!\alpha^{\frac{-z}{2}}\sum_{m=1}^\infty \zeta\left(z, 1+\frac{m}{\alpha}\right),\end{aligned} \tag{3.24}$$

where the change in the order of summation in the second equality is justified by absolute convergence.

Then from (3.24) and (3.1), we obtain the following alternative version of (3.1) when z is a natural number greater than 2:

$$\begin{aligned}&\alpha^{\frac{z}{2}}\sum_{k=1}^\infty \psi^{(z-1)}(1+k\alpha) = \beta^{\frac{z}{2}}\sum_{k=1}^\infty \psi^{(z-1)}(1+k\beta)\\ &= 8(-1)^z(4\pi)^{\frac{z-4}{2}}\\ &\quad\times\int_0^\infty \Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\, dt,\end{aligned} \tag{3.25}$$

since $\Gamma(z) = (z-1)!$. To obtain (3.22), we simply replace $z-1$ by j, α by x and β by $1/x$ in the first equality of (3.25).

4. The second modular relation involving Hurwitz zeta functions

Theorem 4.1. *Let* $0 < \text{Re } z < 2$. *Define* $\varphi(z,x)$ *as*

$$\varphi(z,x) = \zeta(z,x) - \frac{1}{2}x^{-z} + \frac{x^{1-z}}{1-z}, \tag{4.1}$$

where $\zeta(z,x)$ *denotes the Hurwitz zeta function. Then if* α *and* β *are any positive numbers such that* $\alpha\beta = 1$,

$$\alpha^{\frac{z}{2}}\left(\sum_{k=1}^{\infty}\varphi(z,k\alpha) - \frac{\zeta(z)}{2\alpha^z} - \frac{\zeta(z-1)}{\alpha(z-1)}\right) = \beta^{\frac{z}{2}}\left(\sum_{k=1}^{\infty}\varphi(z,k\beta) - \frac{\zeta(z)}{2\beta^z} - \frac{\zeta(z-1)}{\beta(z-1)}\right)$$
$$= \frac{8(4\pi)^{\frac{z-4}{2}}}{\Gamma(z)}\int_0^{\infty}\Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt, \tag{4.2}$$

where $\Xi(t)$ *is defined in* (1.3).

Proof. The asymptotic expansion of $\zeta(z,x)$ [6, p. 25] for large $|x|$ and $|\arg x| < \pi$ is given by

$$\zeta(z,x) = \frac{1}{\Gamma(z)}\left(x^{1-z}\Gamma(z-1) + \frac{1}{2}\Gamma(z)x^{-z} + \sum_{k=1}^{m-1}\frac{B_{2k}}{(2k)!}\Gamma(z+2k-1)x^{-2k-z+1}\right) + O(x^{-2m-z-1}). \tag{4.3}$$

Hence for $0 < \text{Re } z < 2$, the series $\sum_{k=1}^{\infty}\varphi(z,k\alpha)$ as well as $\sum_{k=1}^{\infty}\varphi(z,k\beta)$ are analytic functions. We first prove the result for $1 < \text{Re } z < 2$ and later extend it to $0 < \text{Re } z < 2$ using analytic continuation.

Replacing s by $z-1$ in (1.7), we find that for $0 < \text{Re } z < 2$,

$$\int_0^{\infty}\Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos nt}{z^2+t^2}\,dt$$
$$= \frac{1}{8}(4\pi)^{-\frac{(z-4)}{2}}\int_0^{\infty}x^{z-1}\left(\frac{1}{e^{xe^n}-1} - \frac{1}{xe^n}\right)\left(\frac{1}{e^{xe^{-n}}-1} - \frac{1}{xe^{-n}}\right)dx. \tag{4.4}$$

Multiplying both sides of (4.4) by $8(4\pi)^{\frac{(z-4)}{2}}$ and then letting $n = \frac{1}{2}\log\alpha$, we see that

$$8(4\pi)^{\frac{(z-4)}{2}}\int_0^{\infty}\Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt$$
$$= \int_0^{\infty}x^{z-1}\left(\frac{1}{e^{x\sqrt{\alpha}}-1} - \frac{1}{x\sqrt{\alpha}}\right)\left(\frac{1}{e^{x/\sqrt{\alpha}}-1} - \frac{1}{x/\sqrt{\alpha}}\right)dx. \tag{4.5}$$

Making a change of variable $x = t/\sqrt{\alpha}$ in the integral on the right-hand side of (4.5), we have

$$
\begin{aligned}
&8(4\pi)^{\frac{(z-4)}{2}}\int_0^\infty \Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt\\
&=\alpha^{-z/2}\int_0^\infty t^{z-1}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)\left(\frac{1}{e^{t/\alpha}-1}-\frac{1}{t/\alpha}\right)dt\\
&=\alpha^{-z/2}\int_0^\infty t^{z-1}\left(\frac{1}{(e^{t/\alpha}-1)(e^t-1)}-\frac{\alpha}{t(e^t-1)}-\frac{1}{t(e^{t/\alpha}-1)}+\frac{\alpha}{t^2}\right)dt\\
&=I_1(z,\alpha)+I_2(z,\alpha), \qquad (4.6)
\end{aligned}
$$

where

$$
I_1(z,\alpha)=\alpha^{-z/2}\int_0^\infty t^{z-1}\left(\frac{1}{(e^{t/\alpha}-1)(e^t-1)}-\frac{\alpha}{t(e^t-1)}+\frac{e^{-t/\alpha}}{2t}\right)dt, \qquad (4.7)
$$

and

$$
I_2(z,\alpha)=\alpha^{-z/2}\int_0^\infty t^{z-1}\left(\frac{-1}{t(e^{t/\alpha}-1)}+\frac{\alpha}{t^2}-\frac{e^{-t/\alpha}}{2t}\right)dt. \qquad (4.8)
$$

