PARTITION IMPLICATIONS OF A THREE PARAMETER q-SERIES IDENTITY

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Abstract. A generalization of a beautiful q-series identity found in the unorganized portion of Ramanujan's second and third notebooks is obtained. As a consequence, we derive a three-parameter identity which is a rich source of partition-theoretic information. In particular, we use this identity to obtain a generalization of a recent result of Andrews, Garvan and Liang, which itself generalizes the famous result of Fokkink, Fokkink and Wang. This three-parameter identity also leads to several new weighted partition identities as well as a natural proof of a recent result of Garvan. This natural proof gives interesting number-theoretic information along the way. We also obtain a new result consisting of an infinite series involving a special case of Fine's function $F(a,b;t)$, namely, $F(0,q^n;cq^n)$. For $c=1$, this gives Andrews' famous identity for $\mathrm{spt}(n)$ whereas for $c=-1,0$ and q, it unravels new relations that the divisor function $d(n)$ has with other partition-theoretic functions such as the largest parts function $\mathrm{lpt}(n)$.

1. Introduction

The unorganized portion of Ramanujan's second and third notebooks contains five q-series identities [40, p. 354–355], [42, p. 302–303] that were first proved by Berndt [19, p. 262–265]. The first one is [19, p. 262, Entry 1]

$$\frac{(-aq)_\infty}{(bq)_\infty}=\sum_{n=0}^{\infty}\frac{(-b/a)_n a^n q^{n(n+1)/2}}{(q)_n(bq)_n},$$

where $a\neq 0, 1-bq^n\neq 0, n\geq 1$. It is also recorded in the Lost Notebook [41, p. 370]. Here, and throughout the paper, $|q|<1$, and

$$\begin{aligned}(A)_0 &:= (A;q)_0 = 1,\\ (A)_n &:= (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}), \qquad n\geq 1,\\ (A)_\infty &:= (A;q)_\infty = \lim_{n\to\infty}(A;q)_n.\end{aligned} \tag{1.1}$$

Moreover, the definition in (1.1) can be extended to all integers n by defining

$$(A)_n=\frac{(A)_\infty}{(Aq^n)_\infty}.$$

2010 *Mathematics Subject Classification.* Primary 11P81, 11P84; Secondary 05A17.

Keywords and phrases. partitions, q-series, smallest parts function, largest parts function, divisor function, weighted partition identities.

The third one in the list [40, p. 354], [19, p. 263, Entry 2] states that for $a \neq q^{-n}, n \geq 1$,

$$\sum_{n=1}^{\infty} \frac{na^n q^{n^2}}{(q)_n (aq)_n} = \frac{1}{(aq)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a^n q^{n(n+1)/2}}{1-q^n}.$$

The sequence generated by the special case $a = 1$ of the series on the above left-hand side was posted by Deutsch on the Online Encyclopedia of Integer Sequences (A115995). Andrews, Chan and Kim [14, p. 82] rediscovered the above identity. The second one in Ramanujan's list [40, p. 354], [19, p. 264, Entry 4] is given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}}{(1-q^n)(zq)_n} = \sum_{n=1}^{\infty} \frac{z^n q^n}{1-q^n}, \tag{1.2}$$

where $|zq| < 1, z \neq q^{-n}, n \geq 1$, was rediscovered by Uchimura [43, Equation (3)] and Garvan [29]. The special case $z = 1$ of the above identity is due to Kluyver [36]:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{(1-q^n)(q)_n} = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}. \tag{1.3}$$

It was rediscovered by Fine [27, p. 14, Equations (12.4), (12.42)], Uchimura [43, Theorem 2] and Zudilin [49, p. 4]. Kluyver's identity admits a beautiful partition-theoretic interpretation due to Fokkink, Fokkink and Wang [28]. Before stating this interpretation, we discuss the notation used throughout the paper.

- π: an integer partition,
- $p(n)$: the number of integer partitions of n
- $s(\pi) :=$ the smallest part of π,
- $l(\pi) :=$ the largest part of π,
- $\#(\pi) :=$ the number of parts of π,
- $\mathrm{rank}(\pi) = l(\pi) - \#(\pi)$,
- $L(\pi) :=$ total number of appearances of the largest part of π,
- $\nu_d(\pi) :=$ the number of parts of π not counting multiplicity,
- $\mathcal{P}(n) :=$ collection of all integer partitions of n,
- $\mathcal{D}(n) :=$ collection of all partitions of n into distinct parts,
- $\mathcal{P}_o(n) :=$ collection of all overpartitions of n,
- $\mathcal{P}^*(n) :=$ collection of partitions without gaps (that is, partitions into consecutive integers with smallest part 1).

Then the result of Fokkink, Fokkink and Wang, which was, in fact, obtained by Bressoud and Subbarao [22] much before, states that if $d(n)$ denotes the number of divisors of n,

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} s(\pi) = d(n). \tag{1.4}$$

In his seminal paper [10], Andrews revisited the above interpretation and showed that its generating function version, namely Kluyver's identity (1.3), is simply a corollary of the differentiation of the q-analogue of Gauss' theorem [7, p. 20, Corollary 2.4]. There is a vast literature on such q-series identities related to divisor functions. For developments on this topic since the appearance of Kluyver's identity (1.3), we refer the reader to a paper of Guo and Zeng [32] though we do point out those that are relevant here. As shown in the paper of Ismail and Stanton [35], the genesis of many of such q-series identities lies in the theory of basic hypergeometric functions. For recent developments since the appearance of Guo and Zeng's paper, see [45] and [46].

Andrews, Garvan and Liang [16] denoted the left-hand side of (1.4) by $\mathrm{FFW}(1,n)$ and considered its generalization $\mathrm{FFW}(c,n)$ defined by

$$\mathrm{FFW}(c,n) := \sum_{\pi\in\mathcal{D}(n)} (-1)^{\#(\pi)-1}\left(1+c+\cdots+c^{s(\pi)-1}\right). \tag{1.5}$$

They showed that [16, Theorem 3.5]

$$\sum_{n=1}^{\infty}\mathrm{FFW}(c,n)q^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\frac{n(n+1)}{2}}}{(1-cq^n)(q)_n} = \frac{1}{1-c}\left(1-\frac{(q)_\infty}{(cq)_\infty}\right). \tag{1.6}$$

The above identity is valid for $|cq|<1, c\neq 1$. It is worth mentioning that Yan and Fu [47, p. 117] obtained (1.6) a year before [16] appeared. Another generalization of (1.4) was obtained by Patkowski through a 'sum of tails' identity [39, Corollary 2.4].

In this paper, among other things, we generalize the above result of Andrews, Garvan and Liang by introducing one more variable, namely, z. This is achieved by first obtaining a more general result which generalizes yet another beautiful identity of Ramanujan which is the last one in [40, p. 354], [19, p. 263, Entry 3]. This identity does not seem to have received the attention it should have. It states that for $a\neq 0, |a|<1$, and $|b|<1$,

$$\sum_{n=1}^{\infty}\frac{(b/a)_n a^n}{(1-q^n)(b)_n} = \sum_{n=1}^{\infty}\frac{a^n-b^n}{1-q^n}. \tag{1.7}$$

2. Main results

Our generalization of (1.7) is given in the following theorem.

Theorem 2.1. *Let a,b,c be three complex numbers such that $|a|<1$ and $|cq|<1$. Then*

$$\sum_{n=1}^{\infty}\frac{(b/a)_n a^n}{(1-cq^n)(b)_n} = \sum_{m=0}^{\infty}\frac{(b/c)_m c^m}{(b)_m}\left(\frac{aq^m}{1-aq^m}-\frac{bq^m}{1-bq^m}\right). \tag{2.1}$$

Moreover, for $|a|<1$ and $|b|<\min(|c|,1)$,

$$\sum_{n=1}^{\infty}\frac{(b/a)_n a^n}{(1-cq^n)(b)_n} = \frac{(b/c)_\infty}{(b)_\infty}\sum_{n=0}^{\infty}\frac{(c)_n(b/c)^n}{(q)_n}\sum_{m=1}^{\infty}\frac{a^m-b^m}{1-cq^{m+n}}. \tag{2.2}$$

Remark. We mention here once for all that in the identities in the above theorem and elsewhere in the paper, the general requirement observed is that all values of the variables leading to zero in the denominators of the expressions involved are to be omitted.

It is easy to see that when we let $c = 1$ in (2.2), only the $n = 0$ term survives in the series on the right-hand side resulting in (1.7).

Letting $a \to 0$ and replacing b by zq in (2.1) leads to the aforementioned generalization of Andrews, Garvan and Liang's (1.6) given below.

Theorem 2.2. *For $|cq| < 1$,*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}}{(1 - cq^n)(zq)_n} = \frac{z}{c} \sum_{n=1}^{\infty} \frac{(zq/c)_{n-1}}{(zq)_n} (cq)^n. \tag{2.3}$$

The study of weighted counts of partitions was initiated by Alladi [1], [2], [3]. There have been interesting further studies on weighted partition identities [5], [10], [18], [30], to name a few. Comparing the coefficients of q^n on both sides of (2.3), we obtain the following general weighted partition identity which gives several interesting corollaries, some of which are new and others, well-known.

Theorem 2.3. *If z and c are not functions of q,*

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} z^{l(\pi)+1-s(\pi)} c^{s(\pi)-1} \frac{\left(\frac{z}{c}\right)^{s(\pi)} - 1}{\left(\frac{z}{c}\right) - 1} = \sum_{\pi \in \mathcal{P}(n)} z^{\#(\pi)} c^{l(\pi)-1} \left(1 - \tfrac{1}{c}\right)^{\nu_d(\pi)-1}. \tag{2.4}$$

If we let $z = 1$ in Theorem 2.2, we obtain the following result, of which (2.5) as well as the left-hand side of (2.6) was obtained by Andrews, Garvan and Liang [16].

Corollary 2.4. *For $|cq| < 1, c \neq 1$, we have*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{(1 - cq^n)(q)_n} = \sum_{n=1}^{\infty} \frac{c^{n-1}(q/c)_{n-1} q^n}{(q)_n} = \frac{1}{1-c}\left(1 - \frac{(q)_\infty}{(cq)_\infty}\right). \tag{2.5}$$

Hence if c is not a function of q,

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} \left(\frac{c^{s(\pi)} - 1}{c - 1}\right) = \sum_{\pi \in \mathcal{P}(n)} c^{l(\pi)-1} \left(1 - \frac{1}{c}\right)^{\nu_d(\pi)-1}. \tag{2.6}$$

One of the most important q-series of Ramanujan, and which has been the source of investigation from the point of view of both analytic and algebraic number theory, is

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(-q)_n}.$$

For deep results associated with $\sigma(q)$, the reader is referred to [8], [9] and [15]. In [17], some results associated with one- and two-variable generalization of $\sigma(q)$, namely $\sigma(c, q)$ and $\sigma(c, d, q)$ were obtained. These functions are defined by

$$\sigma(c, q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(1 - cq^n)(-q)_n}, \tag{2.7}$$

$$\sigma(c, d, q) := \sum_{n=0}^{\infty} \frac{(-cd)_n (1 - cdq^{2n}) q^{n(n+1)/2}}{(1 - cq^n)(1 - dq^n)(-q)_n}. \tag{2.8}$$

For example, it was shown that [17, Theorem 1.1] for $|c|<1$,

$$\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(1-cq^n)(-q)_n}=\frac{1}{(-c)_\infty}\left(\sigma(q)+\frac{(-c)_\infty}{c-1}+2\sum_{m,n=0}^{\infty}\frac{(-q)_m}{(q)_m(q)_n}\frac{(-1)^nq^{n(n+1)/2}c^{m+n+1}}{(1-q^{n+m+1})}\right).$$

Letting $z=-1$ in Theorem 2.2 gives us a simpler representation of $\sigma(c,q)$ given below.

