CHARACTER ANALOGUES OF THEOREMS OF RAMANUJAN, KOSHLIAKOV AND GUINAND

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Abstract. We derive analogues of theorems of Ramanujan, Koshliakov, and Guinand involving primitive characters and modified Bessel functions. As particular examples, transformation formulas involving the Legendre symbol and sums-of-divisors functions are established.

Key Words: Character analogues, Koshliakov's formula, Guinand's formula, Ramanujan's Lost Notebook, modified Bessel functions, divisor functions.

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1. Introduction

The study of character analogues perhaps begins with the introduction of Dirichlet L-functions by P.G.L. Dirichlet in the 1830's. In this paper, our starting point is a result (Theorem 2.1) arising from a character analogue of a theorem of G.N. Watson [13] established by the first author in [3] using periodic or character analogues of the Poisson summation formula [2, 3, 7]. We establish character analogues of recent results involving the modified Bessel function $K_\nu(z)$ proved by B.C. Berndt, Y. Lee, and J. Sohn [6] found in Ramanujan's Lost Notebook [12]. These include formulas of N.S. Koshliakov [10] and A.P. Guinand [8]. Koshliakov's formula can be considered an analogue of the familiar transformation formula for the classical theta function [5, p. 122].

Theorem 1.1 (Koshliakov's Formula). *If γ denotes Euler's constant, $d(n)$ denotes the number of divisors of n, $K_\nu(z)$ denotes the modified Bessel function of order ν and $a>0$, then*

$$\gamma-\log\left(\frac{4\pi}{a}\right)+4\sum_{n=1}^{\infty}d(n)K_0(2\pi an)$$
$$=\frac{1}{a}\left(\gamma-\log(4\pi a)+4\sum_{n=1}^{\infty}d(n)K_0\left(\frac{2\pi n}{a}\right)\right). \quad (1.1)$$

Guinand's formula is associated with the Fourier expansions of Eisenstein series and Epstein zeta functions [6].

Theorem 1.2 (Guinand's Formula). *Let $\sigma_k(n)=\sum_{d|n}d^k$, and let $\zeta(s)$ denote the Riemann zeta function. If α and β are positive numbers such that $\alpha\beta=\pi^2$, and if s*

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is any complex number, then

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)$$
$$=\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\}+\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\}. \quad (1.2)$$

In this paper, we prove analogues (Theorems 3.1, 4.1) of Guinand's formula for primitive characters. We further establish character analogues (Theorems 3.3, 4.4) of Koshliakov's formula. Several corollaries and special cases are also given. We could have generalized our results to arbitrary even and odd periodic sequences; but with the restriction that the sequences be completely multiplicative, such sequences must be Dirichlet characters [1, p. 145, Exercise 17(b)].

2. Preliminary Results

We use the well-known fact [9, p. 978, formula 8.469, no. 3]

$$K_{1/2}(z)=\sqrt{\frac{\pi}{2z}}e^{-z}. \quad (2.1)$$

We require the simple asymptotic formula [14, p. 202]

$$K_{\nu}(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z}, \qquad z\to\infty,$$

to ensure the convergence of series and integrals and to also justify the interchange of integration and summation several times in the sequel. We need several integrals of Bessel functions beginning with [9, p. 705, formula 6.544, no. 8]

$$\int_0^{\infty}K_{\nu}\left(\frac{a}{x}\right)K_{\nu}(bx)\frac{dx}{x^2}=\frac{\pi}{a}K_{2\nu}(2\sqrt{ab}), \qquad \text{Re } a>0, \text{Re } b>0. \quad (2.2)$$

Also,

$$\int_0^{\infty}K_0(a/x)K_0(yx)dx=\frac{\pi}{y}K_0(2\sqrt{ay}) \quad (2.3)$$

and

$$\int_0^{\infty}x^{\nu-1}e^{-\beta/x-\gamma x}dx=2\left(\beta/\gamma\right)^{\nu/2}K_{\nu}(2\sqrt{\beta\gamma}), \quad (2.4)$$

for $s\in\mathbb{C}$, Re $\beta>0$ and Re $\gamma>0$. We need the related pair [9, p. 697, formula 6.521, no. 3]