First,

$$
\begin{aligned}
I_1(z,\alpha)&=\alpha^{-z/2}\int_0^\infty t^{z-1}\left(\frac{1}{(e^{t/\alpha}-1)(e^t-1)}-\frac{\alpha}{t(e^t-1)}+\frac{1}{2(e^t-1)}-\frac{1}{2(e^t-1)}+\frac{e^{-t/\alpha}}{2t}\right)dt\\
&=\alpha^{-z/2}\int_0^\infty \frac{t^{z-1}}{(e^t-1)}\left(\frac{1}{(e^{t/\alpha}-1)}-\frac{1}{t/\alpha}+\frac{1}{2}\right)dt\\
&\quad-\frac{\alpha^{-z/2}}{2}\int_0^\infty t^{z-1}\left(\frac{1}{e^t-1}-\frac{e^{-t/\alpha}}{t}\right)dt\\
&=I_3(z,\alpha)+I_4(z,\alpha), \qquad (4.9)
\end{aligned}
$$

where

$$
I_3(z,\alpha)=\alpha^{-z/2}\int_0^\infty \frac{t^{z-1}}{(e^t-1)}\left(\frac{1}{(e^{t/\alpha}-1)}-\frac{1}{t/\alpha}+\frac{1}{2}\right)dt, \qquad (4.10)
$$

and

$$
I_4(z,\alpha)=-\frac{\alpha^{-z/2}}{2}\int_0^\infty t^{z-1}\left(\frac{1}{e^t-1}-\frac{e^{-t/\alpha}}{t}\right)dt. \qquad (4.11)
$$

Now for evaluating $I_3(z,\alpha)$, we make use of the following formula [6, p. 23] valid for $\text{Re}\ z>-1$ and $\text{Re}\ a>0$

$$
\zeta(z,a)=\frac{a^{-z}}{2}-\frac{a^{1-z}}{1-z}+\frac{1}{\Gamma(z)}\int_0^\infty e^{-ax}x^{z-1}\left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}\right)dx. \qquad (4.12)
$$

Since $t > 0$, expanding $\dfrac{1}{(e^t - 1)}$ in terms of its geometric series and then interchanging the summation and integration because of absolute convergence, we find that

$$\begin{aligned} I_3(z,\alpha) &= \alpha^{-z/2}\int_0^\infty \frac{t^{z-1}e^{-t}}{1-e^{-t}}\left(\frac{1}{e^{t/\alpha}-1}-\frac{1}{t/\alpha}+\frac{1}{2}\right)dt \\ &= \alpha^{-z/2}\sum_{k=1}^\infty \int_0^\infty t^{z-1}e^{-kt}\left(\frac{1}{e^{t/\alpha}-1}-\frac{1}{t/\alpha}+\frac{1}{2}\right)dt \\ &= \alpha^{z/2}\sum_{k=1}^\infty \Gamma(z)\left(\zeta(z,k\alpha)-\frac{(k\alpha)^{-z}}{2}+\frac{(k\alpha)^{1-z}}{1-z}\right) \\ &= \alpha^{z/2}\Gamma(z)\sum_{k=1}^\infty \varphi(z,k\alpha), \end{aligned} \tag{4.13}$$

where in the penultimate step, we have made a change of variable $t = \alpha x$ and then used (4.12). Next we evaluate $I_4(z,\alpha)$. Since Re $z > 1$, using (3.3) and the integral representation for $\Gamma(z-1)$, we find that

$$\begin{aligned} I_4(z,\alpha) &= -\frac{\alpha^{-z/2}}{2}\left(\int_0^\infty \frac{t^{z-1}}{e^t-1}\,dt - \int_0^\infty t^{z-2}e^{-t/\alpha}\,dt\right) \\ &= -\frac{\alpha^{-z/2}}{2}\left(\Gamma(z)\zeta(z)-\alpha^{z-1}\Gamma(z-1)\right) \\ &= -\frac{\alpha^{-\frac{z}{2}}}{2}\Gamma(z)\zeta(z)+\frac{\alpha^{\frac{z}{2}-1}}{2}\Gamma(z-1). \end{aligned} \tag{4.14}$$