Corollary 2.5. *For $|cq|<1$, we have*

$$\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(1-cq^n)(-q)_n}=\sum_{n=1}^{\infty}\frac{\left(\frac{-q}{c}\right)_{n-1}c^{n-1}q^n}{(-q)_n}.$$

If c is not a function of q, then

$$\sum_{\pi\in\mathcal{D}(n)}(-1)^{\mathrm{rank}(\pi)}\frac{(1-(-c)^{s(\pi)})}{1+c}=\sum_{\pi\in\mathcal{P}(n)}(-1)^{\#(\pi)-1}c^{l(\pi)-1}\left(1-\frac{1}{c}\right)^{\nu_d(\pi)-1}.$$

When $c=0$, the left-hand side of the above equation readily implies the result in [8] (see Equation (2.1) with $t=-1$) that $\sigma(q)$ is the generating function for the excess number of partition of n into distinct parts with even rank over those with odd rank.

Now let $c=1$ in Theorem 2.2 to get

Corollary 2.6. *Equation* (1.2) *holds. Moreover, if z is not a function of q,*

$$\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)-1}z^{l(\pi)+1}\left(\frac{1-z^{-s(\pi)}}{z-1}\right)=\sum_{d|n}z^d.$$

Again, if we let $c=-1$ in Theorem 2.2, then we get the following corollary.

Corollary 2.7. *We have*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^nq^{\frac{n(n+1)}{2}}}{(1+q^n)(zq)_n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-zq)_{n-1}q^n}{(zq)_n}.$$

Hence if z is not a function of q,

$$\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)+s(\pi)}z^{l(\pi)+1-s(\pi)}\left(\frac{(-z)^{s(\pi)}-1}{-z-1}\right)=\sum_{\pi\in\mathcal{P}(n)}(-1)^{l(\pi)-1}z^{\#(\pi)}2^{\nu_d(\pi)-1}.$$

A few more weighted partition identities resulting from Theorem 2.3 are discussed in Section 5.

Theorem 2.2 also allows us to give a natural proof of a recent identity of Garvan [30, Equation (1.3)] used to prove nice weighted partition identities [30, Corollary 1.3]. It states that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^nq^{n^2}}{(zq;q^2)_n(1-zq^{2n})}=\sum_{n=1}^{\infty}\frac{z^nq^{\frac{n(n+1)}{2}}(q;q)_{n-1}}{(zq;q)_n}.\tag{2.9}$$

Garvan's proof in [30] requires that both sides of the identity be known first, that is, the identity is proved by reducing it to an equivalent one and the latter is proved by showing that the coefficients of z^m of this equivalent identity are equal. Our proof, on the other hand, is direct.

Differentiation of q-series identities often plays a crucial role in deriving new and fundamental results in the theory of partitions and basic hypergeometric series [10], [16], [24], [25], [26]. The q-series identities in this paper are no exception to this.

Differentiating both sides of (2.3) with respect to z and letting $z = 1$ leads to the following new result.

Theorem 2.8. *Let Fine's function $F(a, b; t)$ be defined by* [27, p. 1]

$$F(a, b; t) := \sum_{n=0}^{\infty} \frac{(aq)_n}{(bq)_n} t^n. \tag{2.10}$$

Then for $|cq| < 1$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{n(n+1)/2}}{(1 - cq^n)(q)_n} + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}(q^{n+1})_\infty}{(1 - q^n)(q)_n} F(0, q^n; cq^n)\\ &= \frac{-c}{(1-c)^2} + \frac{(q)_\infty}{(c)_\infty}\left(\frac{c}{1-c} + \sum_{n=1}^{\infty} \frac{(cq)_n}{(q)_n} \frac{q^n}{1-q^n}\right).\end{aligned} \tag{2.11}$$

While we are unable to further simplify the second series on the left-hand side of the above theorem for a general c, the simplification is achieved in the special cases $c = 0, \pm 1$. In fact, it is a pleasant surprise that letting $c \to 1$ in the above result and equating the coefficients of q^n on both sides of the resulting identity, yields Andrews' famous identity [10, Theorem 3] given below. This is proved in Section 7.

Corollary 2.9. *Let* $\mathrm{spt}(n)$ *enumerate the number of smallest parts in all partitions of n and $N_2(n)$ be the second Atkin-Garvan rank moment defined by* [10, Equation (2.13)]

$$N_2(n) := \sum_{m=-\infty}^{\infty} m^2 N(m, n),$$

where $N(m, n)$ is the number of partitions of n with rank m. Then

$$\mathrm{spt}(n) = np(n) - \frac{1}{2} N_2(n). \tag{2.12}$$

The case $c = 0$ of Theorem 2.2 is not interesting. However, the special case $c = 0$ of Theorem 2.8 gives a nice relation between the divisor function $d(n)$ and $\mathrm{lpt}(n)$, that is, the total number of appearances of largest parts in all partitions of n.

Corollary 2.10. *We have*

$$\sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1 - q^n)(q)_n^2} = \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n)(q)_n}. \tag{2.13}$$

Let $\nu(i)$ denote the number of appearances of i in a partition of a positive integer with the understanding that $\nu(l(\pi)) = L(\pi)$. Define $w(n)$ by

$$w(n) := \sum_{\substack{\pi\in\mathcal{P}^*(n)\\ \nu(i)\geq 2, 1\leq i\leq l(\pi)}} \frac{L(\pi)(L(\pi)-1)}{2} \prod_{i=1}^{l(\pi)-1} (\nu(i)-1). \tag{2.14}$$

Then (2.13) *implies*

$$d(n) + w(n) = \mathrm{lpt}(n). \tag{2.15}$$

In his fascinating work on Vassiliev invariants, Zagier [48, Theorem 2] found a nice *sum of tails identity*:

$$\sum_{n=0}^{\infty} ((q)_n - (q)_\infty) = -\frac{1}{2}H(q) + (q)_\infty \left(\frac{1}{2} - E(q)\right), \tag{2.16}$$

where

$$H(q) := \sum_{n=1}^{\infty} n \left(\frac{12}{n}\right) q^{\frac{n^2-1}{24}} \quad \text{and} \quad E(q) := \sum_{n=1}^{\infty} \frac{q^n}{1-q^n},$$

with $\left(\frac{\bullet}{n}\right)$ being the Jacobi symbol.

Using this result as well as our Corollary 2.10, we find an explicit representation for $w(n)$ in terms of the divisor and partition functions.

Corollary 2.11. *Let $w(n)$ be defined in* (2.14)*. Then*

$$w(n) = -2d(n) - \frac{1}{2} \sum_{k=1}^{\lfloor \sqrt{24n+1} \rfloor} k \left(\frac{12}{k}\right) p\left(n - \frac{(k^2-1)}{24}\right), \tag{2.17}$$

with the understanding that $p(x) = 0$ if x is not a positive integer.

As defined in [16], let V denote the set of vector partitions, that is, $V = \mathcal{D}\times\mathcal{P}\times\mathcal{P}$, where $\mathcal{P}$ denotes the set of partitions and $\mathcal{D}$ denotes the set of partitions into distinct parts. Let S denote the following set of vector partitions:

$$S := \{\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V : 1 \leq s(\pi_1) < \infty \text{ and } s(\pi_1) \leq \min(s(\pi_2), s(\pi_3))\}.$$

Let $\omega_1(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$ and define the involution map $\imath : S \to S$ by

$$\imath(\vec{\pi}) = \imath(\pi_1, \pi_2, \pi_3) = \imath(\pi_1, \pi_3, \pi_2).$$

Define an S-partition $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ to be a self-conjugate S-partition if it is a fixed point of $\imath$, that is, if and only if $\pi_2 = \pi_3$. Let $N_{\mathrm{SC}}(n)$ denote the number of self-conjugate S-partitions counted according to the weight ω_1, that is,

$$N_{\mathrm{SC}}(n) = \sum_{\substack{\vec{\pi}\in S, |\vec{\pi}|=n\\ \imath(\vec{\pi})=\vec{\pi}}} \omega_1(\vec{\pi}). \tag{2.18}$$

Andrews, Garvan and Liang [16, Theorem 1.2] showed that the generating function of $N_{\mathrm{SC}}(n)$ equals $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n}$, which is a mock theta function studied by Andrews, Dyson and Hickerson [15].

If we let $c=-1$ in Theorem 2.8, we obtain a result involving the generating function of $N_{\mathrm{SC}}(n)$.

Corollary 2.12. *Let $N_{\mathrm{SC}}(n)$ be defined in (2.18). Then*

$$\begin{aligned}(q)_\infty\sum_{n=1}^{\infty}N_{\mathrm{SC}}(n)q^n+\frac{1}{2}\frac{(q)_\infty}{(-q)_\infty}\sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(1-q^n)(q)_n}\left(\frac{(-q)_n}{(q)_n}-1\right)\\ =\frac{1}{4}-\frac{1}{4}\frac{(q)_\infty}{(-q)_\infty}+\frac{1}{2}\frac{(q)_\infty}{(-q)_\infty}\sum_{n=1}^{\infty}\frac{(-q)_n}{(q)_n}\frac{q^n}{1-q^n}.\end{aligned}\tag{2.19}$$

Using a result conjectured by Beck and proved by Chern [25, Theorem 1.2], we show in Section 7 that the above identity reduces to a new representation for the generating function of $d(n)$.

Corollary 2.13. *We have*

$$\sum_{n=1}^{\infty}\frac{(-q)_n}{(q)_n}\frac{q^n}{1-q^n}-2\sum_{n=1}^{\infty}\frac{(-q)_{n-1}}{(q)_n^2}\frac{q^{n(n+3)/2}}{1-q^n}=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}.\tag{2.20}$$

Hence

$$\sum_{\substack{\pi\in\mathcal{P}_O(n)\\ l(\pi)\,overlined}}(2L(\pi)-1)-\sum_{\substack{\pi\in\mathcal{P}^*(n)\\ L(\pi)\geq 2}}L(\pi)(L(\pi)-1)\prod_{i=1}^{l(\pi)-1}(2\nu(i)-1)=d(n).\tag{2.21}$$

Further special cases of Theorem 2.8 when $c=q$, and more generally, $c=q^m, m\geq 1$, are given in Section 7.

