THE FINITE FOURIER TRANSFORM OF CLASSICAL POLYNOMIALS

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Abstract. The finite Fourier transform of a family of orthogonal polynomials is the usual transform of these polynomials extended by 0 outside their natural domain of orthogonality. Explicit expressions are given for the Legendre, Jacobi, Gegenbauer and Chebyshev families.

1. Introduction

Compendia of formulas, such as the classical *Table of Integrals, Series and Products* by I. S. Gradshteyn and I. M. Ryzhik [6] and the recent *NIST Handbook of Mathematical Functions* [11] do not contain a systematic collection of Fourier transforms of orthogonal polynomials.

Special cases do appear. For instance, [11, formula 18.17.19] contains the identity

$$\int_{-1}^{1} P_n(x) e^{\imath\lambda x} dx = \imath^n \sqrt{\frac{2\pi}{\lambda}} J_{n+\frac{1}{2}}(\lambda), \tag{1.1}$$

for the *finite Fourier transform* of the Legendre polynomial P_n. Here J_α is the Bessel function defined by

$$J_\alpha(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda/2)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}, \qquad \lambda \in \mathbb{C}. \tag{1.2}$$

The use of formula (1.1) in developing algorithms for the convolution involving Legendre polynomials is described in [8].

A second example is [4, formula 3.3(7), page 123]

$$\int_{-1}^{1} P_\nu(x) e^{\imath\lambda x} dx = \frac{2\pi \sin \pi\nu}{\nu(\nu+1)} e^{-\imath\lambda} {}_2F_2 \left(\left. \begin{matrix} 1, 1 \\ -\nu, 2+\nu \end{matrix} \right| 2\imath\lambda \right), \tag{1.3}$$

where $P_\nu(x)$ is the associated Legendre function.

The more natural situation, where the corresponding weight function appears in the integrand, is included in the tables. For instance, for the Jacobi

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polynomial, [11, 18.17.16] gives
(1.4)
$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)e^{\imath\lambda x}\,dx = X_n(\lambda;\alpha,\beta)\,{}_1F_1\left(\begin{matrix} n+\alpha+1 \\ 2n+\alpha+\beta+2\end{matrix}\middle| -2\imath\lambda\right),$$
with

$$X_n(\lambda;\alpha,\beta) = \frac{(\imath\lambda)^n e^{\imath\lambda}}{n!}2^{n+\alpha+\beta+1}\times B(n+\alpha+1,n+\beta+1). \tag{1.5}$$

Here $B(a,b)$ is the classical Euler beta function.

The work presented here was stimulated by a result of A. Fokas et al. [5] involving the Fourier transform of Chebyshev polynomials of the first kind. In turn, this was needed for a project involving Fourier expansions of Zagier polynomials [2]. Properties of these polynomials appear in [1, 3]. It was surprising to the authors that the finite Fourier transform of classical orthogonal polynomials was not readily available in the literature. Some of the results presented here also appear in [5] and [7]. The authors wish to thank A. Fokas and T. Koornwinder for correspondence on the questions discussed here.

The goal of this project is to produce closed-form evaluations of definite integrals of the form

$$\widehat{P}(\lambda) := \int_a^b e^{\imath\lambda x}P(x)\,dx \tag{1.6}$$

for a variety of polynomials P, orthogonal on the interval $[a,b]$. The function $\widehat{P}(\lambda)$ is called the *finite Fourier transform* of the polynomial P. The cases considered here include the Legendre polynomial $P_n(x)$, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, from which the Gegenbauer polynomials $C_n^{(\nu)}(x)$ and both types of Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are derived.

Naturally, depending on the representation given of the polynomial P, it is possible to obtain a variety of expressions for $\widehat{P}$. For instance, if an expression for the coefficients of P is available, the identity in Lemma 1.1 and a simple scaling give directly a double-sum representation for $\widehat{P}(\lambda)$.

It is convenient to introduce the notation

$$E_n(x) = \sum_{j=0}^{n}\frac{x^j}{j!} \tag{1.7}$$

for the partial sums of the exponential function. Many of the results may be expressed in terms of E_n. The following result is elementary and it appears in [6, formula 2.323].

Lemma 1.1. *Let $k \geq 0$ be an integer and λ an indeterminate. Then, for $\lambda \neq 0$,*

$$\int_{-1}^{1}x^k e^{\imath\lambda x}dx = \frac{(-1)^k k!}{(\imath\lambda)^{k+1}}\left[e^{\imath\lambda}E_k(-\imath\lambda) - e^{-\imath\lambda}E_k(\imath\lambda)\right], \tag{1.8}$$

and

$$\int_0^1 x^k e^{\imath\lambda x}dx = \frac{(-1)^k k!}{(\imath\lambda)^{k+1}}\left[e^{\imath\lambda}E_k(-\imath\lambda)-1\right]. \tag{1.9}$$

Proof. Integrate by parts. □

Note 1.2. *The notation employed here is standard. The symbol* $(a)_n$ *denotes the raising factorial, defined by* $(a)_n = a(a+1)\cdots(a+n-1)$ *and* $(a)_0 = 1$. *For* n, $k \in \mathbb{N}$, *the elementary properties*

$$(1)_n = n! \tag{1.10}$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{1.11}$$

$$(a+\tfrac{1}{2})_n = \frac{(2a)_{2n}}{2^{2n}(a)_n} \tag{1.12}$$

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!} \quad \text{for } n > k \text{ and } 0 \text{ otherwise,} \tag{1.13}$$

$$(n+1)_k = \frac{(n+k)!}{n!}, \tag{1.14}$$

$$(-a)_n = (-1)^n(a-n+1)_n, \tag{1.15}$$

are used throughout. These can be found in [14, p. 72].