Hence from (4.9), (4.13) and (4.14), it is seen that

$$I_1(z,\alpha) = \alpha^{z/2}\Gamma(z)\sum_{k=1}^\infty \varphi(z,k\alpha) - \frac{\alpha^{-\frac{z}{2}}}{2}\Gamma(z)\zeta(z)+\frac{\alpha^{\frac{z}{2}-1}}{2}\Gamma(z-1) \tag{4.15}$$

It remains to evaluate $I_2(z,\alpha)$. Now from [12, p. 23, eqn. (2.7.1)] for $0 < \text{Re } z < 1$, we have

$$\Gamma(z)\zeta(z) = \int_0^\infty t^{z-1}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)dt. \tag{4.16}$$

Thus employing a change of variable $u = t/\alpha$ in (4.8) and then using (4.16) and the integral representation for $\Gamma(z-1)$, we see that

$$\begin{aligned} I_2(z,\alpha) &= -\alpha^{\frac{z}{2}-1}\int_0^\infty u^{z-1}\left(\frac{1}{u(e^u-1)}-\frac{1}{u^2}+\frac{e^{-u}}{2u}\right)du \\ &= -\alpha^{\frac{z}{2}-1}\left(\int_0^\infty u^{z-2}\left(\frac{1}{e^u-1}-\frac{1}{u}\right)du+\frac{1}{2}\int_0^\infty e^{-u}u^{z-2}\,du\right) \\ &= -\alpha^{\frac{z}{2}-1}\left(\Gamma(z-1)\zeta(z-1)+\frac{1}{2}\Gamma(z-1)\right). \end{aligned} \tag{4.17}$$

Finally from (4.6), (4.15) and (4.17), we see that after simplification

$$\frac{8(4\pi)^{\frac{(z-4)}{2}}}{\Gamma(z)}\int_0^{\infty}\Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt$$
$$=\alpha^{\frac{z}{2}}\left(\sum_{k=1}^{\infty}\varphi(z,k\alpha)-\frac{\zeta(z)}{2\alpha^z}-\frac{\zeta(z-1)}{\alpha(z-1)}\right). \tag{4.18}$$

Now replacing α by β in (4.18) and then combining the result with (4.18), we arrive at (4.2), since with $\alpha\beta=1$, the left-hand side of (4.18) is invariant under the map $\alpha\to\beta$. This implies (4.2) for $1<\text{Re } z<2$. Now using Stirling's formula on a vertical strip, which states that for $s=\sigma+it$, $\alpha\leq\sigma\leq\beta$ and $|t|\geq 1$, we have

$$|\Gamma(s)|=(2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|}\left(1+O\left(\frac{1}{|t|}\right)\right), \tag{4.19}$$

as $t\to\infty$ and using estimates on the zeta function, it is straightforward to show that the extreme right-hand side of (4.2) is absolutely and uniformly convergent for $0<\text{Re } z<2$ and hence analytic in that strip. Also the first two expressions in (4.2) are analytic for $0<\text{Re } z<2$, except for a possible pole at $z=1$. But it is easily seen that $z=1$ is a removable singularity because the residue of $\zeta(z)$ at $z=1$ is equal to 1 and because $\zeta(0)=-\frac{1}{2}$. Hence by analytic continuation, (4.2) holds for $0<\text{Re } z<2$. □

As a limiting case of (4.2), we obtain Ramanujan's transformation formula, i.e., identity (1.4).

Corollary 4.2. *If*

$$\phi(x):=\psi(x)+\frac{1}{2x}-\log x, \tag{4.20}$$

and α and β are positive numbers such that $\alpha\beta=1$, then

$$\sqrt{\alpha}\left\{\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{k=1}^{\infty}\phi(k\alpha)\right\}=\sqrt{\beta}\left\{\frac{\gamma-\log(2\pi\beta)}{2\beta}+\sum_{k=1}^{\infty}\phi(k\beta)\right\}$$
$$=-\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt. \tag{4.21}$$

Proof. Let $z\to 1$ in (4.2). Then using Lebesgue's dominated convergence theorem, we observe that

$$\lim_{z\to1}\alpha^{\frac{z}{2}}\left(\sum_{k=1}^{\infty}\varphi(z,k\alpha)-\frac{\zeta(z)}{2\alpha^z}-\frac{\zeta(z-1)}{\alpha(z-1)}\right)=\lim_{z\to1}\beta^{\frac{z}{2}}\left(\sum_{k=1}^{\infty}\varphi(z,k\beta)-\frac{\zeta(z)}{2\beta^z}-\frac{\zeta(z-1)}{\beta(z-1)}\right)$$
$$=\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt. \tag{4.22}$$