The last of the five q-series identities of Ramanujan [40, p. 355], [19, p. 265, Entry 5] given by

$$\sum_{n=1}^{\infty}\frac{(q)_{n-1}a^n}{(1-q^n)(a)_n}=\sum_{n=1}^{\infty}\frac{na^n}{1-q^n},\tag{2.22}$$

where $(a)_n\neq 0, n\geq 1$, was proved by Berndt as a special case of (1.7). In the same vein, our Theorem 2.1 gives the following generalization of (2.22):

Theorem 2.14. *For $|zq|<1$ and $|cq|<1$,*

$$\sum_{n=1}^{\infty}\frac{(q)_{n-1}z^nq^n}{(1-cq^n)(zq)_n}=z\sum_{n=1}^{\infty}\frac{(zq/c)_{n-1}}{(zq)_n}\frac{c^{n-1}q^n}{1-zq^n}.\tag{2.23}$$

Moreover, for $|zq|<\min(|c|,1)$,

$$\sum_{n=1}^{\infty}\frac{(q)_{n-1}z^nq^n}{(1-cq^n)(zq)_n}=\frac{(\frac{zq}{c})_\infty}{(zq)_\infty}\sum_{n=0}^{\infty}\frac{(c)_n\left(\frac{zq}{c}\right)^n}{(q)_n}\sum_{m=1}^{\infty}\frac{mz^mq^m}{1-cq^{m+n}}.\tag{2.24}$$

This result gives a nice weighted partition identity given below.

Theorem 2.15. *If z and c are not functions of q, then*

$$\sum_{\pi\in\mathcal{P}(n)} z^{l(\pi)+\#(\pi)-L(\pi)}\left(1-\tfrac{1}{z}\right)^{\nu_d(\pi)-1}\left(\tfrac{z^{L(\pi)}-c^{L(\pi)}}{z-c}\right)=\sum_{\pi\in\mathcal{P}(n)} c^{l(\pi)-1}z^{\#(\pi)}\left(1-\tfrac{1}{c}\right)^{\nu_d(\pi)-1}L(\pi).$$

From the above theorem, we obtain two beautiful weighted partition identities, the first of which is

Corollary 2.16. *The following identity holds:*

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-q)_{n-1}}{(q)_n}\frac{q^n}{1-q^n}. \tag{2.25}$$

Hence if $d_o(n)$ denotes the number of odd divisors of n,

$$d_o(n)=\sum_{\pi\in\mathcal{P}(n)}(-1)^{l(\pi)-1}2^{\nu_d(\pi)-1}L(\pi). \tag{2.26}$$

Though Corteel and Lovejoy [23, p. 1631] have obtained (2.25), their partition-theoretic interpretation of the coefficients of the power series representation of the right-hand side of (2.25) [23, Theorem 4.4] is different from the right-hand side of (2.26). Garvan obtained another weighted partition representation for $d_o(n)$ in [30, Corollary (1.3) (i)], and yet another one is given in Section 5.

The second result that we obtain using (2.23) is

Corollary 2.17. *The following identity holds:*

$$\sum_{n=1}^{\infty}\frac{n(-1)^nq^n}{1-q^n}=\sum_{n=1}^{\infty}\frac{(-1)^n(q)_{n-1}}{(-q)_{n-1}}\frac{q^n}{1-q^{2n}}.$$

Hence

$$\sum_{d|n}(-1)^d d=\sum_{\substack{\pi\in\mathcal{P}(n)\\ L(\pi)\text{ is odd}}}(-1)^{\mathrm{rank}(\pi)-1}2^{\nu_d(\pi)-1}.$$

This paper is organized as follows. In Section 3 we collect some of the fundamental results in q-series that will be used several times in the paper. Section 4 is devoted to the proofs of Theorems 2.1, 2.2 and 2.14. In Section 5, we give proofs of the weighted partition identities mentioned here in Section 2. Many other weighted partition identities are also derived here. Section 6 is reserved for proving Garvan's identity (2.9), obtaining number-theoretic information from its proof, and for giving another proof of Andrews, Garvan and Liang's (1.5) using the Bhargava-Adiga summation. In Section 7, we prove Theorem 2.8 and its corollaries. Finally the concluding remarks and possibilities for future work are addressed in Section 8.

3. Preliminary results

For $|z|<1$, the q-binomial theorem is given by [7, p. 17, Equation (2.2.1)]

$$\sum_{n=0}^{\infty}\frac{(a)_n z^n}{(q)_n}=\frac{(az)_\infty}{(z)_\infty}. \tag{3.1}$$

Let the Gaussian polynomial $\begin{bmatrix} n \\ m \end{bmatrix}$ be defined by [7, p. 35]

$$\begin{bmatrix} n \\ m \end{bmatrix}=\begin{bmatrix} n \\ m \end{bmatrix}_q:=\begin{cases}(q;q)_n(q;q)_m^{-1}(q;q)_{n-m}^{-1}, & \text{if } 0\leq m\leq n,\\ 0, & \text{otherwise.}\end{cases}$$

We have

$$\frac{1}{(z)_N}=\sum_{j=0}^{\infty}\begin{bmatrix} N+j-1 \\ j \end{bmatrix}z^j \tag{3.2}$$

from [7, p. 36, Equation (3.3.7)]. The partial fraction decomposition of $F(a,b;t)$, which is defined in (2.10), is given by [27, p. 18, Equation (16.3)]

$$F(a,b;t)=\frac{(aq)_\infty}{(bq)_\infty}\sum_{n=0}^{\infty}\frac{(b/a)_n}{(q)_n}\frac{(aq)^n}{1-tq^n}. \tag{3.3}$$

Heine's transformation [7, p. 19, Corollary 2.3] gives

$${}_2\phi_1\left(\begin{matrix} a, & b \\ & c \end{matrix};q,z\right)=\frac{(b,az;q)_\infty}{(c,z;q)_\infty}{}_2\phi_1\left(\begin{matrix} \frac{c}{b}, & z \\ & az \end{matrix};q,b\right). \tag{3.4}$$

4. Generalizations of Ramanujan's three q-series identities

We give two proofs of Theorem 2.1 in this section. This is followed by proofs of Theorems 2.2 and 2.14.

Proof of Theorem 2.1. Let

$$G(a,b;c):=\sum_{n=1}^{\infty}\frac{\left(\frac{b}{a}\right)_n a^n}{(1-cq^n)(b)_n}.$$

Note that

$$\begin{aligned}G(a,b;c)-G(aq,bq;c)&=\sum_{n=1}^{\infty}\left(\frac{\left(\frac{b}{a}\right)_n a^n}{(1-cq^n)(b)_n}-\frac{\left(\frac{b}{a}\right)_n (aq)^n}{(1-cq^n)(bq)_n}\right)\\&=\sum_{n=1}^{\infty}\frac{\left(\frac{b}{a}\right)_n a^n}{(1-cq^n)(b)_{n+1}}(1-bq^n-q^n(1-b))\\&=\sum_{n=1}^{\infty}\frac{\left(\frac{b}{a}\right)_n a^n}{(b)_{n+1}}-\frac{(1-c)}{(1-b)}G(aq,bq;c).\end{aligned} \tag{4.1}$$

We now use the result in Berndt [19, p. 264, Equation (3.4)], that is,

$$\sum_{n=1}^{\infty}\frac{\left(\frac{b}{a}\right)_n a^n}{(b)_{n+1}}=\frac{a}{1-a}-\frac{b}{1-b}, \tag{4.2}$$

which is obtained by specializing Heine's transformation and then employing the q-binomial theorem. Thus substituting (4.2) in (4.1), we find that

$$G(a,b;c)-\left(\frac{c-b}{1-b}\right)G(aq,bq;c)=\frac{a}{1-a}-\frac{b}{1-b}. \tag{4.3}$$

We now create a telescoping sum. To that end, replace a and b by aq and bq respectively, then multiply the resulting equation by $(c-b)/(1-b)$ to get

$$\left(\frac{c-b}{1-b}\right)G(aq,bq;c)-\left(\frac{c-b}{1-b}\right)\left(\frac{c-bq}{1-bq}\right)G(aq^2,bq^2;c)=\left(\frac{c-b}{1-b}\right)\left(\frac{aq}{1-aq}-\frac{bq}{1-bq}\right). \tag{4.4}$$

Add the corresponding sides of (4.3) and (4.4) to obtain

$$G(a,b;c)-\frac{(c-b)(c-bq)}{(1-b)(1-bq)}G(aq^2,bq^2;c)=\left(\frac{a}{1-a}-\frac{b}{1-b}\right)+\frac{(c-b)}{(1-b)}\left(\frac{aq}{1-aq}-\frac{bq}{1-bq}\right). \tag{4.5}$$

Repeat this process, that is, first replace a by aq and b by bq in (4.4), multiply both sides by $(c-b)/(1-b)$, and then add the corresponding sides of the resulting identity and (4.5). At the n^{th} step, this gives

$$G(a,b;c)-\frac{c^{n+1}\left(\frac{b}{c}\right)_{n+1}}{(b)_{n+1}}G(aq^{n+1},bq^{n+1};c)=\sum_{k=0}^{n}\frac{c^k\left(\frac{b}{c}\right)_k}{(b)_k}\left(\frac{aq^k}{1-aq^k}-\frac{bq^k}{1-bq^k}\right).$$

Now let $n\to\infty$. Now first assume $|c|\le 1$. Since $G(aq^{n+1},bq^{n+1},c)\to 0$ as $n\to\infty$, the second expression on the left goes to zero. Therefore,

$$G(a,b;c)=\sum_{k=0}^{\infty}\frac{c^k\left(\frac{b}{c}\right)_k}{(b)_k}\left(\frac{aq^k}{1-aq^k}-\frac{bq^k}{1-bq^k}\right), \tag{4.6}$$

which proves (2.1) for $|c|\le 1$. By analytic continuation, it is then easily seen that the result is true for $|cq|<1$. From (4.6), for $|a|<1, |b|<\min(|c|,1)$,

$$\begin{aligned}
G(a,b;c)&=\sum_{k=0}^{\infty}\frac{c^k\left(\frac{b}{c}\right)_k}{(b)_k}\sum_{m=1}^{\infty}(a^m-b^m)q^{km}\\
&=\sum_{m=1}^{\infty}(a^m-b^m)\sum_{k=0}^{\infty}\frac{\left(\frac{b}{c}\right)_k c^k q^{mk}}{(b)_k}\\
&=\frac{\left(\frac{b}{c}\right)_\infty}{(b)_\infty}\sum_{m=1}^{\infty}(a^m-b^m)\sum_{n=0}^{\infty}\frac{(c)_n}{(q)_n}\frac{\left(\frac{b}{c}\right)^n}{1-cq^{m+n}}\\
&=\frac{\left(\frac{b}{c}\right)_\infty}{(b)_\infty}\sum_{n=0}^{\infty}\frac{(c)_n\left(\frac{b}{c}\right)^n}{(q)_n}\sum_{m=1}^{\infty}\frac{a^m-b^m}{1-cq^{m+n}},
\end{aligned}$$

where in the penultimate step, we invoked (3.3). This completes the proof of Theorem 2.1. □