Section 2 contains the results for Legendre polynomials $P_n(x)$ and Section 3 gives explicit formulas for the Fourier transform of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Several special cases of this Fourier transform of Jacobi polynomials are given in Section 4: the first one confirms the values for Legendre polynomials and the other two cases give Fourier transforms of Gegenbauer and Chebyshev polynomials. Section 5 describes consequences of Parseval's identity for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. This is made explicit in the case $\alpha = \beta = 0$, where the result is expressed in terms of Bessel functions and for $\alpha = \beta = -\frac{1}{2}$ where Parseval's identity is given in terms of Ménage polynomials, a class of polynomials connected to the hypergeometric function ${}_3F_1$. Finally, Section 6 presents an alternative procedure for the evaluation of Fourier transforms of polynomials.

2. Legendre polynomials

This section contains a variety of formulas for the finite Fourier transform of the Legendre polynomials $P_n(x)$. These are orthogonal polynomials on the interval $[-1, 1]$, with weight $w(x) \equiv 1$. The next theorem gives all the results.

Theorem 2.1. *The finite Fourier transform of the Legendre polynomial* $P_n(x)$ *is given, for* $\lambda \neq 0$, *by one of the four equivalent forms:*

$$\begin{aligned}
\widehat{P_n}(\lambda) &= 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{\frac{1}{2}(n+k-1)}{n} \frac{(-1)^k k!}{(\imath\lambda)^{k+1}} \left[e^{\imath\lambda} E_k(-\imath\lambda) - e^{-\imath\lambda} E_k(\imath\lambda) \right] \\
&= \imath^n \sqrt{\frac{2\pi}{\lambda}} J_{n+1/2}(\lambda) \\
&= 2 \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\, k!} \frac{\left[e^{-\imath\lambda} E_k(2\imath\lambda) - e^{\imath\lambda} \right]}{(-2\imath\lambda)^{k+1}} \\
&= 2 \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\, k!} \frac{\left[(-1)^{n+k} e^{-\imath\lambda} - e^{\imath\lambda} \right]}{(-2\imath\lambda)^{k+1}}.
\end{aligned}$$

For $\lambda = 0$, the value is

$$\widehat{P_n}(0) = \int_{-1}^{1} P_n(x)\, dx = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \tag{2.1}$$

Proof. The first formula follows from Lemma 1.1 and the explicit representation

$$P_n(x) = 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{\frac{1}{2}(n+k-1)}{n} x^k \tag{2.2}$$

which follows from $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$, the three-terms recurrence satisfied by the Legendre polynomials. The second expression for $\widehat{P}_n(\lambda)$ comes from their Rodrigues formula

$$P_n(x) = \frac{1}{2^n\, n!} \left(\frac{d}{dx} \right)^n (x^2 - 1)^n, \tag{2.3}$$

(see [6, Formula 8.910.2]) and it appears as entry 7.242.5 in [6]. Then

$$\widehat{P}_n(\lambda) = \frac{1}{2^n n!} \int_{-1}^{1} e^{\imath\lambda x} \left(\frac{d}{dx} \right)^n (x^2 - 1)^n \, dx \tag{2.4}$$

and integrating by parts n-times yields

$$\widehat{P}_n(\lambda) = \frac{(-\imath\lambda)^n}{2^n n!} \int_{-1}^{1} (x^2 - 1)^n e^{\imath\lambda x} \, dx. \tag{2.5}$$

Entry 3.387.2 of [6] states that

$$\int_{-1}^{1} (1 - x^2)^{\nu - 1} e^{\imath\mu x} \, dx = \sqrt{\pi} \left(\frac{2}{\mu} \right)^{\nu - \frac{1}{2}} \Gamma(\nu) J_{\nu - \frac{1}{2}}(\mu), \quad \operatorname{Re} \nu > 0. \tag{2.6}$$

The result is obtained by choosing $\mu = \lambda$ and $\nu = n + 1$.

The third form of the finite Fourier transform of the Legendre polynomials is obtained from their hypergeometric representation

$$P_n(x) = {}_2F_1 \left(\left. \begin{matrix} -n & n+1 \\ & 1 \end{matrix} \right| \frac{1-x}{2} \right) = \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(1)_k\, k!} \left(\frac{1-x}{2} \right)^k, \tag{2.7}$$

that gives

$$\widehat{P}_n(\lambda) = \sum_{k=0}^{n} \frac{(-n)_k(n+1)_k}{k!^2} \int_{-1}^{1} e^{\imath\lambda x} \left(\frac{1-x}{2} \right)^k dx. \tag{2.8}$$

The change of variables $x = 1 - 2t$ and the formulas (1.14) and (1.15) give

$$\widehat{P}_n(\lambda) = 2e^{\imath\lambda} \sum_{k=0}^{n} \frac{(-1)^k(n+k)!}{(n-k)!k!^2} \int_0^1 t^k e^{-2\imath\lambda t}\, dt. \tag{2.9}$$

Lemma 1.1 now gives the stated result.