Now since $\sum_{k=1}^{\infty}\varphi(z,k\alpha)$ and $\sum_{k=1}^{\infty}\varphi(z,k\beta)$ converge absolutely and uniformly in a neighborhood of $z=1$, which can be seen from (4.3), we observe that

$$\begin{aligned}&\lim_{z\to1}\alpha^{\frac{z}{2}}\left(\sum_{k=1}^{\infty}\varphi(z,k\alpha)-\frac{\zeta(z)}{2\alpha^z}-\frac{\zeta(z-1)}{\alpha(z-1)}\right)\\&=\left(\lim_{z\to1}\alpha^{\frac{z}{2}}\right)\cdot\left(\sum_{k=1}^{\infty}\left(\lim_{z\to1}\varphi(z,k\alpha)\right)-\lim_{z\to1}\left(\frac{\zeta(z)}{2\alpha^z}+\frac{\zeta(z-1)}{\alpha(z-1)}\right)\right)\\&=\left(\lim_{z\to1}\alpha^{\frac{z}{2}}\right)\cdot\left(\sum_{k=1}^{\infty}\left(\lim_{z\to1}\varphi(z,k\alpha)\right)-\lim_{z\to1}\left(\frac{\zeta(z)}{2\alpha^z}-\frac{1}{2\alpha^z(z-1)}\right)-\lim_{z\to1}\left(\frac{1}{2\alpha^z(z-1)}+\frac{\zeta(z-1)}{\alpha(z-1)}\right)\right).\end{aligned}\tag{4.23}$$

But it is known [12, p. 16] that

$$\lim_{s\to1}\left(\zeta(s)-\frac{1}{s-1}\right)=\gamma.\tag{4.24}$$

Hence

$$\lim_{z\to1}\left(\frac{\zeta(z)}{2\alpha^z}-\frac{1}{2\alpha^z(z-1)}\right)=\frac{\gamma}{2\alpha}.\tag{4.25}$$

Next using L'Hopital's rule, we see that

$$\begin{aligned}\lim_{z\to1}\left(\frac{1}{2\alpha^z(z-1)}+\frac{\zeta(z-1)}{\alpha(z-1)}\right)&=\lim_{z\to1}\frac{1}{2\alpha^z}\cdot\lim_{z\to1}\frac{1+2\alpha^{z-1}\zeta(z-1)}{z-1}\\&=\lim_{z\to1}\frac{1}{2\alpha^z}\cdot\lim_{z\to1}\left(2\alpha^{z-1}\zeta'(z-1)+2\zeta(z-1)\alpha^{z-1}\log\alpha\right)\\&=\frac{1}{2\alpha}\left(2\zeta'(0)+2\zeta(0)\log\alpha\right)\\&=-\frac{\log(2\pi\alpha)}{2\alpha},\end{aligned}\tag{4.26}$$

since $\zeta(0)=-\frac{1}{2}$ and $\zeta'(0)=-\frac{1}{2}\log(2\pi)$ [12, pp. 19-20, eqns. (2.4.3), (2.4.5)]. Now noting that [6, p. 23]

$$\lim_{z\to1}\left(\zeta(z,a)-\frac{1}{z-1}\right)=-\psi(a),\tag{4.27}$$

and using L'Hopital's rule again, we observe that

$$\begin{aligned}&\lim_{z\to1}\varphi(z,k\alpha)\\&=\lim_{z\to1}\left(\zeta(z,k\alpha)-\frac{1}{2}(k\alpha)^{-z}+\frac{(k\alpha)^{1-z}}{1-z}\right)\\&=\lim_{z\to1}\left[\left(\zeta(z,k\alpha)-\frac{1}{z-1}\right)-\frac{(k\alpha)^{-z}}{2}+\frac{(k\alpha)^{1-z}-1}{1-z}\right]\\&=-\psi(k\alpha)-\frac{1}{2k\alpha}+\lim_{z\to1}\frac{-(k\alpha)^{1-z}\log k\alpha}{-1}\end{aligned}$$

$$= -\psi(k\alpha) - \frac{1}{2k\alpha} + \log(k\alpha)$$
$$= -\phi(k\alpha), \tag{4.28}$$

where $\phi(x)$ is as defined in (4.20). Hence from (4.23), (4.25), (4.26) and (4.28), we find that

$$\lim_{z\to1} \alpha^{\frac{z}{2}} \left(\sum_{k=1}^{\infty} \varphi(z, k\alpha) - \frac{\zeta(z)}{2\alpha^z} - \frac{\zeta(z-1)}{\alpha(z-1)} \right) = -\sqrt{\alpha} \left(\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{k=1}^{\infty} \phi(k\alpha) \right), \tag{4.29}$$

Thus from (4.22), (4.29) and (4.29) with α replaced by β, we obtain

$$-\sqrt{\alpha} \left\{ \frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{k=1}^{\infty} \phi(k\alpha) \right\} = -\sqrt{\beta} \left\{ \frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{k=1}^{\infty} \phi(k\beta) \right\}$$
$$= \frac{1}{\pi^{3/2}} \int_0^{\infty} \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\, dt. \tag{4.30}$$

Multiplying (4.30) throughout by -1, we arrive at (4.21). ☐

5. Proof of (1.6)

Here we show that the second term on the right-hand side of identity (19) in [9], namely, $-\frac{1}{4}(4\pi)^{\frac{(s-3)}{2}}\Gamma(s)\zeta(s)\cosh n(1-s)$ is not present and thus indeed that (1.6) is actually the correct version of identity (19) in [9]. Since the exposition in Sections 4 and 5 of [9] is quite terse, we will derive (1.6) giving all the details. We will collect and prove, wherever necessary, several ingredients required for the proof along the way.