Second proof.[1] From [31, p. 72, Equation (3.2.7)], for $|DE/(ABC)|<1$ and $|E/A|<1$, we have

$$ {}_3\phi_2\left(\begin{matrix}A, B, C\\ D, E\end{matrix}; q, \frac{DE}{ABC}\right)=\frac{\left(\frac{E}{A}\right)_\infty\left(\frac{DE}{BC}\right)_\infty}{(E)_\infty\left(\frac{DE}{ABC}\right)_\infty}{}_3\phi_2\left(\begin{matrix}A, \frac{D}{B}, \frac{D}{C}\\ D, \frac{DE}{BC}\end{matrix}; q, \frac{E}{A}\right). $$

Let $A=q, B=\frac{bq}{a}, C=cq, D=bq$ and $E=cq^2$ in the above transformation so that for $|a|<1$ and $|cq|<1$,

$$ \frac{1}{(1-cq)}{}_3\phi_2\left(\begin{matrix}\frac{bq}{a}, q, cq\\ bq, cq^2\end{matrix}; q, a\right)=\frac{1}{(1-a)}{}_3\phi_2\left(\begin{matrix}\frac{b}{c}, q, a\\ bq, cq\end{matrix}; q, cq\right). \tag{4.7} $$

However, one can observe that

$$ \begin{aligned} \frac{1}{(1-cq)}{}_3\phi_2\left(\begin{matrix}\frac{bq}{a}, q, cq\\ bq, cq^2\end{matrix}; q, a\right)&=\frac{(1-b)}{(a-b)}\sum_{n=1}^{\infty}\frac{\left(\frac{b}{a}\right)_n a^n}{(1-cq^n)(b)_n},\\ \frac{1}{(1-a)}{}_3\phi_2\left(\begin{matrix}\frac{b}{c}, q, a\\ bq, cq\end{matrix}; q, cq\right)&=(1-b)\sum_{n=0}^{\infty}\frac{\left(\frac{b}{c}\right)_n (cq)^n}{(b)_n(1-bq^n)(1-aq^n)}, \end{aligned} $$

which, together with (4.7), gives (2.1) upon simplification. □

Proof of Theorem 2.2. Let $a\to 0$ in (2.1), use the elementary identity $\lim_{a\to0}(b/a)_n a^n=(-b)^n q^{n(n-1)/2}$, and then replace b by zq to arrive at (2.3). □

We now prove the generalization of (2.22) given in Theorem 2.14.

Proof of Theorem 2.14. To obtain (2.23), divide both sides of (2.1) by $1-b/a$, let $b\to a$, then replace a by zq and simplify. Now, additionally if $|zq|<|c|$, perform the same operations on (2.2) to obtain (2.24). □

5. Weighted partition identities

5.1. **Weighted partition identities resulting from Theorem 2.2 and its corollaries.**

We begin with a lemma which will be used several times in the sequel.

Lemma 5.1.

$$ \sum_{\pi\in\mathcal{P}(n)}\lim_{z\to1}\left(1-\frac{1}{z}\right)^{\nu_d(\pi)-1}=d(n). $$

Proof. Note that

$$ \lim_{z\to1}\left(1-\frac{1}{z}\right)^{\nu_d(\pi)-1}=\begin{cases}1, & \text{if} \quad \nu_d(\pi)=1,\\ 0, & \text{if} \quad \nu_d(\pi)>1,\end{cases} $$

[1] The authors are indebted to George E. Andrews [12] for this proof.

where, as remarked in the introduction, $\nu_d(\pi)$ the number of parts of π not counting multiplicity. Therefore we need to consider only those partitions π of n for which $\nu_d(\pi) = 1$. If m is a divisor of n then we have $n = m\ell$, for some positive integer ℓ, so we can write n as

$$n = \underbrace{m + m + \cdots + m}_{\ell \text{ times}}.$$

Therefore corresponding to every divisor of n we can obtain a partition whose number of parts without multiplicity is 1 and conversely, any partition π of n with $\nu_d(\pi) = 1$ corresponds to a divisor of n. $\square$

Proof of Theorem 2.3. To prove that the left-hand side of (2.3) is the generating function of that of (2.4) we first write the former as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}}{(1-cq^n)(zq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{1+2+\cdots+n}}{(1-zq)(1-zq^2)\cdots(1-zq^n)(1-cq^n)} \tag{5.1}$$

and use

$$\frac{1}{1-zq^i} = \sum_{k_i=0}^{\infty} z^{k_i} q^{i\,k_i},$$

for $1 \le i \le n-1$, as well as [16, p. 214]

$$\frac{q^n}{(1-zq^n)(1-cq^n)} = \sum_{k_n=1}^{\infty} \frac{\left(\frac{z}{c}\right)^{k_n} - 1}{\left(\frac{z}{c}\right) - 1} c^{k_n - 1} q^{n\,k_n}$$

in the summand of the series on the right-hand side of (5.1). Thus

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}}{(1-cq^n)(zq)_n} \\
&= \sum_{n=1}^{\infty} \sum_{\substack{k_i=0, 1\le i\le n-1\\ k_n=1}}^{\infty} (-1)^{n-1} z^{n+k_1+k_2+\cdots+k_{n-1}} c^{k_n-1} \frac{\left(\frac{z}{c}\right)^{k_n} - 1}{\left(\frac{z}{c}\right) - 1} q^{\frac{n(n-1)}{2} + k_1 + 2k_2 + \cdots + nk_n}.
\end{aligned}$$

Now write the exponent of q in the form $a_1 + a_2 + \cdots + a_n$ where

$$\begin{aligned}
a_1 &:= (k_1 + 1) + (k_2 + 1) + \cdots + (k_{n-1} + 1) + k_n \\
a_2 &:= (k_2 + 1) + (k_3 + 1) + \cdots + (k_{n-1} + 1) + k_n \\
&\cdots\cdots\cdots \\
a_{n-1} &:= k_{n-1} + 1 + k_n \\
a_n &:= k_n.
\end{aligned}$$

Thus $a_1 + a_2 + \cdots + a_n$ is a partition of a number into distinct parts, where the largest part is a_1, smallest part is a_n and n is the number of parts. This proves that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}}{(1-cq^n)(zq)_n} = \sum_{n=1}^{\infty} \left(\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)-1} z^{l(\pi)+1-s(\pi)} c^{s(\pi)-1} \frac{\left(\frac{z}{c}\right)^{s(\pi)} - 1}{\left(\frac{z}{c}\right) - 1} \right) q^n. \tag{5.2}$$

Next we write the right-hand side of (2.3) in the form

$$\frac{z}{c}\sum_{n=1}^{\infty}\frac{(zq/c)_{n-1}}{(zq)_n}(cq)^n=\frac{z}{c}\sum_{n=1}^{\infty}\frac{\left(1-\frac{z}{c}q\right)\left(1-\frac{z}{c}q^2\right)\cdots\left(1-\frac{z}{c}q^{n-1}\right)}{(1-zq)(1-zq^2)\cdots(1-zq^{n-1})}\frac{c^nq^n}{(1-zq^n)}.$$

For $1\le i\le n-1$, we have

$$\frac{\left(1-\frac{z}{c}q^i\right)}{(1-zq^i)}=1+\sum_{k_i=1}^{\infty}\left(1-\frac{1}{c}\right)z^{k_i}q^{i\,k_i}=\sum_{k_i=0}^{\infty}a_{k_i}z^{k_i}q^{i\,k_i},$$

where

$$a_{k_i}=\begin{cases}1, & \text{if}\quad k_i=0,\\ 1-\frac{1}{c}, & \text{if}\quad k_i\ge 1.\end{cases}$$

Furthermore, write

$$\frac{zq^n}{1-zq^n}=\sum_{k_n=1}^{\infty}z^{k_n}q^{n\,k_n}.$$

Using these series expansions in the above identity, we get

$$\frac{z}{c}\sum_{n=1}^{\infty}\frac{(zq/c)_{n-1}}{(zq)_n}(cq)^n=\sum_{n=1}^{\infty}c^{n-1}\prod_{i=1}^{n-1}\left(\sum_{k_i=0}^{\infty}a_{k_i}z^{k_i}q^{i\,k_i}\right)\sum_{k_n=1}^{\infty}z^{k_n}q^{n\,k_n}.$$

Here the typical exponent of q is $k_1+2k_2+\cdots+nk_n$, where $k_i\ge 0$ for $1\le i\le n-1$ and $k_n\ge 1$. This corresponds to an ordinary partition of the exponent where, each part i, $1\le i\le n$, appears k_i times, the largest part is n and the number of parts is $k_1+k_2+\cdots+k_n$. Thus

$$\frac{z}{c}\sum_{n=1}^{\infty}\frac{(zq/c)_{n-1}}{(zq)_n}(cq)^n=\sum_{n=1}^{\infty}\left(\sum_{\pi\in\mathcal{P}(n)}c^{l(\pi)-1}\left(1-\frac{1}{c}\right)^{\nu_d(\pi)-1}z^{\#(\pi)}\right)q^n. \tag{5.3}$$

From (2.3), (5.2) and (5.3), we obtain (2.4). □

When $c=z$, Theorems 2.2 and 2.3 give the following corollary.

Corollary 5.2. *For* $|zq|<1$,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^nq^{\frac{n(n+1)}{2}}}{(1-zq^n)(zq)_n}=\sum_{n=1}^{\infty}\frac{(q)_{n-1}}{(zq)_n}(zq)^n. \tag{5.4}$$

Hence, if z *is not a function of* q,

$$\sum_{\pi\in\mathcal{D}(n)}(-1)^{\#(\pi)-1}z^{l(\pi)}s(\pi)=\sum_{\pi\in\mathcal{P}(n)}z^{\#(\pi)+l(\pi)-1}\left(1-\frac{1}{z}\right)^{\nu_d(\pi)-1}.$$

We note that (5.4) can also be obtained by letting $c=0$ in the first equality of Theorem 2.14.

The above corollary, in turn, specializes to Kluyver's identity and the result of Fokkink, Fokkink and Wang:

Corollary 5.3. *The identities* (1.3) *and* (1.4) *hold.*

Proof. Let $z = 1$ in Corollary 5.2, and employ Lemma 5.1 to get the second equality. □

Corollary 5.2 also gives a new weighted partition identity, namely, when $z = -1$,

Corollary 5.4. *We have*

$$\sum_{n=1}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(1+q^n)(-q)_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(q)_{n-1}}{(-q)_n} q^n.$$

Hence,

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\mathrm{rank}(\pi)}\, s(\pi) = \sum_{\pi \in \mathcal{P}(n)} (-1)^{\mathrm{rank}(\pi)}\, 2^{\nu_d(\pi)-1}. \tag{5.5}$$

Example 1. *Let $n = 5$. In the first table below, we list the partitions of n into distinct parts, and in the other, all partitions of* 5.

$\pi \in \mathcal{D}(5)$	$\mathrm{rank}(\pi)$	$(-1)^{\mathrm{rank}(\pi)} s(\pi)$
5	4	5
$4+1$	2	1
$3+2$	1	-2

$\pi \in \mathcal{P}(5)$	$\mathrm{rank}(\pi)$	$(-1)^{\mathrm{rank}(\pi)} 2^{\nu_d(\pi)-1}$
5	4	1
$4+1$	2	2
$3+2$	1	-2
$3+1+1$	0	2
$2+2+1$	-1	-2
$2+1+1+1$	-2	2
$1+1+1+1+1+1$	-4	1

It can be checked that the sum of the last columns in each of the tables is equal to 4, *which verifies* (5.5) *for $n = 5$.*

Similarly when $c = -z$, Theorems 2.2 and 2.3 result in the following.