To produce the last form for $\widehat{P}_n(\lambda)$, let $t = 2\imath\lambda$ in the third expression for this transform. Then, after multiplication by t^n and some simplification, the claim is equivalent to the polynomial identity

$$\sum_{k=0}^{n} \frac{(2n-k)!}{k!(n-k)!} (-1)^k t^k \sum_{j=0}^{n-k} \frac{t^j}{j!} = \sum_{k=0}^{n} \frac{(2n-k)!}{k!(n-k)!} t^k. \tag{2.10}$$

To simplify the sum, let $\nu = k + j$ on the left-hand side to show that the desired identity is equivalent to

$$\sum_{\nu=0}^{n} \left[\sum_{k=0}^{\nu} \frac{(-1)^k(2n-k)!}{k!\,(n-k)!\,(\nu-k)!} \right] t^\nu = \sum_{k=0}^{n} \frac{(2n-k)!}{k!\,(n-k)!} t^k. \tag{2.11}$$

Matching coefficients, the result follows from

$$\sum_{j=0}^{k} \frac{(-1)^j(2n-j)!}{j!\,(n-j)!\,(k-j)!} = \frac{(2n-k)!}{k!\,(n-k)!} \tag{2.12}$$

for every $0 \leq k \leq n$. This is equivalent to the binomial identity given in Lemma 2.2 below. The proof is complete. □

Lemma 2.2. *For $n \in \mathbb{N}$ and $0 \leq k \leq n$*

$$\sum_{j=0}^{k} (-1)^j \binom{n}{j} \binom{2n-j}{2n-k} = \binom{n}{k}. \tag{2.13}$$

Proof. The proof uses $\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$ to write

$$\binom{2n-j}{2n-k} = \binom{2n-j}{k-j} = (-1)^{k-j} \binom{k-2n-1}{k-j} \tag{2.14}$$

and then (2.13) is converted into Vandermonde identity

$$\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}. \tag{2.15}$$

□

3. Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{\alpha+n}{k} \binom{\beta+n}{n-k} (x-1)^{n-k}(x+1)^k \tag{3.1}$$

are orthogonal on $[-1,1]$ with respect to the weight

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\, \beta > -1. \tag{3.2}$$

This section contains expressions for their finite Fourier transform. The hypergeometric representation

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n, \quad n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \Bigg| \frac{1-x}{2} \right), \tag{3.3}$$

is used in the calculations.

Theorem 3.1. *The finite Fourier transform of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ is given by one of the two equivalent forms*

$$\begin{aligned} \widehat{P_n^{(\alpha,\beta)}}(\lambda) &= 2e^{\imath\lambda}(\alpha+1)_n \sum_{k=0}^{n} \frac{(n+\alpha+\beta+1)_k}{(n-k)!(\alpha+1)_k} \left[\frac{e^{-2\imath\lambda}E_k(2\imath\lambda)-1}{(-2\imath\lambda)^{k+1}} \right] \\ &= 2\sum_{k=0}^{n} \frac{(n+\alpha+\beta+1)_k}{(-2\imath\lambda)^{k+1}(n-k)!} \\ &\quad \times \left[(-1)^{n-k}e^{-\imath\lambda}(\beta+k+1)_{n-k} - e^{\imath\lambda}(\alpha+k+1)_{n-k} \right] \end{aligned}$$

for $\lambda \neq 0$. For $\lambda = 0$,

$$\widehat{P_n^{(\alpha,\beta)}}(0) = \frac{2}{(n+\alpha+\beta)} \left[\binom{\alpha+n}{n+1} + (-1)^n \binom{\beta+n}{n+1} \right]. \tag{3.4}$$

Proof. The first statement comes from the hypergeometric form

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n, \quad n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \Bigg| \frac{1-x}{2} \right) \\ &= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k k! 2^k} (1-x)^k \end{aligned} \tag{3.5}$$

and use Lemma 1.1 to produce

$$\int_{-1}^{1} (1-x)^k e^{\imath\lambda x} dx = -e^{\imath\lambda} \frac{k!}{(\imath\lambda)^{k+1}} \left[e^{-2\imath\lambda}E_k(2\imath\lambda) - 1 \right] \tag{3.6}$$

and then (1.13) to simplify the result.

Now use identity (the case $m=1$ of [6, 8.961.4]):

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{3.7}$$

and integrate by parts to obtain

$$\widehat{P_n^{(\alpha,\beta)}}(\lambda) = \frac{e^{\imath\lambda x}}{\imath\lambda} P_n^{(\alpha,\beta)}(x)\Big|_{-1}^{1} - \frac{(n+\alpha+\beta+1)}{2\imath\lambda} \widehat{P_{n-1}^{(\alpha+1,\beta+1)}}(\lambda).$$

Introduce the notation for the boundary term

$$a_n^{(\alpha,\beta)}(\lambda) = \frac{e^{\imath\lambda x}}{\imath\lambda} P_n^{(\alpha,\beta)}(x)\Big|_{-1}^{1}, \tag{3.8}$$

to write the previous computation as the recurrence

$$\widehat{P_n^{(\alpha,\beta)}}(\lambda) = a_n^{(\alpha,\beta)}(\lambda) - \frac{(n+\alpha+\beta+1)}{2\imath\lambda} \widehat{P_{n-1}^{(\alpha+1,\beta+1)}}(\lambda). \tag{3.9}$$