First, identity(15) in [9] states that for Re $s > -1$ and $\alpha\beta = 4\pi^2$, we have

$$G(s;\alpha) := \frac{\zeta(1-s)}{4\cos(\pi s/2)}\alpha^{(s-1)/2} + \frac{\zeta(-s)}{8\sin(\pi s/2)}\alpha^{(s+1)/2} + \alpha^{(s+1)/2} \int_0^{\infty}\int_0^{\infty} \frac{x^s \sin(\alpha xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\, dxdy$$
$$= \frac{\zeta(1-s)}{4\cos(\pi s/2)}\beta^{(s-1)/2} + \frac{\zeta(-s)}{8\sin(\pi s/2)}\beta^{(s+1)/2} + \beta^{(s+1)/2} \int_0^{\infty}\int_0^{\infty} \frac{x^s \sin(\beta xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\, dxdy. \tag{5.1}$$

This relation can be proved by obtaining integral representations for $\frac{\zeta(1-s)}{4\cos(\pi s/2)}$ and $\frac{\zeta(-s)}{8\sin(\pi s/2)}$ and by using the identity (see [9, p. 253])

$$\int_0^{\infty} \frac{\sin(\alpha xy)}{e^{2\pi y}-1}\, dy = \frac{1}{2}\left(\frac{1}{e^{\alpha x}-1} - \frac{1}{\alpha x} + \frac{1}{2} \right). \tag{5.2}$$

Since we are concerned with the case Re $s > 1$ as far as identity (19) in [9] is concerned, we prove (5.1) for Re $s > 1$ only. Other cases can be similarly proved.

Now the functional equation for $\zeta(s)$ in its non-symmetric form [12, p. 13, eqn. (2.1.1)] states that

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s)\zeta(1-s)\sin\left(\tfrac{1}{2}\pi s\right). \tag{5.3}$$

Using this and (3.3), one can easily show that for Re $s>1$,

$$\frac{\zeta(1-s)}{4\cos(\frac{1}{2}\pi s)}=\frac{1}{2}\int_0^\infty \frac{x^{s-1}dx}{e^{2\pi x}-1}, \tag{5.4}$$

and

$$\frac{\zeta(-s)}{8\sin(\frac{1}{2}\pi s)}=\frac{-1}{4}\int_0^\infty \frac{x^{s}dx}{e^{2\pi x}-1}. \tag{5.5}$$

Hence using (5.2), (5.4) and (5.5) in (5.1), we see that

$$G(s;\alpha)=\frac{\alpha^{(s+1)/2}}{2}\int_0^\infty \frac{x^s}{(e^{2\pi x}-1)(e^{\alpha x}-1)}\,dx. \tag{5.6}$$

So (5.1) will be proved for Re $s>1$ if we can show that

$$\frac{\alpha^{(s+1)/2}}{2}\int_0^\infty \frac{x^s}{(e^{2\pi x}-1)(e^{\alpha x}-1)}\,dx=\frac{\beta^{(s+1)/2}}{2}\int_0^\infty \frac{x^s}{(e^{2\pi x}-1)(e^{\beta x}-1)}\,dx. \tag{5.7}$$

But this is easily seen by making the substitution $x=\dfrac{2\pi y}{\alpha}$ on the left-hand side of (5.7) and using the fact that $\alpha\beta=4\pi^2$. Thus (5.1) is proved for Re $s>1$.

Next, identity (17) in [9] states that when $\alpha\beta=4\pi^2$ and Re $s>-1$, we have

$$\begin{aligned}
&\frac{\zeta(1-s)}{4\cos(\pi s/2)}\frac{s-1}{(s-1)^2+t^2}\left(\alpha^{(s-1)/2}+\beta^{(s-1)/2}\right)+\frac{\zeta(-s)}{8\sin(\pi s/2)}\frac{s+1}{(s+1)^2+t^2}\left(\alpha^{(s+1)/2}+\beta^{(s+1)/2}\right)\\
&+\alpha^{(s+1)/2}\int_0^\infty\int_0^\infty\left(\frac{\alpha xy}{1!}\frac{s+3}{(s+3)^2+t^2}-\frac{(\alpha xy)^3}{3!}\frac{s+7}{(s+7)^2+t^2}+\cdots\right)\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\\
&+\beta^{(s+1)/2}\int_0^\infty\int_0^\infty\left(\frac{\beta xy}{1!}\frac{s+3}{(s+3)^2+t^2}-\frac{(\beta xy)^3}{3!}\frac{s+7}{(s+7)^2+t^2}+\cdots\right)\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\\
&=\frac{2^{(s-3)/2}}{\pi}\frac{\Gamma\left(\frac{1}{4}(s-1+it)\right)\Gamma\left(\frac{1}{4}(s-1-it)\right)}{(s+1)^2+t^2}\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\cos\left(\frac{1}{4}t\log\frac{\alpha}{\beta}\right).
\end{aligned} \tag{5.8}$$