$$\int_0^{\infty}xK_{\nu}(ax)K_{\nu}(bx)dx=\frac{\pi(ab)^{-\nu}(a^{2\nu}-b^{2\nu})}{2\sin(\pi\nu)(a^2-b^2)}, \qquad |\text{Re } \nu|<1, \ \text{Re } (a+b)>0, \quad (2.5)$$

and

$$\int_0^{\infty}xK_0(ax)K_0(bx)dx=\frac{\log(a/b)}{a^2-b^2}, \qquad a,b>0, \quad (2.6)$$

which can be obtained by letting $v \to 0$ in (2.5). Also, we need the evaluation [9, p. 708, formula 6.561, no. 16], for Re $a > 0$ and $\mathrm{Re}(\mu + 1 \pm \nu) > 0$,

$$\int_0^\infty x^\mu K_\nu(ax)dx = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right). \tag{2.7}$$

We now state a theorem of Berndt specialized to characters [3, p. 171], which generalizes a theorem of Watson [13], and which will be used in Theorems 3.1 and 4.1 below. First, let

$$G(\chi) := \sum_{n=1}^{k} \chi(n)e^{2\pi in/k}, \tag{2.8}$$

where χ is a primitive character with period k. We make use of the fact [1, p. 168] that $|G(\chi)|^2 = k$.

Theorem 2.1. *Let $x > 0$. If χ is even with period k and* Re $\nu > 0$*, then*

$$\sum_{n=1}^\infty \chi(n)n^\nu K_\nu(2\pi nx/k) = \frac{\pi^{\frac{1}{2}}}{2xG(\overline{\chi})}\left(\frac{kx}{\pi}\right)^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)\sum_{n=1}^\infty \overline{\chi}(n)(n^2+x^2)^{-\nu-\frac{1}{2}}; \tag{2.9}$$

if χ is odd with period k and Re $\nu > -1$*, then*

$$\sum_{n=1}^\infty \chi(n)n^{\nu+1} K_\nu(2\pi nx/k) = \frac{i\pi^{\frac{1}{2}}}{2x^2G(\overline{\chi})}\left(\frac{kx}{\pi}\right)^{\nu+2}\Gamma\left(\nu+\frac{3}{2}\right)\sum_{n=1}^\infty \overline{\chi}(n)n(n^2+x^2)^{-\nu-\frac{3}{2}}. \tag{2.10}$$

Lastly, we state Frullani's integral theorem [11, p. 612, Equation (1)].

Theorem 2.2. *Let $f(x)$ be a Lebesgue integrable function over any interval $0 < A \le x \le B < \infty$. Assuming that the limits exist, write $f(0) = \lim_{x\to 0^+} f(x)$ and $f(\infty) = \lim_{x\to\infty} f(x)$. Then, for $a, b > 0$,*

$$\int_0^\infty \frac{f(at)-f(bt)}{t}dt = (f(\infty)-f(0))\log\left(\frac{a}{b}\right). \tag{2.11}$$

3. Character analogues of theorems of Ramanujan, Koshliakov and Guinand for even primitive characters

Throughout this section, $\chi(n)$ denotes an even primitive character of modulus k.

Theorem 3.1. *If α and β are positive numbers such that $\alpha\beta = \pi^2$ and s is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^\infty \chi(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2}K_{s/2}(2n\alpha/k) = \frac{k\sqrt{\beta}}{G(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2}K_{s/2}(2n\beta/k). \tag{3.1}$$

Proof. Invoking (2.9) in the third equality below, we find that

$$\begin{aligned}
&\sqrt{\alpha}\sum_{n=1}^{\infty}\chi(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2}K_{s/2}(2n\alpha/k)\\
&=\sqrt{\alpha}\sum_{n=1}^{\infty}\chi(n)\sum_{d|n}d^{-s}\left(\frac{n}{k}\right)^{s/2}K_{s/2}(2n\alpha/k)\\
&=\sqrt{\alpha}\sum_{d=1}^{\infty}\chi(d)\sum_{m=1}^{\infty}\chi(m)\left(\frac{m}{dk}\right)^{s/2}K_{s/2}(2dm\alpha/k)\\
&=\sqrt{\alpha}k^{-s/2}\sum_{d=1}^{\infty}\chi(d)d^{-s/2}\left(\frac{\pi^{3/2}}{2d\alpha}\left(\frac{kd\alpha}{\pi^2}\right)^{s/2+1}\Gamma\left(\frac{s+1}{2}\right)\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{G(\overline{\chi})(n^2+(d\alpha/\pi)^2)^{(s+1)/2}}\right)\\
&=\frac{k\sqrt{\pi}}{2G(\overline{\chi})}\alpha^{(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{\chi(d)\overline{\chi}(n)}{(n^2\pi^2+d^2\alpha^2)^{(s+1)/2}}. \qquad (3.2)
\end{aligned}$$