Corollary 5.5. *For $|zq| < 1$,*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}}{(1+zq^n)(zq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-q)_{n-1}}{(zq)_n}(zq)^n. \tag{5.6}$$

Hence if z is not a function of q,

$$\sum_{\substack{\pi \in \mathcal{D}(n) \\ s(\pi)\ odd}} (-1)^{\#(\pi)-1} z^{l(\pi)} = \sum_{\pi \in \mathcal{P}(n)} (-1)^{l(\pi)-1} z^{\#(\pi)+l(\pi)-1} \left(1+\frac{1}{z}\right)^{\nu_d(\pi)-1}. \tag{5.7}$$

Proof. Equation (5.6) is immediate after we let $c = -z$ in Theorem 2.2. To obtain (5.7), let $c = -z$ in Theorem 2.3 and note that the resulting left-hand side is non-zero only when $s(\pi)$ is odd. □

Now if we let $z = 1$ in the above corollary, we obtain

Corollary 5.6.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\frac{n(n+1)}{2}}}{(1+q^n)(q)_n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-q)_{n-1}}{(q)_n}q^n.$$

Hence

$$\sum_{\substack{\pi\in\mathcal{D}(n)\\ s(\pi)\text{ odd}}}(-1)^{\#(\pi)-1}=\sum_{\pi\in\mathcal{P}(n)}(-1)^{l(\pi)-1}2^{\nu_d(\pi)-1}. \tag{5.8}$$

We note that in Andrews, Garvan and Liang's notation in (1.5), the left-hand side of the above weighted partition identity is nothing but $\mathrm{FFW}(-1,n)$. As mentioned in [16, Remark 3.7], Alladi [4, Theorem 2] has obtained a simpler representation for the left-hand side of (5.8), namely, it is equal to $(-1)^{\sqrt{n}-1}$ if n is a square and zero otherwise.

Lastly if we let $z=-1$ in Corollary 5.5, we obtain

Corollary 5.7.

$$\sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(1-q^n)(-q)_n}=\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}. \tag{5.9}$$

Hence if $d_e(n)$ and $d_o(n)$ denote the number of even divisors and the number of odd divisors of n respectively,

$$\sum_{\substack{\pi\in\mathcal{D}(n)\\ s(\pi)\text{ odd}}}(-1)^{\mathrm{rank}(\pi)-1}=d_e(n)-d_o(n). \tag{5.10}$$

Proof. Letting $z=-1$ in (5.6) immediately results in (5.9). To obtain (5.10), let $z\to-1$ in (5.7) and observe that analogous to the result in Lemma 5.1, the following limit holds:

$$\lim_{z\to-1}\sum_{\pi\in\mathcal{P}(n)}z^{\#(\pi)}\left(1+\frac{1}{z}\right)^{\nu_d(\pi)-1}=d_e(n)-d_o(n).$$

□

Example 2. *Suppose $n=10$. Below we list the partitions of 10 into distinct parts such that their smallest parts are odd.*

π	$\mathrm{rank}(\pi)$	$(-1)^{\mathrm{rank}(\pi)}$
$9+1$	7	-1
$7+3$	5	-1
$7+2+1$	4	$+1$
$6+3+1$	3	-1
$5+4+1$	2	$+1$
$4+3+2+1$	0	$+1$

It can be easily checked that $d_o(10)=d_e(10)=2$ and that the sum of the last column equals zero. This verifies (5.10) for $n=10$.

Remark 1. *Recall the definitions of $\mathcal{D}(n)$ and $\mathcal{P}^*(n)$, namely, the collections of partitions of n into distinct parts, and into those without gaps respectively. In all of the weighted partition representations of expressions in the above theorems involving partitions into distinct parts, one can replace $\mathcal{D}(n)$ by $\mathcal{P}^*(n)$, and interchange $l(\pi)$ with $\#(\pi)$, and $s(\pi)$ with $L(\pi)$. This simply follows from conjugation of partitions. See, for example, Ando's representation* [6, p. 2] *of the left-hand side* (1.4).

5.2. **Weighted partition identities resulting from Theorem 2.14 and its corollaries.** The details of its derivation of Theorem 2.15 from Theorem 2.14 are similar to that of Theorem 2.3 from Theorem 2.2, hence a proof is omitted.

Note that letting $c = z$ in (2.23) does not produce anything interesting, however, $c = -z$ gives the following result.

Corollary 5.8. *For $|zq| < 1$,*

$$\sum_{n=1}^{\infty} \frac{(q)_{n-1} z^n q^n}{(1+zq^n)(zq)_n} = z \sum_{n=1}^{\infty} \frac{(-q)_{n-1}}{(zq)_n} \frac{(-z)^{n-1} q^n}{1-zq^n}.$$

Hence

$$\sum_{\substack{\pi \in \mathcal{P}(n) \\ L(\pi) \ odd}} z^{l(\pi)+\#(\pi)-1} \left(1 - \frac{1}{z}\right)^{\nu_d(\pi)-1} = \sum_{\pi \in \mathcal{P}(n)} (-z)^{l(\pi)-1} z^{\#(\pi)} \left(1 + \tfrac{1}{z}\right)^{\nu_d(\pi)-1} L(\pi).$$

Let $z = 1$ in Theorems 2.14 and 2.15. This gives

Corollary 5.9. *For $|cq| < 1$, we have*

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-cq^n)(1-q^n)} = \sum_{n=1}^{\infty} \frac{(q/c)_{n-1}}{(q)_n} \frac{c^{n-1} q^n}{1-q^n}.$$

Hence if c is not a function of q,

$$\sum_{d|n} \frac{c^d - 1}{c - 1} = \sum_{\pi \in \mathcal{P}(n)} c^{l(\pi)-1} \left(1 - \frac{1}{c}\right)^{\nu_d(\pi)-1} L(\pi).$$

Proof of Corollary 2.16. Let $c = -1$ in the above corollary. □

Remark 2. *One more representation of $d_o(n)$ that one can obtain is*

$$d_o(n) = \frac{1}{2} \sum_{\pi \in \mathcal{P}_o(n)} (-1)^{l(\pi)-1} L(\pi).$$

To see this, write the right-hand side of (2.25) *as*

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-q)_{n-1}}{(q)_{n-1}} \frac{q^n}{(1-q^n)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}(-q)_{n-1}}{(q)_{n-1}} \sum_{k=1}^{\infty} k q^{nk}. \tag{5.11}$$

It is well-known that the generating function for overpartitions is

$$\sum_{n=0}^{\infty}\bar{p}(n)q^n=\frac{(-q)_\infty}{(q)_\infty}=1+2\sum_{n=1}^{\infty}\frac{(-q)_{n-1}}{(q)_{n-1}}\frac{q^n}{1-q^n},$$

where $\bar{p}(n)$ *denotes the number of overpartitions of* n*. Thus, the presence of* $(1-q^n)^2$ *in the denominator of* (5.11) *suggests that the series is the generating function of the total number of appearances of the largest part* $L(\pi)$ *of a partition* π *weighted by* $(-1)^{l(\pi)-1}$*.*

Now let $z=-1$ in Theorems 2.14 and 2.15 to obtain

Corollary 5.10. *For* $|cq|<1$*,*

$$\sum_{n=1}^{\infty}\frac{(q)_{n-1}(-1)^nq^n}{(1-cq^n)(-q)_n}=-\sum_{n=1}^{\infty}\frac{(-q/c)_{n-1}}{(-q)_{n-1}}\frac{c^{n-1}q^n}{(1+q^n)^2}.$$

Hence if c *is not a function of* q*,*

$$\sum_{\pi\in\mathcal{P}(n)}(-1)^{\mathrm{rank}(\pi)-L(\pi)}2^{\nu_d(\pi)-1}\tfrac{(-1)^{L(\pi)}-c^{L(\pi)}}{-1-c}=\sum_{\pi\in\mathcal{P}(n)}c^{l(\pi)-1}(-1)^{\#(\pi)}\left(1-\tfrac{1}{c}\right)^{\nu_d(\pi)-1}L(\pi).$$

Proof of Corollary 2.17. Let $c=1$ in the above corollary and observe that

$$-\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}=\sum_{n=1}^{\infty}\frac{n(-1)^nq^n}{1-q^n}.$$

□

6. Some more applications of Theorem 2.2

6.1. **Proof of Garvan's identity** (2.9)**.** Replace q by q^2, z by z/q, and then c by z in (2.3). This gives

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^nq^{n^2}}{(zq;q^2)_n(1-zq^{2n})}=\sum_{m=1}^{\infty}z^mq^{2m-1}\frac{(q;q^2)_{m-1}}{(zq;q^2)_m}. \tag{6.1}$$

By (3.2),

$$\frac{1}{(zq;q^2)_m}=\sum_{j=0}^{\infty}\frac{(q^2;q^2)_{m+j-1}}{(q^2;q^2)_j(q^2;q^2)_{m-1}}(zq)^j.$$

Employing this on the right-hand side of (6.1) gives

$$\begin{aligned}\sum_{m=1}^{\infty}z^mq^{2m-1}\frac{(q;q^2)_{m-1}}{(zq;q^2)_m}&=\sum_{m=1}^{\infty}z^mq^{2m-1}(q;q^2)_{m-1}\sum_{j=0}^{\infty}\frac{(q^2;q^2)_{m+j-1}}{(q^2;q^2)_j(q^2;q^2)_{m-1}}(zq)^j\\&=zq\sum_{j=0}^{\infty}(zq)^j{}_2\phi_1\left(\begin{matrix}q, & q^{2j+2}\\ & 0\end{matrix};q^2,zq^2\right),\end{aligned} \tag{6.2}$$

where in the last step we interchanged the order of summation, and then replaced the index of summation of the inner sum, that is, m by $m+1$. Thus using the above equation along with Heine's transformation (3.4) in the first step below results in

$$\begin{aligned}\sum_{m=1}^{\infty} z^m q^{2m-1}\frac{(q;q^2)_{m-1}}{(zq;q^2)_m} &= zq\sum_{j=0}^{\infty}(zq)^j\frac{(q^{2j+2};q^2)_\infty(zq^3;q^2)_\infty}{(zq^2;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(zq^2;q^2)_k(q^{2j+2})^k}{(zq^3;q^2)_k(q^2;q^2)_k}\\ &= zq\frac{(q^2;q^2)_\infty(zq^3;q^2)_\infty}{(zq^2;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(zq^2;q^2)_k q^{2k}}{(zq^3;q^2)_k(q^2;q^2)_k}\sum_{j=0}^{\infty}\frac{(zq^{2k+1})^j}{(q^2;q^2)_j}\\ &= zq\frac{(q^2;q^2)_\infty(zq^3;q^2)_\infty}{(zq^2;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(zq^2;q^2)_k q^{2k}}{(zq^3;q^2)_k(q^2;q^2)_k}\frac{1}{(zq^{2k+1};q^2)_\infty}\\ &= zq\frac{(q^2;q^2)_\infty}{(zq^2;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(zq^2;q^2)_k}{(q^2;q^2)_k}\frac{q^{2k}}{1-zq^{2k+1}}, \qquad (6.3)\end{aligned}$$

where in the penultimate step, we applied (3.1). Now replace q by q^2, t by zq, a by 1 and b by zq^2 in (3.3) so that the right-hand side of (6.3) can be simplified to

$$\begin{aligned}zq\frac{(q^2;q^2)_\infty}{(zq^2;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(zq^2;q^2)_k}{(q^2;q^2)_k}\frac{q^{2k}}{1-zq^{2k+1}} &= \frac{zq}{1-zq^2}\sum_{k=0}^{\infty}\frac{(q^2;q^2)_k}{(zq^4;q^2)_k}(zq)^k\\ &= \sum_{k=1}^{\infty}\frac{(q^2;q^2)_{k-1}}{(zq^2;q^2)_k}(zq)^k. \qquad (6.4)\end{aligned}$$