Iteration yields

$$\begin{aligned} \widehat{P_n^{(\alpha,\beta)}}(\lambda) &= \sum_{k=1}^{n} (-1)^{n-k} \frac{(n+\alpha+\beta+1)_{n-k}}{(2\imath\lambda)^{n-k}} a_k^{(\alpha+n-k,\beta+n-k)}(\lambda) \\ &+ (-1)^n \frac{(n+\alpha+\beta+1)_n}{(2\imath\lambda)^n} \widehat{P_0^{(\alpha+n,\beta+n)}}(\lambda). \end{aligned}$$

Evaluate the last term as $a_0^{(\alpha,\beta)}(\lambda)$ and use

$$P_n^{(\alpha,\beta)}(1) = \binom{\alpha+n}{n} \text{ and } P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{\beta+n}{n} \tag{3.10}$$

from (3.1) to obtain

$$a_n^{(\alpha,\beta)} = \frac{1}{\imath\lambda}\left[e^{\imath\lambda}\binom{\alpha+n}{n} - (-1)^n e^{-\imath\lambda}\binom{\beta+n}{n}\right]. \tag{3.11}$$

Some algebraic simplification now gives the stated result. The value for $\lambda = 0$ comes directly from (3.7). □

The next statement represents a hypergeometric rewrite of the last formula in Theorem 3.1.

Theorem 3.2. *The finite Fourier transform of the Jacobi polynomial, for $\lambda \neq 0$, is given by*

$$\begin{aligned} \widehat{P_n^{(\alpha,\beta)}}(\lambda) &= \frac{(\beta+1)_n}{\imath\lambda n!}(-1)^{n+1} e^{-\imath\lambda} {}_3F_1\left(\begin{matrix} n+\alpha+\beta+1, -n, 1 \\ \beta+1 \end{matrix} \,\Bigg|\, \frac{-1}{2\imath\lambda}\right) + \\ &+ \frac{(\alpha+1)_n}{\imath\lambda n!} e^{\imath\lambda} {}_3F_1\left(\begin{matrix} n+\alpha+\beta+1, -n, 1 \\ \alpha+1 \end{matrix} \,\Bigg|\, \frac{1}{2\imath\lambda}\right). \end{aligned}$$

Proof. The first term in the expression of the last formula in Theorem 3.1 is simplified using (1.13) and $(\beta+k+1)_{n-k} = \frac{(\beta+1)_n}{(\beta+1)_k}$ to obtain

$$\begin{aligned} &\frac{(-1)^{n-k}(n+\alpha+\beta+1)_k(\beta+k+1)_{n-k}}{(-2\imath\lambda)^{k+1}(n-k)!} = \\ &\qquad\qquad \frac{(-1)^{n+1}(\beta+1)_n}{2\imath\lambda n!} \frac{(n+\alpha+\beta+1)_k(-n)_k(1)_k}{(\beta+1)_k} \frac{t^k}{k!} \end{aligned}$$

with $t = -1/2\imath\lambda$. Summing from $k = 0$ to n gives the first term in the answer. A similar argument simplifies the second term in Theorem 3.1. $\square$

Note 3.3. *Define*

$$A_n^{(\alpha,\beta)}(t) = \frac{(\alpha+1)_n}{n!} {}_3F_1\left(\left. \begin{matrix} n+\alpha+\beta+1, -n, 1 \\ \alpha+1 \end{matrix} \right| \frac{1}{t} \right). \tag{3.12}$$

then the finite Fourier transform of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, for $\lambda \neq 0$, is given by

$$\widehat{P_n^{(\alpha,\beta)}}(\lambda) = \frac{1}{\imath\lambda}\left[(-1)^{n+1} e^{-\imath\lambda} A_n^{(\beta,\alpha)}(-2\imath\lambda) + e^{\imath\lambda} A_n^{(\alpha,\beta)}(2\imath\lambda) \right]. \tag{3.13}$$

4. A collection of special cases

This section presents a collection of special cases of the Jacobi polynomials and their respective finite Fourier transforms.

4.1. **Legendre polynomials.** These polynomials were discussed in Section 3 and correspond to the special case $\alpha = \beta = 0$; that is,

$$P_n(x) = P_n^{(0,0)}(x). \tag{4.1}$$

The first formula in Theorem 3.1 reproduces the third formula in Theorem 2.1. Similarly, the second formula in Theorem 3.1 gives the last expression for the finite Fourier transform of Legendre polynomials in Theorem 2.1.