By symmetry,

$$\begin{aligned}
&\sqrt{\beta}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2}K_{s/2}(2n\beta/k)\\
&=\frac{\sqrt{\pi}}{2}\beta^{(s+1)/2}G(\overline{\chi})\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{\overline{\chi}(d)\chi(n)}{(n^2\pi^2+d^2\beta^2)^{(s+1)/2}}\\
&=\frac{\sqrt{\pi}}{2}\beta^{-(s+1)/2}G(\overline{\chi})\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{\overline{\chi}(d)\chi(n)}{(n^2\pi^2/\beta^2+d^2)^{(s+1)/2}}\\
&=\frac{\sqrt{\pi}}{2}\alpha^{(s+1)/2}G(\overline{\chi})\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\chi(d)}{(d^2\alpha^2+\pi^2n^2)^{(s+1)/2}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\chi(d)}{(d^2\alpha^2+\pi^2n^2)^{(s+1)/2}}\\
&\qquad=\frac{2\sqrt{\beta}}{\sqrt{\pi}\alpha^{(s+1)/2}G(\overline{\chi})\Gamma((s+1)/2)}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2}K_{s/2}(2n\beta/k). \qquad (3.3)
\end{aligned}$$

Substituting (3.3) in (3.2), we deduce (3.1) to complete the proof. □

For example, let $\chi(n)=\left(\frac{n}{p}\right)$, the Legendre symbol modulo p, where p is a prime such that $p\equiv 1\,(\mathrm{mod}\,4)$. Then from a well-known theorem of Gauss [4], we know that for any prime $p\equiv 1\,(\mathrm{mod}\,4)$, $G(\chi)=\sqrt{p}$, where $\chi(n)=\left(\frac{n}{p}\right)$. Thus, substituting $\left(\frac{n}{p}\right)$

for $\chi(n)$ in (3.1) and simplifying, we find that

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{-s}(n)\left(\frac{n}{p}\right)^{s/2}K_{s/2}(2n\alpha/p)$$
$$=\sqrt{\beta}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{-s}(n)\left(\frac{n}{p}\right)^{s/2}K_{s/2}(2n\beta/p). \tag{3.4}$$

Now let $s=1$ in (3.4). Then using (2.1), we deduce the following corollary.

Corollary 3.2. *If α and β are positive numbers, $\alpha\beta=\pi^2$ and $\left(\frac{n}{p}\right)$ is the Legendre symbol, where p is a prime with $p\equiv 1\,(\operatorname{mod} 4)$, then*

$$\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{-1}(n)e^{-2n\alpha/p}=\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{-1}(n)e^{-2n\beta/p}. \tag{3.5}$$

Next we state and prove a character analogue of Koshliakov's formula.

Theorem 3.3. *If α and β are positive numbers, $\alpha\beta=\pi^2$ and s is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\chi(n)d(n)K_0(2n\alpha/k)=\frac{k\sqrt{\beta}}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)d(n)K_0(2n\beta/k). \tag{3.6}$$

Proof. Setting $s=0$ in (3.1) and noting that $\sigma_0(n)=d(n)$, the number of divisors of n, we arrive at (3.6). □

Letting $\chi(n)=\left(\frac{n}{p}\right)$ in (3.6), we find that

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)d(n)K_0(2n\alpha/p)=\sqrt{\beta}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)d(n)K_0(2n\beta/p). \tag{3.7}$$

Next we give analogues of theorems on page 254 of Ramanujan's Lost Notebook [6, pp. 30–34], [12].