Thus, from (6.3) and (6.4), we obtain

$$\sum_{m=1}^{\infty} z^m q^{2m-1}\frac{(q;q^2)_{m-1}}{(zq;q^2)_m} = \sum_{k=1}^{\infty}\frac{(q^2;q^2)_{k-1}}{(zq^2;q^2)_k}(zq)^k. \qquad (6.5)$$

By a result of Ramanujan [13, Entry 1.7.2, p. 29], if $|b| < 1$ and a is an arbitrary complex number, then

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-q;q)_n(-\frac{aq}{b};q)_n\, b^n}{(aq;q^2)_{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n(-\frac{aq}{b};q)_n\, b^n q^{\frac{n(n+1)}{2}}}{(-b;q)_{n+1}}. \qquad (6.6)$$

Replace a by zq and b by $-zq$ in (6.6) to get

$$\sum_{n=1}^{\infty}\frac{(q^2;q^2)_{n-1}}{(zq^2;q^2)_n}(zq)^n = \sum_{n=1}^{\infty}\frac{z^n q^{\frac{n(n+1)}{2}}(q;q)_{n-1}}{(zq;q)_n}. \qquad (6.7)$$

From (6.1), (6.5) and (6.7), we obtain (2.9).

Remark 3. *Let $z=1$ in (6.1), (6.5) and (6.7). This gives*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n(1-q^{2n})} = \sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}} = \sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}} = \sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{1-q^n}.$$

The equalities between the last three expressions are well-known; see, for example, [37, p. 28].

Now if we let $z=-1$ in (6.1), (6.5) and (6.7), we get

$$-\sum_{n=1}^{\infty}\frac{q^{n^2}}{(-q;q^2)_n(1+q^{2n})}=\sum_{n=1}^{\infty}(-1)^nq^{2n-1}\frac{(q;q^2)_{n-1}}{(-q;q^2)_n}=\sum_{n=1}^{\infty}\frac{(q^2;q^2)_{n-1}}{(-q^2;q^2)_n}(-q)^n$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^nq^{\frac{n(n+1)}{2}}(q;q)_{n-1}}{(-q;q)_n}. \tag{6.8}$$

Equation (6.8) encodes interesting number-theoretic and partition-theoretic information. To see this, let $\mathcal{Q}(n)$ be the set of partitions of n in which all parts except possibly the largest part are odd and all odd positive integers less than or equal to the largest part occur as parts. Also let $\ell_O(\pi)$ denote the largest odd part in a partition. Then, Garvan [30, Corollary 1.3(ii)] has obtained the weighted partition identity resulting from equating the extreme sides of (6.8), which can be stated as

$$\sum_{\pi\in\mathcal{Q}(n)}(-1)^{\frac{\ell_O(\pi)+1}{2}+\#(\pi)}=(-1)^{\frac{n(n-1)}{2}}d_{8,1}(n), \tag{6.9}$$

where $d_{8,1}(n)$ is the number of divisors of n congruent to $\pm1 \pmod 8$ minus the number of divisors of n congruent to $\pm3 \pmod 8$.

With the two new expressions linking the extreme sides in (6.8) that we have obtained, further information can be extracted, namely, if $\mathcal{P}_{\mathrm{odd}}(n)$ denotes the number of partitions of n into odd parts, then the expressions on the extreme sides of (6.8) are also equal to

$$\sum_{\pi\in\mathcal{P}_{\mathrm{odd}}(n)}2^{\nu_d(\pi)-1}(-1)^{\frac{l(\pi)+1}{2}+\#(\pi)} \tag{6.10}$$

since

$$\sum_{n=1}^{\infty}(-1)^nq^{2n-1}\frac{(q;q^2)_{n-1}}{(-q;q^2)_n}=\sum_{n=1}^{\infty}\left(\sum_{\pi\in\mathcal{P}_{\mathrm{odd}}(n)}(-1)^{\frac{l(\pi)+1}{2}+\#(\pi)}2^{\nu_d(\pi)-1}\right)q^n,$$

as can be seen using similar techniques employed in Section 5.

Moreover, we note that the third expression in (6.8) occurs in a recent work of Wang and Yee [44]. From their Theorem 1.3 and Corollary 5.2, we have

$$\sum_{n=1}^{\infty}\frac{(q^2;q^2)_{n-1}}{(-q^2;q^2)_n}(-q)^n=\sum_{k=1}^{\infty}\sum_{m=-\left\lfloor\frac{k}{2}\right\rfloor}^{\left\lfloor\frac{k-1}{2}\right\rfloor}(-1)^{m-1}q^{k^2-2m^2}.$$

Thus, if

$$r^*(n):=\sum_{\substack{k^2-2m^2=n\\ k\geq1,-\left\lfloor\frac{k}{2}\right\rfloor\leq m\leq\left\lfloor\frac{k-1}{2}\right\rfloor}}(-1)^{m-1},$$

then the expressions in (6.9) and (6.10) are also equal to $r^*(n)$.

6.2. **Another proof of a result of Andrews, Garvan and Liang.** As we have seen, identity (1.6) follows from (2.3) by substituting $z = 1$. Another proof of it can be obtained through the Bhargava-Adiga summation [20, Equation (1.1)] (see also [11, Equation (1.3)]), which states that for $|a| < 1, |d| < 1$ and $b \neq q^m$, $m \in \mathbb{Z}$,

$$\sum_{n=-\infty}^{\infty} \frac{(q/a)_n a^n}{(d)_n(1-bq^n)} = \frac{(d/b)_\infty (ab)_\infty (q)_\infty^2}{(q/b)_\infty (d)_\infty (a)_\infty (b)_\infty}. \tag{6.11}$$

Replace d by q and b by c in (6.11) to get

$$\sum_{n=-\infty}^{\infty} \frac{\left(\frac{q}{a}\right)_n a^n}{(q)_n(1-cq^n)} = \frac{(ac)_\infty (q)_\infty}{(c)_\infty (a)_\infty}.$$

Now split the sum on the left into two sums, one over $n \geq 0$ and the other over $n < 0$. Since $1/(q)_n = 0$ for $n < 0$, the second sum vanishes. Separate the $n = 0$ term to get

$$\frac{1}{1-c} + \sum_{n=1}^{\infty} \frac{\left(\frac{q}{a}\right)_n a^n}{(q)_n(1-cq^n)} = \frac{(ac)_\infty (q)_\infty}{(c)_\infty (a)_\infty}.$$

Now let $a \to 0$ to deduce (1.6) upon simplification.

Remark 4. *Let $c = q$ in* (1.6) *and simplify so as to obtain*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{(1-q^{n+1})(q)_n} = \frac{q}{1-q}.$$

If we let $a \to 0$, $b = zq$ and $c = q$ in (2.2) *and then let $z \to 1$, we obtain*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{(1-q^{n+1})(q)_n} = \lim_{z \to 1} (1-z) \sum_{n=0}^{\infty} z^n \sum_{m=1}^{\infty} \frac{z^m q^m}{1-q^{m+n+1}}.$$

Comparing the right-hand sides of the above two equations, we see that the double series $\displaystyle\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{q^m}{1-q^{m+n+1}}$ *diverges.*

7. Important q-series identities through differentiation

Here we first prove Theorem 2.8 and then proceed on deriving Corollaries 2.9 and 2.12.

Proof of Theorem 2.8. Differentiate both sides of (2.3) with respect to z so as to obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{n-1} q^{n(n+1)/2}}{(1-cq^n)(zq)_n} \left(n + \sum_{k=1}^{n} \frac{zq^k}{1-zq^k} \right) \\
&= \frac{1}{c} \sum_{n=1}^{\infty} \frac{(\frac{zq}{c})_{n-1} (cq)^n}{(zq)_n} + \frac{z}{c} \sum_{n=1}^{\infty} \frac{(\frac{zq}{c})_{n-1} (cq)^n}{(zq)_n} \left(\sum_{k=1}^{n-1} \frac{-q^k/c}{1-zq^k/c} + \sum_{k=1}^{n} \frac{q^k}{1-zq^k} \right).
\end{aligned}$$

Now let $z=1$ and employ (3.1) to simplify the first series on the right-hand side so that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^{n(n+1)/2}}{(1-cq^n)(q)_n}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}}{(1-cq^n)(q)_n}\sum_{k=1}^{n}\frac{q^k}{1-q^k}$$
$$=\frac{1}{c-1}\left(\frac{(q)_\infty}{(cq)_\infty}-1\right)+\frac{1}{c}\sum_{n=1}^{\infty}\frac{(\frac{q}{c})_{n-1}(cq)^n}{(q)_n}\left(\sum_{k=1}^{n}\frac{q^k}{1-q^k}-\sum_{k=1}^{n-1}\frac{q^k/c}{1-q^k/c}\right). \tag{7.1}$$

To simplify the second series on the left, we employ van Hamme's identity [34]

$$\sum_{k=1}^{n}\frac{q^k}{1-q^k}=\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^{k-1}q^{k(k+1)/2}}{(1-q^k)}, \tag{7.2}$$

which gives

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}}{(1-cq^n)(q)_n}\sum_{k=1}^{n}\frac{q^k}{1-q^k}&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}}{(1-cq^n)}\sum_{k=1}^{n}\frac{(-1)^{k-1}q^{k(k+1)/2}}{(1-q^k)(q)_k(q)_{n-k}}\\&=\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^m q^{\frac{(m+k)(m+k+1)}{2}}}{1-cq^{m+k}}\frac{q^{k(k+1)/2}}{(q)_k(q)_m(1-q^k)}\\&=\sum_{k=1}^{\infty}\frac{q^{k^2+k}}{(q)_k(1-q^k)}\sum_{m=0}^{\infty}\frac{(-1)^m q^{\frac{m(m+1)}{2}+mk}}{(q)_m(1-cq^{m+k})}\\&=\sum_{k=1}^{\infty}\frac{q^{k^2+k}(q^{k+1})_\infty}{(q)_k(1-q^k)}F(0,q^k;cq^k),\end{aligned} \tag{7.3}$$

where in the last step, we used the $a\to 0$ case of (3.3).