4.2. **Gegenbauer polynomials.** These polynomials are also special cases of $P_n^{(\alpha,\beta)}(x)$ where $\alpha = \beta = \nu - \frac{1}{2}$:

$$C_n^{(\nu)}(x) = \frac{(2\nu)_n}{(\nu+1/2)_n} P_n^{(\nu-1/2,\nu-1/2)}(x). \tag{4.2}$$

Theorem 4.1. *The finite Fourier transform of the Gegenbauer polynomial $C_n^{(\nu)}(x)$ is given, for $\lambda \neq 0$, by one of the three equivalent forms*

$$\begin{aligned} \widehat{C_n^{(\nu)}}(\lambda) &= 2(2\nu)_n e^{\imath\lambda} \sum_{k=0}^{n} 2^{2k} \frac{(n+2\nu)_k (\nu)_k}{(n-k)!(2\nu)_{2k}} \left[\frac{e^{-2\imath\lambda} E_k(2\imath\lambda) - 1}{(-2\imath\lambda)^{k+1}} \right] \\ &= 2(2\nu)_n \sum_{k=0}^{n} 2^{2k} \frac{(n+2\nu)_k (\nu)_k}{(n-k)!(2\nu)_{2k}} \left[\frac{(-1)^{n-k} e^{-\imath\lambda} - e^{\imath\lambda}}{(-2\imath\lambda)^{k+1}} \right] \end{aligned}$$

and

$$\begin{aligned} \widehat{C_n^{(\nu)}}(\lambda) = \frac{(2\nu)_n}{\imath\lambda n!} \times \Bigg[& (-1)^{n+1} e^{-\imath\lambda} {}_3F_1\left(\left. \begin{matrix} n+2\nu, -n, 1 \\ \nu+\frac{1}{2} \end{matrix} \right| -\frac{1}{2\imath\lambda} \right) + \\ & e^{\imath\lambda} {}_3F_1\left(\left. \begin{matrix} n+2\nu, -n, 1 \\ \nu+\frac{1}{2} \end{matrix} \right| \frac{1}{2\imath\lambda} \right) \Bigg]. \end{aligned}$$

For $\lambda = 0$, the Fourier transform is

$$\widehat{C_n^{(\nu)}}(0) = \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} \frac{2(1+(-1)^n)}{n+2\nu-1} \binom{n+\nu-\frac{1}{2}}{n+1}. \tag{4.3}$$

4.3. **Chebyshev polynomials.** The Chebyshev polynomials are related to Gegenbauer polynomials by

$$U_n(x) = C_n^{(1)}(x) \text{ and } T_n(x) = \lim_{\nu \to 0} \frac{nC_n^{(\nu)}(x)}{2\nu}, \text{ for } n \geq 1. \tag{4.4}$$

These formulas are now used to evaluate the finite Fourier transform of Chebyshev polynomials. The proof is omitted.

Theorem 4.2. *The finite Fourier transform of the Chebyshev polynomials is given, for $\lambda \neq 0$, by*

$$\begin{aligned} \widehat{U_n}(\lambda) &= e^{\imath\lambda} \sum_{k=0}^{n} 2^{2k+1} k! \binom{n+k+1}{n-k} \left[\frac{e^{-2\imath\lambda} E_k(2\imath\lambda) - 1}{(-2\imath\lambda)^{k+1}} \right] \\ &= \sum_{k=0}^{n} \frac{2^{2k+1}(n+k+1)!\, k!}{(2k+1)!\,(n-k)!} \frac{\left[(-1)^{n-k} e^{-\imath\lambda} - e^{\imath\lambda}\right]}{(-2\imath\lambda)^{k+1}} \end{aligned}$$

and

$$\widehat{T_n}(\lambda) = \sum_{k=0}^{n} (-1)^{k+1} \frac{n 2^k (n+k)! k!}{(n-k)!(2k)!(n+k)} \frac{\left[(-1)^{n-k} e^{-\imath\lambda} - e^{\imath\lambda}\right]}{(\imath\lambda)^{k+1}}$$

For $\lambda = 0$, the values are

$$\widehat{T_n(0)} = \int_{-1}^{1} T_n(x)\, dx = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ \frac{1+(-1)^n}{1-n^2} & \text{if } n > 1, \end{cases} \tag{4.5}$$

and

$$\widehat{U_n(0)} = \int_{-1}^{1} U_n(x)\, dx = \frac{1+(-1)^n}{1+n}. \tag{4.6}$$

5. Biorthogonality for the Jacobi polynomials

The sequence of functions $\{\frac{1}{\sqrt{2}} e^{\pi\imath jx} : j \in \mathbb{Z}\}$ forms an orthonormal family on the Hilbert space $L^2([-1,1])$. Therefore, every $f \in L^2([-1,1])$ may be expanded in the form

$$f(x) = \frac{1}{\sqrt{2}} \sum_{j=-\infty}^{\infty} a_j(f) e^{\pi\imath jx}, \tag{5.1}$$

where the Fourier coefficients are given by

$$a_j(f) = \frac{1}{\sqrt{2}} \int_{-1}^{1} f(x) e^{-\pi\imath jx}\, dx. \tag{5.2}$$

Parseval's identity [9, Theorem 14] states that

$$\int_{-1}^{1} f(x)\overline{g(x)}\, dx = \sum_{j=-\infty}^{\infty} a_j(f)\overline{a_j(g)}. \tag{5.3}$$

This identity is now made explicit for the case

$$f(x) = P_n^{(\alpha,\beta)}(x) \text{ and } g(x) = Q_n^{(\alpha,\beta)}(x) := (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x).$$

