A SIMPLE PROOF OF A CONGRUENCE FOR A SERIES INVOLVING THE LITTLE q-JACOBI POLYNOMIALS

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Dedicated to Professor George E. Andrews on the occasion of his 80th birthday

Abstract. We give a simple and a more explicit proof of a mod 4 congruence for a series involving the little q-Jacobi polynomials which arose in a recent study of a certain restricted overpartition function.

1. Introduction

In [3], Andrews, Schultz, Yee and the author studied the overpartition function $\overline{p}_\omega(n)$, namely, the number of overpartitions of n such that all odd parts are less than twice the smallest part, and in which the smallest part is always overlined. In the same paper, they obtained a representation for the generating function of $\overline{p}_\omega(n)$ in terms of a ${}_3\phi_2$ basic hypergeometric series and an infinite series involving the little q-Jacobi polynomials. The latter are given by [2, Equation (3.1)]

$$p_n(x;\alpha,\beta:q):={}_2\phi_1\left(\begin{matrix}q^{-n}, & \alpha\beta q^{n+1}\\ & \alpha q\end{matrix};qx\right), \tag{1.1}$$

where the basic hypergeometric series ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_{r+1}\\ b_1,b_2,\ldots,b_r\end{matrix};q,z\right):=\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_{r+1};q)_n}{(q;q)_n(b_1;q)_n\cdots(b_r;q)_n}z^n,$$

and where we use the notation

$$(A;q)_0=1;\quad (A;q)_n=(1-A)(1-Aq)\cdots(1-Aq^{n-1}),\quad n\geq 1,$$
$$(A;q)_\infty=\lim_{n\to\infty}(A;q)_n\quad (|q|<1).$$

The precise representation for the generating function of $\overline{p}_\omega(n)$ obtained in [3] is as follows.

Theorem 1.1. *The following identity holds for $|q|<1$:*

$$\overline{P}_\omega(q):=\sum_{n=1}^{\infty}\overline{p}_\omega(n)q^n=-\frac{1}{2}\frac{(q;q)_\infty(q;q^2)_\infty}{(-q;q)_\infty(-q;q^2)_\infty}\,{}_3\phi_2\left(\begin{matrix}-1, & iq^{1/2}, & -iq^{1/2}\\ q^{1/2}, & -q^{1/2} & \end{matrix};q,q\right)$$
$$+\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}p_{2n}(-1;q^{-2n-1},-1:q). \tag{1.2}$$

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Later, Bringmann, Jennings-Shaffer and Mahlburg [4, Theorem 1.1] showed that $\overline{P}_\omega(q)+\frac{1}{4}-\frac{\eta(4\tau)}{2\eta(2\tau)^2}$, where $q=e^{2\pi i\tau}$ and $\eta(\tau)$ is the Dedekind eta function, can be completed to a function $\hat{P}_\omega(\tau)$, which transforms like a weight 1 modular form. They called the function $\overline{P}_\omega(q)+\frac{1}{4}-\frac{\eta(4\tau)}{2\eta(2\tau)^2}$ a *higher depth mock modular form.*

While the series involving the little q-Jacobi polynomials in Theorem 1.1 itself looks formidable, it was shown in [3, Theorem 1.3] that modulo 4, it is a simple q-product. The mod 4 congruence proved in there is given below.

Theorem 1.2. *The following congruence holds:*

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}p_{2n}(-1;q^{-2n-1},-1:q)\equiv\frac{1}{2}\frac{(q;q^2)_\infty}{(-q;q^2)_\infty}\pmod{4}. \tag{1.3}$$

The proof of this congruence in [3] is beautiful but somewhat involved. The objective of this short note is to give a very simple proof of it. In fact, we derive it as a trivial corollary of the following result.

Theorem 1.3. *For $|q|<1$, we have*

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}p_{2n}(-1;q^{-2n-1},-1:q)$$
$$=\frac{1}{2}\frac{(q;q^2)_\infty}{(-q;q^2)_\infty}+\frac{4q^2}{(1+q)}\sum_{n=0}^{\infty}\frac{(q^3;q^2)_n(-q)^n}{(-q^3;q^2)_n(1+q^{2n+2})}\sum_{j=0}^{n}\frac{(-q;q)_{2j}q^{2j}}{(q^2;q)_{2j}}. \tag{1.4}$$

The presence of 4 in front of the series on the right-hand side in the above equation immediately implies that Theorem 1.2 holds.

2. Proof of Theorem 1.3

Observe that from (1.1),

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}p_{2n}(-1;q^{-2n-1},-1:q)=\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}\sum_{j=0}^{2n}\frac{(-1;q)_j}{(q;q)_j}(-q)^j. \tag{2.1}$$

However, let us first consider

$$A(q):=\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}\sum_{j=0}^{2n}\frac{(-1;q)_j}{(q;q)_j}q^j. \tag{2.2}$$

The only difference in the series on the right-hand side of (2.1) and the series in (2.2) is the presence of $(-1)^j$ inside the finite sum in the former.

To simplify $A(q)$, we start with a result of Alladi [1, p. 215, Equation (2.6)]:

$$\frac{(abq;q)_n}{(bq;q)_n}=1+b(1-a)\sum_{j=1}^{n}\frac{(abq;q)_{j-1}q^j}{(bq;q)_j}. \tag{2.3}$$

Let $a=-1, b=1$ and replace n by $2n$ so that

$$\sum_{j=0}^{2n} \frac{(-1;q)_j q^j}{(q;q)_j} = \frac{(-q;q)_{2n}}{(q;q)_{2n}}. \tag{2.4}$$

Substitute (2.4) in (2.2) to see that

$$\begin{aligned} A(q) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})} \frac{(-q;q)_{2n}}{(q;q)_{2n}} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}}{(q^2;q^2)_n}(-q)^n \\ &= \frac{1}{2}\frac{(q;q^2)_\infty}{(-q;q^2)_\infty}, \end{aligned} \tag{2.5}$$

where in the last step we used the q-binomial theorem $\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}$, valid for $|z|<1$ and $|q|<1$.

From (2.1) and (2.2),

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})} p_{2n}(-1;q^{-2n-1},-1:q) - A(q) \\ &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})} \sum_{j=0}^{2n}((-1)^j-1)\frac{(-1;q)_j q^j}{(q;q)_j} \\ &= -2\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})} \sum_{j=1}^{n} \frac{(-1;q)_{2j-1}q^{2j-1}}{(q;q)_{2j-1}} \\ &= \frac{4q^2}{(1+q)} \sum_{n=0}^{\infty} \frac{(q^3;q^2)_n(-q)^n}{(-q^3;q^2)_n(1+q^{2n+2})} \sum_{j=0}^{n} \frac{(-q;q)_{2j}q^{2j}}{(q^2;q)_{2j}}. \end{aligned} \tag{2.6}$$

Invoking (2.5), we see that the proof of Theorem 1.3 is complete.

References

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