Letting $\alpha=\beta=2\pi$ in (5.8) and simplifying, we find that

$$\begin{aligned}
&\frac{\zeta(1-s)}{\pi\cos(\pi s/2)}\frac{s-1}{(s-1)^2+t^2}+\frac{\zeta(-s)}{\sin(\pi s/2)}\frac{s+1}{(s+1)^2+t^2}\\
&+8\int_0^\infty\int_0^\infty\left(\frac{2\pi xy}{1!}\frac{s+3}{(s+3)^2+t^2}-\frac{(2\pi xy)^3}{3!}\frac{s+7}{(s+7)^2+t^2}+\cdots\right)\frac{x^s\,dx\,dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\\
&=\frac{1}{\pi^{(s+3)/2}}\frac{\Gamma\left(\frac{1}{4}(s-1+it)\right)\Gamma\left(\frac{1}{4}(s-1-it)\right)}{(s+1)^2+t^2}\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right).
\end{aligned} \tag{5.9}$$

Now we know that for Re $a>0$,

$$\int_0^\infty e^{-au}\cos bu\,du=\frac{a}{a^2+b^2}. \tag{5.10}$$

Using (5.10), we wish to replace the fractions of the form $\frac{s+j}{(s+j)^2+t^2}$ in (5.9) by integrals. Since Re $s > 1$, for $j \geq -1$, we have

$$\frac{s+j}{(s+j)^2+t^2} = \int_0^\infty e^{-(s+j)u} \cos tu \, du. \tag{5.11}$$

Hence using (5.11) in (5.9), inverting the order of integration because of absolute convergence and simplifying, we see that

$$\begin{aligned}
&\frac{1}{\pi^{(s+3)/2}} \frac{\Gamma\left(\frac{1}{4}(s-1+it)\right)\Gamma\left(\frac{1}{4}(s-1-it)\right)}{(s+1)^2+t^2} \Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right) \\
&= \frac{\zeta(1-s)}{\pi\cos(\pi s/2)} \int_0^\infty e^{-(s-1)u}\cos tu\,du + \frac{\zeta(-s)}{\sin(\pi s/2)} \int_0^\infty e^{-(s+1)u}\cos tu\,du \\
&\quad + 8\int_0^\infty\int_0^\infty \left(\frac{2\pi xy}{1!}\int_0^\infty e^{-(s+3)u}\cos tu\,du - \frac{(2\pi xy)^3}{3!}\int_0^\infty e^{-(s+7)u}\cos tu\,du + \cdots\right) \\
&\qquad \times \frac{x^s\,dxdy}{(e^{2\pi x}-1)(e^{2\pi y}-1)} \\
&= \int_0^\infty \left[\frac{\zeta(1-s)}{\pi\cos(\pi s/2)}e^{-(s-1)u} + \frac{\zeta(-s)}{\sin(\pi s/2)}e^{-(s+1)u}\right. \\
&\quad \left. + 8e^{-(s+1)u}\int_0^\infty\int_0^\infty \left(\frac{2\pi xye^{-2u}}{1!} - \frac{(2\pi xye^{-2u})^3}{3!} + \cdots\right)\frac{x^s\,dxdy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\right]\cos tu\,du \\
&= \int_0^\infty \left[\frac{\zeta(1-s)}{\pi\cos(\pi s/2)}e^{-(s-1)u} + \frac{\zeta(-s)}{\sin(\pi s/2)}e^{-(s+1)u} + 8e^{-(s+1)u}\int_0^\infty\int_0^\infty \frac{x^s\sin\left(2\pi xye^{-2u}\right)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\,dxdy\right] \\
&\qquad \times \cos tu\,du.
\end{aligned} \tag{5.12}$$

Now let

$$f(u,s) := \frac{\zeta(1-s)}{\pi\cos(\pi s/2)}e^{-(s-1)u} + \frac{\zeta(-s)}{\sin(\pi s/2)}e^{-(s+1)u} + 8e^{-(s+1)u}\int_0^\infty\int_0^\infty \frac{x^s\sin\left(2\pi xye^{-2u}\right)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\,dxdy, \tag{5.13}$$

and

$$\widehat{f}(t,s) := \frac{1}{\pi^{(s+3)/2}} \frac{\Gamma\left(\frac{1}{4}(s-1+it)\right)\Gamma\left(\frac{1}{4}(s-1-it)\right)}{(s+1)^2+t^2} \Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right). \tag{5.14}$$

Then from (5.12), (5.13) and (5.14), we have

$$\widehat{f}(t,s) = \int_0^\infty f(u,s)\cos tu\,du. \tag{5.15}$$

Now we show that f is an even function in u.

If we let $\alpha = 2\pi e^{-2u}$ and $\beta = 2\pi e^{2u}$ in (5.1), upon simplification, we find that

$$\frac{\zeta(1-s)}{\pi\cos(\pi s/2)}e^{-(s-1)u} + \frac{\zeta(-s)}{\sin(\pi s/2)}e^{-(s+1)u} + 8e^{-(s+1)u}\int_0^\infty\int_0^\infty \frac{x^s\sin\left(2\pi xye^{-2u}\right)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\,dxdy$$
$$= \frac{\zeta(1-s)}{\pi\cos(\pi s/2)}e^{(s-1)u} + \frac{\zeta(-s)}{\sin(\pi s/2)}e^{(s+1)u} + 8e^{(s+1)u}\int_0^\infty\int_0^\infty \frac{x^s\sin\left(2\pi xye^{2u}\right)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\,dxdy. \tag{5.16}$$

This proves that f is an even function in u. Also using the fact that $\Xi(-t) = \Xi(t)$, we readily observe that $\widehat{f}$ is an even function in t.