The Fourier coefficients $a_j(Q_m^{(\alpha,\beta)}(x))$ are given in (1.4) and $a_j(P_n^{(\alpha,\beta)}(x))$ have been evaluated in Theorem 3.2. Parseval's identity and the orthogonality of Jacobi polynomials give

$$\sum_{j=-\infty}^{\infty} a_j(P_n^{(\alpha,\beta)}(x))\overline{a_j(Q_m^{(\alpha,\beta)}(x))} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}\delta_{n,m},$$

where $\delta_{n,m}$ is Kronecker's delta (1 if $n = m$ and 0 if $n \neq m$). Only the case $n \neq m$ leads to an interesting relation. A direct calculation shows that $a_0(Q_m^{(\alpha,\beta)}(x)) = 0$, so that Parseval's identity is written as

$$\sum_{j\in\mathbb{Z}, j\neq 0} a_j(P_n^{(\alpha,\beta)}(x))\overline{a_j(Q_m^{(\alpha,\beta)}(x))} = 0, \qquad \text{for } n \neq m.$$

Now replace $\lambda = -\pi j$ in (1.4) to obtain

$$\begin{aligned}\overline{a_j(Q_m^{(\alpha,\beta)})} = \frac{(-1)^j (\pi\imath)^m j^m}{m!} & 2^{m+\alpha+\beta+1/2} B(m+\alpha+1, m+\beta+1) \\ & {}_1F_1\left(\left. \begin{matrix} m+\alpha+1 \\ 2m+\alpha+\beta+2 \end{matrix} \right| -2\pi\imath j \right).\end{aligned}$$

Similarly, Theorem 3.2 with $\lambda = -\pi j$ gives

$$\begin{aligned} a_j(P_n^{(\alpha,\beta)}) = \frac{(-1)^j}{\sqrt{2\pi\imath j} n!} & \\ & \left[(-1)^n (\beta+1)_n \, {}_3F_1\left(\left. \begin{matrix} n+\alpha+\beta+1, -n, 1 \\ \beta+1 \end{matrix} \right| \frac{1}{2\pi\imath j} \right) - \right. \\ & \left. (\alpha+1)_n \, {}_3F_1\left(\left. \begin{matrix} n+\alpha+\beta+1, -n, 1 \\ \alpha+1 \end{matrix} \right| -\frac{1}{2\pi\imath j} \right) \right].\end{aligned}$$

Parseval's identity now produces the next result. Kummer's identity

$$(5.4) \qquad {}_1F_1\left(\left. \begin{matrix} u \\ u+v \end{matrix} \right| z \right) = e^z \, {}_1F_1\left(\left. \begin{matrix} v \\ u+v \end{matrix} \right| -z \right)$$

is used in the simplification.

Theorem 5.1. *Define*

$$\begin{aligned} W_{n,m}^{(\alpha,\beta)}(j) = (\alpha+1)_n j^{m-1} & \\ & {}_3F_1\left(\left. \begin{matrix} n+\alpha+\beta+1, -n, 1 \\ \alpha+1 \end{matrix} \right| \frac{1}{2\pi\imath j} \right) {}_1F_1\left(\left. \begin{matrix} m+\alpha+1 \\ 2m+\alpha+\beta+2 \end{matrix} \right| 2\pi\imath j \right).\end{aligned}$$

Then, for $n \neq m$,

$$(-1)^{n+m+1} \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}}^{\infty} W_{n,m}^{(\beta,\alpha)}(j) = \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}}^{\infty} W_{n,m}^{(\alpha,\beta)}(j). \tag{5.5}$$

In particular, if n and m have opposite parity, then

$$\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}}^{\infty} W_{n,m}^{(\beta,\alpha)}(j) = \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}}^{\infty} W_{n,m}^{(\alpha,\beta)}(j). \tag{5.6}$$

Two special cases of the result in Theorem 5.1 are described next. Both examples have $\alpha = \beta$ and $n \equiv m \bmod 2$. Theorem 5.1 reduces to the next statement.

Corollary 5.2. *Assume $\alpha > -1$, $n \neq m$ and $n \equiv m \bmod 2$. Then*

$$\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}}^{\infty} W_{n,m}^{(\alpha,\alpha)}(j) = 0, \tag{5.7}$$

where

$$W_{n,m}^{(\alpha,\alpha)}(j) = (\alpha+1)_n j^{m-1} {}_3F_1\left(\begin{matrix} n+2\alpha+1, -n, 1 \\ \alpha+1 \end{matrix} \middle| \frac{1}{2\pi\imath j}\right) {}_1F_1\left(\begin{matrix} m+\alpha+1 \\ 2m+2\alpha+2 \end{matrix} \middle| 2\pi\imath j\right).$$

To simplify these expressions, use the identity

$${}_1F_1\left(\begin{matrix} a \\ 2a \end{matrix} \middle| z\right) = 2^{2a-1}\Gamma(a+1/2) z^{1/2-a} e^{z/2} I_{a-\frac{1}{2}}\left(\frac{z}{2}\right), \tag{5.8}$$

where $I_\alpha(z)$ is the modified Bessel function of the first kind (see [13, p. 487, formula 7.11.1.5]), to obtain

$${}_1F_1\left(\begin{matrix} m+\alpha+1 \\ 2m+2\alpha+2 \end{matrix} \middle| 2\pi\imath j\right) = \frac{2^{m+\alpha+1/2}\Gamma(m+\alpha+3/2)}{(\pi\imath)^{m+\alpha+1/2}} \times \frac{(-1)^j I_{m+\alpha+\frac{1}{2}}(\pi\imath j)}{j^{m+\alpha+1/2}}.$$

In the general case, the term involving ${}_3F_1$ does not simplify. Moreover, the usual tables do not contain many occurrences of this function.