To simplify the series on the right-hand side of (7.1), we use the result of Guo and Zhang [33, Corollary 3.1] which states that if $n\geq 0$ and $0\leq m\leq n$,

$$\sum_{\substack{k=0\\k\neq m}}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(q/w)_k(w)_{n-k}}{1-q^{k-m}}w^k$$
$$=(-1)^m q^{\frac{m(m+1)}{2}}\begin{bmatrix}n\\m\end{bmatrix}(wq^{-m})_n\left(\sum_{k=0}^{n-1}\frac{wq^{k-m}}{1-wq^{k-m}}-\sum_{\substack{k=0\\k\neq m}}^{n}\frac{q^{k-m}}{1-q^{k-m}}\right).$$

Let $m=0$ in the above identity and simplify to obtain

$$\sum_{k=1}^{n}\frac{q^k}{1-q^k}-\sum_{k=1}^{n-1}\frac{wq^k}{1-wq^k}=\frac{w}{1-w}-\frac{1}{(w)_n}\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(q/w)_k(w)_{n-k}w^k}{1-q^k}. \tag{7.4}$$

That this is a generalization of (7.2) is easily seen by letting $w\to 0$ on both sides.

Invoking (7.4) with $w = 1/c$ in the series on the right-hand side of (7.1), we see that

$$\begin{aligned}
&\frac{1}{c}\sum_{n=1}^{\infty}\frac{(\frac{q}{c})_{n-1}(cq)^n}{(q)_n}\left(\sum_{k=1}^{n}\frac{q^k}{1-q^k}-\sum_{k=1}^{n-1}\frac{q^k/c}{1-q^k/c}\right)\\
&=\frac{1}{(c-1)^2}\left(\frac{(q)_\infty}{(cq)_\infty}-1\right)-\frac{1}{c-1}\sum_{n=1}^{\infty}\frac{(cq)^n}{(q)_n}\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(cq)_k(1/c)_{n-k}c^{-k}}{1-q^k}\\
&=\frac{1}{(c-1)^2}\left(\frac{(q)_\infty}{(cq)_\infty}-1\right)-\frac{1}{c-1}\sum_{j=0}^{\infty}\frac{(1/c)_j(cq)^j}{(q)_j}\sum_{k=1}^{\infty}\frac{(cq)_k}{(q)_k}\frac{q^k}{1-q^k}. \qquad (7.5)
\end{aligned}$$

Finally, substitute (7.3) and (7.5) in (7.1) so as to obtain (2.11) upon simplification. □

We now give various corollaries that follow from Theorem 2.8.

It is first shown that Andrews' famous identity for the spt-function, namely (2.12), can be derived from Theorem 2.8. This identity was the main ingredient in obtaining his beautiful congruences for $\mathrm{spt}(n)$ modulo $5, 7$ and 13 [10, Theorem 2]. The generating function version [10, Theorem 4, p. 137] is

$$\sum_{n=1}^{\infty}\mathrm{spt}(n)q^n=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2}.$$

Note that [10, Equations (3.3), (3.4)]

$$\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}=\sum_{n=1}^{\infty}np(n)q^n \qquad (7.6)$$

$$\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2}=\sum_{n=0}^{\infty}\frac{-1}{2}N_2(n)q^n. \qquad (7.7)$$

Andrews proved (7.7) by applying the operator $\frac{d^2}{dz^2}\Big|_{z=1}$ to a representation of the generating function of $N(m,n)$, namely [10, Equations (2.15), (2.17)]

$$R(z;q):=1+\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}N(m,n)z^mq^n=\frac{(1-z)}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}}{1-zq^n}. \qquad (7.8)$$

Here we require another representation of $R(z;q)$, that is [10, Equation (2.16)],

$$R(z;q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}. \qquad (7.9)$$

Proof of Corollary 2.9. Let $c\to 1$ on both sides of (2.11) to get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^{n(n+1)/2}}{(1-q^n)(q)_n}+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}(q^{n+1})_\infty}{(1-q^n)(q)_n}F(0,q^n;q^n)\\
&=\lim_{c\to1}\left\{\frac{-c}{(1-c)^2}+\frac{(q)_\infty}{(c)_\infty}\left(\frac{c}{1-c}+\sum_{n=1}^{\infty}\frac{(cq)_n}{(q)_n}\frac{q^n}{1-q^n}\right)\right\}. \qquad (7.10)
\end{aligned}$$

From [16, Theorem 3.8, Equation (3.24)],

$$\sum_{n=1}^{\infty}\mathrm{spt}(n)q^n=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^{n(n+1)/2}}{(1-q^n)(q)_n}. \tag{7.11}$$

We first show that

$$\sum_{n=1}^{\infty}\frac{q^{n(n+1)}(q^{n+1})_\infty}{(1-q^n)(q)_n}F(0,q^n;q^n)=(q)_\infty\sum_{n=0}^{\infty}\frac{1}{2}N_2(n)q^n. \tag{7.12}$$

From [27, p. 13, Equation (12.33)],

$$F(0,t;t)=\frac{1}{1-t}\sum_{j=0}^{\infty}\frac{t^{2j}q^{j^2}}{(tq)_j^2}. \tag{7.13}$$

Invoking (7.13) with $t=q^k$ in (7.3), we see that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^{n(n+1)}(q^{n+1})_\infty}{(1-q^n)(q)_n}F(0,q^n;q^n)&=(q)_\infty\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(q)_n^2(1-q^n)^2}\sum_{j=0}^{\infty}\frac{q^{j^2+2nj}}{(q^{n+1})_j^2}\\&=(q)_\infty\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\sum_{j=n}^{\infty}\frac{q^{j^2}}{(q)_j^2}\\&=(q)_\infty\sum_{j=1}^{\infty}\frac{q^{j^2}}{(q)_j^2}\sum_{n=1}^{j}\frac{q^n}{(1-q^n)^2}\\&=\frac{1}{2}(q)_\infty\,\frac{\mathrm{d}^2}{\mathrm{d}z^2}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}\Bigg|_{z=1},\end{aligned} \tag{7.14}$$

as can be seen from a laborious but straightforward calculation. From (7.9),

$$\begin{aligned}\frac{1}{2}(q)_\infty\,\frac{\mathrm{d}^2}{\mathrm{d}z^2}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}\Bigg|_{z=1}&=\frac{1}{2}(q)_\infty\,\frac{\mathrm{d}^2}{\mathrm{d}z^2}R(z;q)\Bigg|_{z=1}\\&=\frac{1}{2}(q)_\infty\sum_{n=0}^{\infty}N_2(n)q^n,\end{aligned} \tag{7.15}$$

where the last step follows easily from the first equality in (7.8) or from [10, Equation (3.4)]. Equation (7.12) now follows from (7.14) and (7.15).

Lastly, we show that

$$\lim_{c\to1}\left\{\frac{-c}{(1-c)^2}+\frac{(q)_\infty}{(c)_\infty}\left(\frac{c}{1-c}+\sum_{n=1}^{\infty}\frac{(cq)_n}{(q)_n}\frac{q^n}{1-q^n}\right)\right\}=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}. \tag{7.16}$$

Let L denote the above limit. Then employing (3.1) in the first step below, we see that

$$\begin{aligned}L&=\lim_{c\to1}\frac{(q)_\infty}{(cq)_\infty}\lim_{c\to1}\frac{1}{1-c}\left\{\frac{c}{1-c}-\frac{c}{1-c}\sum_{n=0}^{\infty}\frac{(c)_n}{(q)_n}q^n+\sum_{n=1}^{\infty}\frac{(cq)_n}{(q)_n}\frac{q^n}{1-q^n}\right\}\\&=\lim_{c\to1}\frac{1}{1-c}\left\{-c\sum_{n=1}^{\infty}\frac{(cq)_{n-1}}{(q)_n}q^n+\sum_{n=1}^{\infty}\frac{(cq)_n}{(q)_n}\frac{q^n}{1-q^n}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{c\to 1}\frac{1}{1-c}\sum_{n=1}^{\infty}\frac{(cq)_{n-1}}{(q)_n}q^n\left(-c+\frac{1-cq^n}{1-q^n}\right)\\
&= \sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}. \qquad (7.17)
\end{aligned}
$$

From (7.10), (7.6), (7.11), (7.12) and (7.16), we arrive at (2.12). □

Remark 5. *Note that one can equivalently start from Ramanujan's identity* (1.2)*, differentiate both sides with respect to z and then let $z = 1$ to arrive at* (2.12)*. This way, the calculation involving the right-hand side of* (1.2) *would be straightforward in comparison with that in* (7.17)*. However, our intention to derive* (2.12) *from* (2.11) *is to show the uniformity in the approach in deriving* (2.12)*,* (2.13) *and* (2.19)*.*

Proof of Corollary 2.10. Let $c = 0$ in Theorem 2.8 and divide both sides of the resulting identity by $(q)_\infty$. This gives

$$
\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^{n(n+1)/2}}{(q)_n}+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q^n)(q)_n^2}=\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)(q)_n}. \qquad (7.18)
$$

By a recent result of Merca [38, Theorem 1],

$$
\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^{n(n+1)/2}}{(q)_n}=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}. \qquad (7.19)
$$

Substituting (7.19) in (7.18), we arrive at (2.13).

In order to prove (2.15), we first observe that

$$
\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)(q)_n}=\sum_{n=1}^{\infty}\frac{1}{(q)_{n-1}}\frac{q^n}{(1-q^n)^2}=\sum_{n=1}^{\infty}\mathrm{lpt}(n)q^n. \qquad (7.20)
$$

In order to show that

$$
\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q^n)(q)_n^2}=\sum_{n=1}^{\infty}w(n)q^n,
$$

where $w(n)$ is defined in (2.14), we write

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q^n)(q)_n^2}\\
&=\sum_{n=1}^{\infty}\left(\frac{q}{1-q}\right)^2\cdots\left(\frac{q^{n-1}}{1-q^{n-1}}\right)^2\cdot\frac{q^{2n}}{(1-q^n)^3}\\
&=\sum_{n=1}^{\infty}\left(\sum_{k_1=2}^{\infty}(k_1-1)q^{k_1}\right)\cdots\left(\sum_{k_{n-1}=2}^{\infty}(k_{n-1}-1)q^{(n-1)k_{n-1}}\right)\left(\sum_{k_n=2}^{\infty}\frac{1}{2}k_n(k_n-1)q^{nk_n}\right)\\
&=\sum_{n=1}^{\infty}\sum_{k_i\geq 2,1\leq i\leq n}\frac{1}{2}k_n(k_n-1)\prod_{i=1}^{n-1}(k_i-1)q^{k_1+2k_2+\cdots+nk_n}
\end{aligned}
$$

$$= \sum_{n=1}^{\infty} w(n)q^n. \tag{7.21}$$

From (2.13), (7.20) and (7.21), we deduce (2.15). □

Proof of Corollary 2.11. We begin with [2]

$$\begin{aligned}\sum_{n=1}^{\infty} \mathrm{lpt}(n)q^n &= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q)_n} = \sum_{m,n=1}^{\infty} \frac{q^{mn}}{(q)_n}\\ &= \sum_{m=1}^{\infty} \left(\frac{1}{(q^m)_\infty} - 1 \right)\\ &= \frac{1}{2} - \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} - \frac{1}{2(q)_\infty} \sum_{n=1}^{\infty} n \left(\frac{12}{n} \right) q^{\frac{n^2-1}{24}}, \end{aligned} \tag{7.22}$$

where in the last step we applied (2.16). Combining this with (2.13), we see that

$$\sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1-q^n)(q)_n^2} = \frac{1}{2} - 2\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} - \frac{1}{2(q)_\infty} \sum_{n=1}^{\infty} n \left(\frac{12}{n} \right) q^{\frac{n^2-1}{24}}. \tag{7.23}$$