Then from Fourier's integral theorem and (5.15), we deduce that for n real,

$$f(n,s) = \frac{2}{\pi}\int_0^\infty \widehat{f}(t,s)\cos nt\,dt. \tag{5.17}$$

Now define

$$F(n,s) := \frac{\pi^{\frac{s+5}{2}}}{2}f(n,s). \tag{5.18}$$

Then from (5.13), (5.14) and (5.17), we find that

$$F(n,s) = \int_0^\infty \frac{\Gamma\left(\frac{1}{4}(s-1+it)\right)\Gamma\left(\frac{1}{4}(s-1-it)\right)}{(s+1)^2+t^2}\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\cos nt\,dt$$
$$= \frac{\pi^{\frac{s+5}{2}}}{2}\left(\frac{\zeta(1-s)}{\pi\cos(\pi s/2)}e^{-(s-1)n} + \frac{\zeta(-s)}{\sin(\pi s/2)}e^{-(s+1)n}\right.$$
$$\left. + 8e^{-(s+1)n}\int_0^\infty\int_0^\infty \frac{t^s\sin\left(2\pi tye^{-2n}\right)}{(e^{2\pi t}-1)(e^{2\pi y}-1)}\,dtdy\right). \tag{5.19}$$

Substituting (5.2), (5.4) and (5.5) in (5.19), we find that

$$F(n,s) = \frac{\pi^{\frac{s+5}{2}}}{2}\left[\frac{2e^{-(s-1)n}}{\pi}\int_0^\infty \frac{t^{s-1}dt}{e^{2\pi t}-1} - 2e^{-(s+1)n}\int_0^\infty \frac{t^sdt}{e^{2\pi t}-1}\right.$$
$$\left. + 4e^{-(s+1)n}\int_0^\infty \frac{t^sdt}{e^{2\pi t}-1}\left(\frac{1}{e^{2\pi te^{-2n}}-1} - \frac{1}{2\pi te^{-2n}} + \frac{1}{2}\right)\right]. \tag{5.20}$$

Letting $t = \dfrac{e^n x}{2\pi}$ in (5.20), we find that

$$F(n,s) = \frac{\pi^{\frac{s+5}{2}}}{2}\left[\frac{e^n}{2^{s-1}\pi^{s+1}}\int_0^\infty \frac{x^{s-1}\,dx}{(e^{xe^n}-1)} - \frac{1}{2^s\pi^{s+1}}\int_0^\infty \frac{x^s\,dx}{(e^{xe^n}-1)}\right.$$
$$\left. + \frac{1}{2^{s-1}\pi^{s+1}}\int_0^\infty \frac{x^sdx}{(e^{xe^n}-1)}\left(\frac{1}{e^{xe^{-n}}-1} - \frac{1}{xe^{-n}} + \frac{1}{2}\right)\right]$$
$$= \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\left[e^n\int_0^\infty \frac{x^{s-1}\,dx}{(e^{xe^n}-1)} - \frac{1}{2}\int_0^\infty \frac{x^s\,dx}{(e^{xe^n}-1)}\right.$$
$$\left. + \int_0^\infty \frac{x^sdx}{(e^{xe^n}-1)}\left(\frac{1}{e^{xe^{-n}}-1} - \frac{1}{xe^{-n}} + \frac{1}{2}\right)\right]$$
$$= \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\int_0^\infty \frac{x^s\,dx}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}. \tag{5.21}$$

Finally, we obtain (1.6) from (5.19) and (5.21).

6. Proof of (1.8)

We will be brief with our exposition here. We note that we cannot prove identity (1.8) using the above method of Ramanujan, since (5.8), which was crucially employed in that method, is true only when Re $s > -1$. Instead, we use a reverse route in the sense that we obtain (1.8) through a corresponding transformation formula valid for $-3 < \text{Re } s < -1$, which was established in [3] and which is as follows.

Theorem 6.1. *Let* $-3 < \text{Re } s < -1$. *Define* $\Lambda(s,x)$ *by*

$$\Lambda(s,x) = \zeta(s+1,x) - \frac{x^{-s}}{s} - \frac{1}{2}x^{-s-1} - \frac{(s+1)x^{-s-2}}{12}, \tag{6.1}$$

where $\zeta(s,x)$ *denotes the Hurwitz zeta function. Then if* α *and* β *are any positive numbers such that* $\alpha\beta = 1$,

$$\begin{aligned}
&\alpha^{\frac{s+1}{2}}\left(\sum_{n=1}^{\infty}\Lambda(s,n\alpha) - \frac{\zeta(s)}{\alpha s} + \frac{\zeta(s+1)}{2} + \frac{(s+1)\zeta(s+2)}{12\alpha^{s+2}}\right)\\
&= \beta^{\frac{s+1}{2}}\left(\sum_{n=1}^{\infty}\Lambda(s,n\beta) - \frac{\zeta(s)}{\beta s} + \frac{\zeta(s+1)}{2} + \frac{(s+1)\zeta(s+2)}{12\beta^{s+2}}\right)\\
&= \frac{8(4\pi)^{\frac{s-3}{2}}}{\Gamma(s+1)}\int_0^{\infty}\Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(s+1)^2+t^2}\,dt,
\end{aligned} \tag{6.2}$$

where $\Xi(t)$ *is defined in* (1.3).