Example A. Take $\alpha = \beta = 0$ and let $z = 2\pi\imath j$. Then

$$W_{n,m}^{(0,0)}(j) = n! j^{m-1} {}_3F_1\left(\begin{matrix} n+1, -n, 1 \\ 1 \end{matrix} \middle| \frac{1}{z}\right) {}_1F_1\left(\begin{matrix} m+1 \\ 2m+2 \end{matrix} \middle| z\right). \tag{5.9}$$

The reduction of the ${}_1F_1$ described above gives

$${}_1F_1\left(\begin{matrix} m+1 \\ 2m+2 \end{matrix} \middle| 2\pi\imath j\right) = \frac{2^{m+1/2}\Gamma(m+3/2)}{(\pi\imath)^{m+1/2}} \times \frac{(-1)^j I_{m+1/2}(\pi\imath j)}{j^{m+1/2}}. \tag{5.10}$$

For $\alpha = 0$, there is the reduction

$$ {}_3F_1\left(\left. \begin{matrix} n+1, -n, 1 \\ 1 \end{matrix} \right| \frac{1}{z} \right) = {}_2F_0\left(\left. \begin{matrix} n+1, -n \\ - \end{matrix} \right| \frac{1}{z} \right) \tag{5.11} $$

and now use [13, p. 614, Formula 7.17.1.4]

$$ {}_2F_0\left(\left. \begin{matrix} n+1, -n \\ - \end{matrix} \right| w \right) = \frac{e^{-1/(2w)}}{\sqrt{-\pi w}} K_{n+1/2}\left(-\frac{1}{2w} \right). \tag{5.12} $$

This needs to be evaluated at $w = 1/2\pi\imath j$. Here $K_{n+\frac{1}{2}}$ is the modified Bessel function of the second kind.

In order to simplify these expressions, the values $I_{m+1/2}(\pi\imath j)$ and $K_{n+1/2}(-\pi\imath j)$, are now expressed in terms of the Bessel J function.

Lemma 5.3. *Let j, n, $m \in \mathbb{N}$, then*

$$ I_{m+\frac{1}{2}}(\pi\imath j) = \frac{1+\imath}{\sqrt{2}} e^{\imath\pi m/2} J_{m+\frac{1}{2}}(\pi j) $$

and

$$ K_{n+\frac{1}{2}}(-\pi\imath j) = (-1)^n \frac{\pi}{2\sqrt{2}} \left[(1+i)e^{\pi\imath n/2} J_{-n-\frac{1}{2}}(\pi j) - (1-i)e^{-\pi\imath n/2} J_{n+\frac{1}{2}}(\pi j) \right]. $$

Proof. This follows directly from the power series representation of J_ν, I_ν and the identity

$$ K_\nu(z) = \frac{\pi}{2} \frac{I_\nu(z) - I_\nu(z)}{\sin \pi\nu}, \quad \nu \notin \mathbb{Z} \tag{5.13} $$

that appears in [6, formula 8.485]. □

Corollary 5.4. *For $n \neq m$ and $n \equiv m \bmod 2$*

$$ \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}}^{\infty} \frac{1}{j} I_{m+\frac{1}{2}}(\pi\imath j) K_{n+\frac{1}{2}}(-\pi\imath j) = 0. \tag{5.14} $$

Proof. The term $W_{n,m}^{(0,0)}(j)$ in (5.9) is given by

$$ W_{n,m}^{(0,0)}(j) = \frac{C_{n,m}}{j} I_{m+\frac{1}{2}}(\pi\imath j) K_{n+\frac{1}{2}}(-\pi\imath j) \tag{5.15} $$

with a constant $C_{n,m}$ independent of j. The result now follows from (5.5). □

Using the values given in Lemma 5.3 the identity (5.14) reduces to

$$ \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}}^{\infty} \frac{1}{j} J_{m+\frac{1}{2}}(\pi j) \left[(1+\imath)e^{\pi\imath n/2} J_{-n-\frac{1}{2}}(\pi j) - (1-\imath)e^{-\pi\imath n/2} J_{n+\frac{1}{2}}(\pi j) \right] = 0. $$

Now split the sum over $j > 0$ and $j < 0$ and use $J_\alpha(-x) = (-1)^\alpha J_\alpha(x)$ and the fact that $n \equiv m \bmod 2$ to obtain

$$\sum_{j=1}^{\infty} \frac{J_{m+\frac{1}{2}}(\pi j)}{j} \left[(1+\imath)e^{\pi \imath n/2} J_{-n-\frac{1}{2}}(\pi j) - (1-\imath)e^{-\pi \imath n/2} J_{n+\frac{1}{2}}(\pi j) \right] =$$
$$\sum_{j=1}^{\infty} \frac{J_{m+\frac{1}{2}}(\pi j)}{j} \left[(1+\imath)e^{\pi \imath n/2} J_{-n-\frac{1}{2}}(\pi j) + (1-\imath)e^{-\pi \imath n/2} J_{n+\frac{1}{2}}(\pi j) \right].$$

The terms containing $J_{-n-\frac{1}{2}}(\pi j)$ cancel and the remaining ones lead to the following identity. It is the limiting case of entry 5.7.24.4 in [12].