Theorem 3.4. *If $a>0$, then*

$$\int_0^{\infty}\left(\sum_{m=1}^{\infty}\frac{\overline{\chi}(m)}{G(\overline{\chi})}e^{-2\pi mx}\right)\left(\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{G(\overline{\chi})}e^{-2\pi na/x}\right)\frac{dx}{x}$$
$$=\frac{1}{G(\overline{\chi})^2}\int_0^{\infty}\left(\frac{\sum_{s=0}^{k-1}\overline{\chi}(s)e^{-2\pi sx}}{1-e^{-2\pi kx}}\right)\left(\frac{\sum_{v=0}^{k-1}\overline{\chi}(v)e^{-2\pi av/x}}{1-e^{-2\pi ak/x}}\right)\frac{dx}{x}$$
$$=\frac{2}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)d(n)K_0(4\pi\sqrt{an})$$
$$=\frac{ak^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\chi(n)d(n)\log(n/ak^2)}{n^2-a^2k^4}. \tag{3.8}$$

Proof. Writing $m = kr + s$, $0 \le r < \infty$, $0 \le s \le k-1$, and $n = ku + v$, $0 \le u < \infty$, $0 \le v \le k-1$, we find that, since χ has period k,

$$
\begin{aligned}
&\int_0^\infty \left(\sum_{m=1}^\infty \frac{\overline{\chi}(m)}{G(\overline{\chi})} e^{-2\pi m x} \right) \left(\sum_{n=1}^\infty \frac{\overline{\chi}(n)}{G(\overline{\chi})} e^{-2\pi n a/x} \right) \frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2} \int_0^\infty \left(\sum_{r=0}^\infty \sum_{s=0}^{k-1} \overline{\chi}(kr+s) e^{-2(kr+s)\pi x} \right) \left(\sum_{u=0}^\infty \sum_{v=0}^{k-1} \overline{\chi}(ku+v) e^{-2(ku+v)\pi a/x} \right) \frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2} \int_0^\infty \left(\sum_{r=0}^\infty e^{-2\pi k r x} \sum_{s=0}^{k-1} \overline{\chi}(s) e^{-2\pi s x} \right) \left(\sum_{u=0}^\infty e^{-2\pi a k u/x} \sum_{v=0}^{k-1} \overline{\chi}(v) e^{-2\pi a v/x} \right) \frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2} \int_0^\infty \left(\frac{\sum_{s=0}^{k-1} \overline{\chi}(s) e^{-2\pi s x}}{1 - e^{-2\pi k x}} \right) \left(\frac{\sum_{v=0}^{k-1} \overline{\chi}(v) e^{-2\pi a v/x}}{1 - e^{-2\pi a k/x}} \right) \frac{dx}{x},
\end{aligned}
\tag{3.9}
$$

which proves the first equality.

Interchanging the order of summation and integration by absolute convergence, we find that

$$
\begin{aligned}
&\int_0^\infty \left(\sum_{m=1}^\infty \frac{\overline{\chi}(m)}{G(\overline{\chi})} e^{-2\pi m x} \right) \left(\sum_{n=1}^\infty \frac{\overline{\chi}(n)}{G(\overline{\chi})} e^{-2\pi n a/x} \right) \frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2} \sum_{m=1}^\infty \sum_{n=1}^\infty \overline{\chi}(mn) \int_0^\infty e^{-2\pi(mx + an/x)} \frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2} \sum_{m=1}^\infty \sum_{n=1}^\infty \overline{\chi}(mn) \int_0^\infty e^{-2\pi(u + amn/u)} \frac{du}{u} \\
&= \frac{2}{G(\overline{\chi})^2} \sum_{m=1}^\infty \sum_{n=1}^\infty \overline{\chi}(mn) K_0(4\pi\sqrt{amn}) \\
&= \frac{2}{G(\overline{\chi})^2} \sum_{n=1}^\infty \overline{\chi}(n) d(n) K_0(4\pi\sqrt{an}),
\end{aligned}
\tag{3.10}
$$

which proves the second equality.