We wish to emphasize that this does not lead us to circular reasoning since (6.2) is proved in [3] using contour integration and Mellin transforms. To establish (1.8), we do not need the full strength of (6.2), rather just the equality of the first and the third expressions. We now establish the result first for $-2 < \text{Re } s < -1$.

First of all, it is easy to adapt (4.12) to see that, for Re $s > -2$,

$$\Gamma(s+1)\Lambda(s,n\alpha) = \int_0^{\infty} e^{-ax}x^s\left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} - \frac{x}{12}\right)dx. \tag{6.3}$$

Then following a similar procedure as in (4.13), we observe that

$$\alpha^{(s+1)/2}\Gamma(s+1)\sum_{n=1}^{\infty}\Lambda(s,n\alpha) = \alpha^{-(s+1)/2}\int_0^{\infty}\frac{t^s}{(e^t-1)}\left(\frac{1}{e^{t/\alpha}-1} - \frac{1}{t/\alpha} + \frac{1}{2} - \frac{t/\alpha}{12}\right)dt. \tag{6.4}$$

Since $-2 < \text{Re } s < -1$, we also see that

$$\Gamma(s)\zeta(s) = \int_0^{\infty} t^{s-1}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right)dt,$$

$$\Gamma(s+1)\zeta(s+1) = \int_0^{\infty} t^{s}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)dt,$$

$$\Gamma(s+2)\zeta(s+2) = \int_0^\infty t^{s+1}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)dt. \tag{6.5}$$

Then from the equality of the first and the third expressions in (6.2), and from (6.4) and (6.5), we see that after simplification

$$\begin{aligned}
&8(4\pi)^{\frac{s-3}{2}}\int_0^\infty \Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(s+1)^2+t^2}\,dt\\
&=\alpha^{-(s+1)/2}\int_0^\infty t^s\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)\left(\frac{1}{e^{t/\alpha}-1}-\frac{1}{t/\alpha}+\frac{1}{2}\right)dt\\
&=\int_0^\infty x^s\left(\frac{1}{e^{x\sqrt{\alpha}}-1}-\frac{1}{x\sqrt{\alpha}}+\frac{1}{2}\right)\left(\frac{1}{e^{x/\sqrt{\alpha}}-1}-\frac{1}{x/\sqrt{\alpha}}+\frac{1}{2}\right)dx,
\end{aligned} \tag{6.6}$$

where in the ultimate step, we made a change of variable $t = x\sqrt{\alpha}$. Finally, we obtain (1.8) for $-2 < \text{Re } s < -1$ by letting $\alpha = e^{2n}$ in the last step in (6.6). Now using (4.19) and estimates on zeta function, we can show that the left-hand side of (1.8) is absolutely and uniformly convergent for $-3 < \text{Re } s < -1$ and hence analytic in that strip. Similarly, by analysing the behavior of the integrand on the right-hand side of (1.8) at 0 and at ∞, one easily sees that the right-hand side of (1.8) is analytic for $-3 < \text{Re } s < -1$. Thus by analytic continuation, (1.8) holds for $-3 < \text{Re } s < -1$.

Remarks. 1. Identity (1.7), for $-1 < \text{Re } s < 1$, is derived in a very similar manner as the derivation of (1.6) in Section 5, except that since $\text{Re } s < 1$, the first expression on the left-hand side of (5.9) is written as $-\dfrac{\zeta(1-s)}{\pi\cos(\pi s/2)}\dfrac{1-s}{(1-s)^2+t^2}$ and then we use the evaluation,

$$\frac{1-s}{(1-s)^2+t^2} = \int_0^\infty e^{(s-1)u}\cos tu\,du. \tag{6.7}$$

This along with an analysis similar to that in Section 5 gives (1.7). Now it turns out that if we use (6.7) instead of (5.11) with $j=-1$ when $\text{Re } s > 1$, then we *do* get the second term on the right-hand side of identity (19) in [9], i.e., $-\dfrac{1}{4}(4\pi)^{\frac{(s-3)}{2}}\Gamma(s)\zeta(s)\cosh n(1-s)$, as given by Ramanujan. This explains how Ramanujan was erroneously led to his identity.

Acknowledgements. The author wishes to express his gratitude to Professor Bruce C. Berndt for his constant support, careful reading of this manuscript and for helpful comments. The author would also like to thank Professor Harold G. Diamond, Boonrod Yuttanan and Jonah Sinick for their help.

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