Corollary 5.5. *Let $n \neq m$ and $n \equiv m \bmod 2$. Then*

$$\sum_{j=1}^{\infty} \frac{1}{j} J_{m+\frac{1}{2}}(\pi j) J_{n+\frac{1}{2}}(\pi j) = 0. \tag{5.16}$$

Example B. The elusive function ${}_3F_1$ appears in the next special case of the Fourier transform of the Jacobi polynomials, for $\alpha = \beta = -\frac{1}{2}$, in terms of the polynomials

$$\phi_n(t) = 2(t-1)^n {}_3F_1 \left(\left. \begin{matrix} -n, n, 1 \\ \frac{1}{2} \end{matrix} \right| \frac{1}{4(1-t)} \right), \tag{5.17}$$

named *Ménage polynomials* in [10]. These polynomials extend the classical combinatorial problem of counting the number of ways in which n married couples can sit at a circular table so that no wife sits next to her husband. These numbers are given by Touchard [15] with the so-called *Ménage numbers*

$$M_n = 2n! \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!, \quad n \geq 1. \tag{5.18}$$

The Ménage polynomials are also given by

$$\phi_n(t) = 2n! \sum_{k=0}^{n} (t-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!, \quad t \in \mathbb{R},\ n \geq 1. \tag{5.19}$$

Observe that $M_n = \phi_n(0)$. The result in Corollary 5.2 now gives the identity

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j^{n+1}} \phi_n\left(1 - \tfrac{1}{2}\pi \imath j\right) I_m(\pi \imath j) = 0, \tag{5.20}$$

for $n \equiv m \bmod 2$.

6. An operator point of view

The previous results may also be obtained via a different approach outlined here. To obtain the finite Fourier transform of a polynomial start with

$$\int_{-1}^{1} x^k e^{\imath \lambda x}\, dx = (-\imath D)^k (2 \text{ sinc } \lambda) \tag{6.1}$$

where the *sinc* function is

$$\text{sinc } \lambda = \frac{\sin \lambda}{\lambda} \tag{6.2}$$

and $D = \frac{d}{d\lambda}$. The action is extended by linearity to obtain

$$\widehat{P}(\lambda) = P(-\imath D)(2 \text{ sinc } \lambda). \tag{6.3}$$

For instance, for the Chebyshev polynomial

$$U_n(x) = \sum_{k=0}^{n} (-2)^k \binom{n+k+1}{n-k} (1-x)^k \tag{6.4}$$

leads to

$$\begin{aligned} \widehat{U_n}(\lambda) &= \sum_{k=0}^{n} (-2)^k \binom{n+k+1}{n-k} (1+\imath D)^k (2 \text{ sinc } \lambda) \\ &= U_n(-\imath D)(2 \text{ sinc } \lambda). \end{aligned} \tag{6.5}$$

It is elementary to check that

$$\left(\frac{d}{d\lambda} \right)^n \text{ sinc } \lambda = A_n(\lambda) \sin \lambda + B_n(\lambda) \cos \lambda \tag{6.6}$$

where A_n, B_n are polynomials in $1/\lambda$ that satisfy the recurrences

$$\begin{aligned} A_{n+1}(\lambda) &= A_n'(\lambda) - B_n(\lambda) \\ B_{n+1}(\lambda) &= A_n(\lambda) + B_n'(\lambda), \end{aligned}$$

with initial values $A_0(\lambda) = 1/\lambda$ and $B_0(\lambda) = 0$. An explicit expression for these polynomials can be obtained from

$$\left(\frac{d}{d\lambda} \right)^n \text{ sinc } \lambda = \sum_{j=0}^{n} \frac{n!}{(n-j)!} \frac{\sin(\lambda + (n+j)\frac{\pi}{2})}{\lambda^{j+1}}. \tag{6.7}$$

It follows from here that

$$\begin{aligned} A_n(\lambda) &= (-1)^n \frac{n!}{2\lambda^{n+1}} \left(E_n(\imath\lambda) + E_n(-\imath\lambda) \right) \\ B_n(\lambda) &= (-1)^n \frac{\imath n!}{2\lambda^{n+1}} \left(E_n(\imath\lambda) - E_n(-\imath\lambda) \right). \end{aligned} \tag{6.8}$$

The use of this method is illustrated with the evaluation of the finite Fourier transform of Legendre polynomials:

$$\begin{aligned}
\int_{-1}^{1} P_n(x)e^{\imath\lambda x}\,dx &= P_n(-\imath D)(2\,\text{sinc }\lambda) \\
&= 2^{n+1}\sum_{k=0}^{n}\binom{n}{k}\binom{\frac{1}{2}(n+k-1)}{n}(-\imath)^k D^k(\text{ sinc }\lambda) \\
&= 2^{n+1}\sum_{k=0}^{n}\binom{n}{k}\binom{\frac{1}{2}(n+k-1)}{n}\left[A_k(\lambda)\sin\lambda + B_k(\lambda)\cos\lambda\right],
\end{aligned}$$

and (6.8) then gives the second formula in Theorem 2.1.

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