Now letting $\alpha = \pi a$ and $\beta = \pi/a$ for $a > 0$ in (3.6) and simplifying, we deduce that

$$
\sum_{n=1}^\infty \chi(n) d(n) K_0(2n\pi a/k) = \frac{k}{aG(\overline{\chi})^2} \sum_{n=1}^\infty \overline{\chi}(n) d(n) K_0(2n\pi/ak). \tag{3.11}
$$

Multiplying both sides by $aK_0(2\pi ayk)$ and integrating with respect to a from 0 to ∞, we have

$$\int_0^\infty a\sum_{n=1}^\infty \chi(n)d(n)K_0(2n\pi a/k)K_0(2\pi ayk)da$$
$$= \frac{k}{G(\overline{\chi})^2}\int_0^\infty \sum_{n=1}^\infty \overline{\chi}(n)d(n)K_0(2n\pi/ak)K_0(2\pi ayk)da. \tag{3.12}$$

Interchanging the order of summation and integration and using (2.3) and (2.6), we find that

$$\frac{k^2}{4\pi^2}\sum_{n=1}^\infty \frac{\chi(n)d(n)\log(n/yk^2)}{n^2-y^2k^4} = \frac{1}{2yG(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)d(n)K_0(4\pi\sqrt{ny}). \tag{3.13}$$

Multiplying both sides by $4y$ and then letting $y=a$, we find that

$$\frac{ak^2}{\pi^2}\sum_{n=1}^\infty \frac{\chi(n)d(n)\log(n/ak^2)}{n^2-a^2k^4} = \frac{2}{G(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)d(n)K_0(4\pi\sqrt{na}). \tag{3.14}$$

This proves the last equality in Theorem 3.4 and completes the proof. □

Theorem 3.5. *If $a>0$, then*

$$\sum_{n=1}^\infty \chi(n)\sigma_{-1/2}(n)e^{-4\pi\sqrt{an}} = \frac{ak^{7/2}}{\pi G(\overline{\chi})^2}\sum_{n=1}^\infty \frac{\overline{\chi}(n)\sigma_{-1/2}(n)}{(n+ak^2)(\sqrt{n}+k\sqrt{a})}. \tag{3.15}$$

Proof. Let $s=1/2$ in (3.1) and set $\alpha=x$ and $\beta=\pi^2/x$. Then

$$\sqrt{x}\sum_{n=1}^\infty \chi(n)\sigma_{-1/2}(n)\left(\frac{n}{k}\right)^{1/4}K_{1/4}(2nx/k)$$
$$= \frac{k\pi}{\sqrt{x}G(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)\sigma_{-1/2}(n)\left(\frac{n}{k}\right)^{1/4}K_{1/4}(2n\pi^2/xk). \tag{3.16}$$

Now multiplying both sides by $x^{-5/2}K_{1/4}(2a\pi^2k/x)$, integrating with respect to x from 0 to ∞, and interchanging the order of summation and integration by absolute convergence, we find that

$$k^{-1/4}\sum_{n=1}^\infty \chi(n)\sigma_{-1/2}(n)n^{1/4}\int_0^\infty \frac{1}{x^2}K_{1/4}(2nx/k)K_{1/4}(2a\pi^2k/x)dx$$
$$= \frac{k^{3/4}\pi}{G(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)\sigma_{-1/2}(n)n^{1/4}\int_0^\infty \frac{1}{x^3}K_{1/4}(2n\pi^2/xk)K_{1/4}(2a\pi^2k/x)dx. \tag{3.17}$$

Now using (2.1), (2.2) and (2.5) and simplifying, we have

$$\frac{1}{4\sqrt{2}k^{5/4}a^{5/4}\pi}\sum_{n=1}^\infty \chi(n)\sigma_{-1/2}(n)e^{-4\pi\sqrt{an}} = \frac{\sqrt{2}\pi k^{9/4}a^{-1/4}}{8\pi^3G(\overline{\chi})^2}\sum_{n=1}^\infty \frac{\overline{\chi}(n)\sigma_{-1/2}(n)}{(n+ak^2)(\sqrt{n}+k\sqrt{a})}, \tag{3.18}$$

which upon further simplification yields (3.15). □

Finally we give a character analogue of the last theorem on page 254 of Ramanujan's Lost Notebook [12].