Now

$$\frac{1}{(q)_\infty} \sum_{n=1}^{\infty} n \left(\frac{12}{n} \right) q^{\frac{n^2-1}{24}} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\lfloor \sqrt{24n+1} \rfloor} k \left(\frac{12}{k} \right) p\left(n - \frac{(k^2-1)}{24} \right) \right) q^n. \tag{7.24}$$

Thus comparing the coefficients of q^n on both sides of (7.23) and employing (7.21) and (7.24), we get (2.17). □

Proof of Corollary 2.12. Recall the definition of $N_{\mathrm{SC}}(n)$ in (2.18). Let $c = -1$ in (2.11) and use [16, Theorem 3.8, Equation (3.25)],

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{n(n+1)/2}}{(q)_n(1+q^n)} = (q)_\infty \sum_{n=1}^{\infty} N_{\mathrm{SC}}(n)q^n,$$

to get

$$\begin{aligned}&(q)_\infty \sum_{n=1}^{\infty} N_{\mathrm{SC}}(n)q^n + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}(q^{n+1})_\infty}{(1-q^n)(q)_n} F(0, q^n; -q^n)\\ &= \frac{1}{4} - \frac{1}{4}\frac{(q)_\infty}{(-q)_\infty} + \frac{1}{2}\frac{(q)_\infty}{(-q)_\infty} \sum_{n=1}^{\infty} \frac{(-q)_n}{(q)_n} \frac{q^n}{1-q^n}. \end{aligned} \tag{7.25}$$

□

[2]It was during the international conference on the occasion of Ramanujan's 125th birth anniversary at University of Delhi in December 2012 that the first author learned from George E. Andrews about the connection between lpt(n) and Zagier's identity in Equation (7.22). Both the authors thank Andrews for the same.

We now simplify the second series on the left using [27, p. 17, Equation (15.51)],

$$F(0,q^n;-q^n)=\frac{1-q^n}{(-q^n)_\infty}\sum_{j=0}^{\infty}\frac{q^{nj+j(j+1)/2}}{(1-q^{n+j})(q)_j}.$$

Thus

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^{n(n+1)}(q^{n+1})_\infty}{(1-q^n)(q)_n}F(0,q^n;-q^n)&=\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(q)_n}\frac{(q^{n+1})_\infty}{(-q^n)_\infty}\sum_{j=0}^{\infty}\frac{q^{nj+j(j+1)/2}}{(1-q^{n+j})(q)_j}\\&=\frac{1}{2}\frac{(q)_\infty}{(-q)_\infty}\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}(-1)_n}{(q)_n^2}\sum_{j=0}^{\infty}\frac{q^{(n+j)(n+j+1)/2}}{(q)_j(1-q^{n+j})}\\&=\frac{1}{2}\frac{(q)_\infty}{(-q)_\infty}\sum_{k=1}^{\infty}\frac{q^{k(k+1)/2}}{(q)_k(1-q^k)}\sum_{n=1}^{k}\begin{bmatrix}k\\n\end{bmatrix}\frac{(-1)_n}{(q)_n}q^{n(n+1)/2}\\&=\frac{1}{2}\frac{(q)_\infty}{(-q)_\infty}\sum_{k=1}^{\infty}\frac{q^{k(k+1)/2}}{(q)_k(1-q^k)}\left(\frac{(-q)_k}{(q)_k}-1\right),\end{aligned}\tag{7.26}$$

where the last step follows from the q-Chu-Vandermonde summation [23, Corollary 1.2]. Finally, (2.19) follows from (7.25) and (7.26).

Before proving Corollary 2.13, we begin with a lemma.

Lemma 7.1. *We have*

$$2(-q)_\infty\sum_{n=1}^{\infty}N_{\mathrm{SC}}(n)q^n-\sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(1-q^n)(q)_n}=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}.$$

Proof of Corollary 2.13. From [16, Equation (1.12)],

$$\sum_{n=1}^{\infty}N_{\mathrm{SC}}(n)q^n=\frac{1}{(-q)_\infty}\sum_{n=1}^{\infty}\frac{(-q)_{n-1}q^n}{1-q^n}.\tag{7.27}$$

Let

$$\sum_{n=1}^{\infty}\frac{(-q)_{n-1}q^n}{1-q^n}=\sum_{n=1}^{\infty}a(n)q^n.\tag{7.28}$$

Then $a(n)$ equals the number of partitions of n in which only the largest part is allowed to repeat. From Andrews [11, p. 153], these partitions are conjugates of compact partitions. Hence if $c(n)$ denotes the number of compact partitions of n, then $a(n)=c(n)$. However, by a result conjectured by Beck and recently proved by Chern [25, Theorem 1.2], $c(n)=\mathrm{ssptd}_o(n)$, where $\mathrm{ssptd}_o(n)$ denotes the sum of the smallest parts in all partitions of n into distinct parts which are odd in number. Hence

$$a(n)=\mathrm{ssptd}_o(n).\tag{7.29}$$

Let $\mathrm{ssptd}(n)$ denote the sum of the smallest parts in all partitions of n into distinct parts, and $\mathrm{ssptd}_e(n)$, the same with the added restriction that the parts be even in number. As can be conjured by observing that the left-hand side of (1.3) is the generating function of that of (1.4),

$$\sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(1-q^n)(q)_n}=\sum_{n=1}^{\infty}\mathrm{ssptd}(n)q^n=\sum_{n=1}^{\infty}\mathrm{ssptd}_o(n)q^n+\sum_{n=1}^{\infty}\mathrm{ssptd}_e(n)q^n. \tag{7.30}$$

Thus, from (7.27), (7.28), (7.29) and (7.30),

$$\begin{aligned}2(-q)_\infty\sum_{n=1}^{\infty}N_{\mathrm{SC}}(n)q^n-\sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(1-q^n)(q)_n}&=\sum_{n=1}^{\infty}\left(\mathrm{ssptd}_o(n)-\mathrm{ssptd}_e(n)\right)q^n\\&=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n},\end{aligned}$$

where the last step follows from the Fokkink, Fokkink and Wang identity (1.4). This proves the lemma. □

Proof of Corollary 2.13. Multiply both sides of (2.19) by $2(-q)_\infty/(q)_\infty$, employ Lemma 7.1 and simplify to obtain

$$\sum_{n=1}^{\infty}\frac{(-q)_n}{(q)_n}\frac{q^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{(-q)_n}{(q)_n^2}\frac{q^{n(n+1)/2}}{1-q^n}+\frac{1}{2}\frac{(-q)_\infty}{(q)_\infty}-\frac{1}{2}=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}. \tag{7.31}$$

Now from [23, Equation (1.3)],

$$\frac{(-q)_\infty}{(q)_\infty}=\sum_{n=0}^{\infty}\frac{(-1)_n}{(q)_n^2}q^{n(n+1)/2}. \tag{7.32}$$

Invoke (7.32) in (7.31) and simplify to obtain (2.20).

To prove (2.21), note that

$$\sum_{n=1}^{\infty}\frac{(-q)_{n-1}}{(q)_n^2}\frac{q^{n(n+3)/2}}{1-q^n}=\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{\pi\in\mathcal{P}^*(n)\\ L(\pi)\geq 2}}\frac{L(\pi)(L(\pi)-1)}{2}\prod_{i=1}^{l(\pi)-1}(2\nu(i)-1)\Bigg)q^n \tag{7.33}$$

can be proved along exact similar lines as (7.21). As far as proving

$$\sum_{n=1}^{\infty}\frac{(-q)_n}{(q)_n}\frac{q^n}{1-q^n}=\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{\pi\in\mathcal{P}_O(n)\\ l(\pi)\text{overlined}}}(2L(\pi)-1)\Bigg)q^n \tag{7.34}$$

is concerned, it is easily seen that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-q)_n}{(q)_n}\frac{q^n}{1-q^n}&=\sum_{n=1}^{\infty}\frac{(-q)_{n-1}}{(q)_{n-1}}\frac{q^n(1+q^n)}{(1-q^n)^2}\\&=\sum_{n=1}^{\infty}\frac{(-q)_{n-1}}{(q)_{n-1}}\sum_{k_n=1}^{\infty}(2k_n-1)q^{nk_n},\end{aligned}$$

which proves (7.34). Together, (2.20), (7.33) and (7.34) give (2.21). □

Another special case of (2.11) when $c = q^m, m \geq 1$, is stated below without proof.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{n(n+1)/2}}{(1-q^{n+m})(q)_n} + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}(q^{n+1})_\infty}{(1-q^n)(q)_n} F(0, q^n; q^{n+m})$$
$$= \frac{q^m((q)_m - 1)}{(1-q^m)^2} + (q)_{m-1} \sum_{n=1}^{\infty} \begin{bmatrix} n+m \\ n \end{bmatrix} \frac{q^n}{1-q^n}.$$

When $m = 1$, it simplifies to

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{n(n+1)/2}}{(1-q^{n+1})(q)_n} + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}(q^{n+1})_\infty}{(1-q^n)(q)_n} F(0, q^n; q^{n+1}) = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}. \tag{7.35}$$

8. Concluding remarks

We hope to have demonstrated the richness of partition-theoretic information embedded in our generalizations (2.1), (2.3) and (2.23) of Ramanujan's (1.7), (1.2) and (2.22) respectively.

In Corollary 2.5, we have shown that the one-variable generalization of $\sigma(q)$ defined in (2.7), namely $\sigma(c, q)$, admits a simpler representation. We have also given the partition-theoretic meaning of its coefficients. Thus we may analogously ask if $\sigma(c, d, q)$, the two-variable generalization of $\sigma(q)$ defined in (2.8) also admits a simpler representation than the complicated one given in [17, Theorem 1.2]. However, following the approach in our paper to accomplish this would first require finding a generalization of (2.3) with one more variable d. Without doubt, this would be interesting in itself.

Except for the special case $c \to 1$ of Theorem 2.8, that is, the one which gives Andrews' identity for $\mathrm{spt}(n)$, each of the cases $c = 0, -1$ and q in (2.13), (2.20) and (7.35) respectively involves the divisor generating function $\displaystyle\sum_{n=1}^{\infty} \frac{q^n}{1-q^n}$. This is intriguing, to say the least, and certainly merits further study.

Acknowledgements

The authors thank Christian Krattenthaler who also informed them about the short proof of (2.1) which is the same as that of Andrews [12]. They also thank Bruce C. Berndt for his suggestions which improved the exposition of the paper. The first author's research is supported by the SERB-DST grant ECR/2015/000070 whereas the second author is a SERB National Post Doctoral Fellow (NPDF) supported by the fellowship PDF/2017/000370. Both sincerely thank SERB-DST for the support.

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