Theorem 3.6. *For $a > 0$,*

$$2\sqrt{a}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an}) = \frac{ak^3}{2\pi G(\overline{\chi})^2}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\sigma_{-1}(n)}{(n+ak^2)} = \frac{-ak^2}{2\pi G(\overline{\chi})^2}S(a,k), \tag{3.19}$$

where

$$S(a,k) = \sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^2}\sum_{m=0}^{k-1}\overline{\chi}(m)\sum_{t=1}^{\infty}\frac{(ak^2-mn)}{t(tnk+ak^2-mn)}. \tag{3.20}$$

Proof. Letting $s = 1$ in (3.1) and setting $\alpha = x$ and $\beta = \pi^2/x$, we arrive at

$$\sqrt{x}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\left(\frac{n}{k}\right)^{1/2}K_{1/2}(2nx/k) = \frac{k\pi}{\sqrt{x}G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1}(n)\left(\frac{n}{k}\right)^{1/2}K_{1/2}(2n\pi^2/xk). \tag{3.21}$$

Multiplying both sides of (3.21) by $x^{-5/2}K_{1/2}(2a\pi^2k/x)$ and integrating with respect to x from 0 to ∞, and interchanging the order of summation and integration by absolute convergence, we obtain

$$\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\left(\frac{n}{k}\right)^{1/2}\int_0^{\infty}\frac{1}{x^2}K_{1/2}(2nx/k)K_{1/2}(2a\pi^2k/x)dx = \frac{k\pi}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1}(n)\left(\frac{n}{k}\right)^{1/2}\int_0^{\infty}\frac{1}{x^3}K_{1/2}(2n\pi^2/kx)K_{1/2}(2a\pi^2k/x)dx. \tag{3.22}$$

Making the substitution $x = 1/u$ in the integral on the right-hand side of (3.22) and then using (2.5) for this integral, and (2.2) for the integral on the left side, we find after simplifying that

$$\frac{1}{2a\pi k}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an}) = \frac{k^2}{8\pi^2\sqrt{a}G(\overline{\chi})^2}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\sigma_{-1}(n)}{(n+ak^2)}, \tag{3.23}$$

which after simplification yields the first equality.

Next, to show that the first and third expressions of (3.19) are equal, we use (2.1) on the right-hand side of (3.21). After simplification, (3.21) becomes

$$\frac{2}{\sqrt{k\pi}}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\sqrt{nx}K_{1/2}(2nx/k) = \frac{k}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1}(n)e^{-2n\pi^2/kx}. \tag{3.24}$$

Now multiplying both sides of (3.24) by $x^{-5/2}K_{1/2}(2a\pi^2k/x)$ and integrating with respect to x from 0 to ∞, interchanging the order of summation and integration by absolute convergence, and using (2.2), we find that

$$\begin{aligned}
&\frac{2}{\sqrt{k\pi}}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\sqrt{n}\int_0^{\infty}\frac{1}{x^2}K_{1/2}(2nx/k)K_{1/2}(2a\pi^2k/x)dx\\
&=\frac{1}{ak^{3/2}\pi^{3/2}}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an})\\
&=\int_0^{\infty}\left(\frac{k}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1}(n)e^{-2n\pi^2/kx}\right)x^{-5/2}K_{1/2}(2a\pi^2k/x)dx. \qquad (3.25)
\end{aligned}$$

Since

$$\sum_{m=1}^{\infty}\overline{\chi}(m)e^{-2md\pi^2/kx}=\frac{1}{(1-e^{-2d\pi^2/x})}\sum_{s=0}^{k-1}\overline{\chi}(s)e^{-2ds\pi^2/kx}, \qquad (3.26)$$

with d replaced by n, we deduce that

$$\begin{aligned}
\frac{k}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1}(n)e^{-2n\pi^2/kx}&=k\sum_{d=1}^{\infty}\sum_{m=1}^{\infty}\frac{\overline{\chi}(m)\overline{\chi}(d)}{dG(\overline{\chi})^2}e^{-2md\pi^2/kx}\\
&=k\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{nG(\overline{\chi})}\left(\frac{\sum_{s=0}^{k-1}\overline{\chi}(s)e^{-2ns\pi^2/kx}}{G(\overline{\chi})(1-e^{-2n\pi^2/x})}\right). \qquad (3.27)
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^{\infty}\left(\frac{k}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1}(n)e^{-2n\pi^2/kx}\right)x^{-5/2}K_{1/2}(2a\pi^2k/x)dx\\
&=\frac{\sqrt{k}}{2\sqrt{a\pi}}\int_0^{\infty}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{nG(\overline{\chi})}\left(\frac{\sum_{s=0}^{k-1}\overline{\chi}(s)e^{-2ns\pi^2/kx}}{G(\overline{\chi})(1-e^{-2n\pi^2/x})}\right)e^{-2a\pi^2k/x}\frac{dx}{x^2}\\
&=\frac{\sqrt{k}}{4\sqrt{a}\pi^{5/2}}\int_0^{\infty}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^2G(\overline{\chi})}\left(\frac{\sum_{s=0}^{k-1}\overline{\chi}(s)e^{-us/k}}{G(\overline{\chi})(1-e^{-u})}\right)e^{-aku/n}du. \qquad (3.28)
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\sum_{s=0}^{k-1}\overline{\chi}(s)e^{-us/k}}{G(\overline{\chi})(1-e^{-u})}&=\frac{\sum_{s=0}^{k-1}\overline{\chi}(s)e^{u(k-s)/k}}{G(\overline{\chi})(e^{u}-1)}\\
&=\frac{\sum_{m=1}^{k-1}\overline{\chi}(m)e^{um/k}}{G(\overline{\chi})(e^{u}-1)}-\frac{\sum_{m=0}^{k-1}\overline{\chi}(m)}{uG(\overline{\chi})}\\
&=\sum_{m=0}^{k-1}\frac{\overline{\chi}(m)}{G(\overline{\chi})}\left(\frac{e^{um/k}}{e^{u}-1}-\frac{1}{u}\right), \qquad (3.29)
\end{aligned}$$

for $\sum_{m=0}^{k-1}\overline{\chi}(m)=0$, since χ is non-principal. Thus we need to evaluate

$$\int_0^{\infty}\left(\frac{e^{um/k}}{e^{u}-1}-\frac{1}{u}\right)e^{-aku/n}du, \qquad 0\le m\le k-1. \qquad (3.30)$$

Thus,

$$\int_0^\infty \left(\frac{e^{um/k}}{e^u-1}-\frac{1}{u}\right)e^{-aku/n}du = -\int_0^\infty \left(\frac{e^{-aku/n}}{u}-\frac{e^{-(ak/n-m/k+1)u}}{1-e^{-u}}\right)du$$
$$= -\left(\int_0^\infty \frac{e^{-aku/n}-e^{-(ak/n-m/k+1)u}}{u}du + \int_0^\infty \left(\frac{1}{u}-\frac{1}{1-e^{-u}}\right)e^{-(ak/n-m/k+1)u}du\right). \tag{3.31}$$

The first integral in (3.31) can be evaluated with the help of Theorem 2.2 with $f(u) = e^{-u}$. Since $ak/n > 0$,

$$\int_0^\infty \frac{e^{-aku/n}-e^{-(ak/n-m/k+1)u}}{u}du = \log\left(\frac{ak/n-m/k+1}{ak/n}\right). \tag{3.32}$$

This integral can also be evaluated by differentiation under the integral sign.

To evaluate the second integral, we use the following representation for $\psi(z)$, the logarithmic derivative of the gamma function $\Gamma(z)$ [9, p. 952, formula 8.361, no. 8]:

$$\psi(z) = \log(z) + \int_0^\infty \left(\frac{1}{t}-\frac{1}{1-e^{-t}}\right)e^{-zt}dt, \tag{3.33}$$

for Re $(z) > 0$.

Thus from (3.31), (3.33) and (3.32),

$$\int_0^\infty \left(\frac{e^{um/k}}{e^u-1}-\frac{1}{u}\right)e^{-aku/n}du$$
$$= -\log\left(\frac{ak}{n}-\frac{m}{k}+1\right)+\log\left(\frac{ak}{n}\right)-\psi\left(\frac{ak}{n}-\frac{m}{k}+1\right)+\log\left(\frac{ak}{n}-\frac{m}{k}+1\right)$$
$$= \log\left(\frac{ak}{n}\right)+\gamma-\sum_{t=1}^\infty \frac{(ak/n-m/k)}{t\,(ak/n-m/k+t)}$$
$$= \log\left(\frac{ak}{n}\right)+\gamma-\sum_{t=1}^\infty \frac{(ak^2-mn)}{t(tnk+ak^2-mn)}, \tag{3.34}$$

since

$$\psi(z) = -\gamma+\sum_{n=1}^\infty\left(\frac{1}{n}-\frac{1}{z+n-1}\right) = -\gamma+\sum_{n=1}^\infty \frac{z-1}{n(z+n-1)}. \tag{3.35}$$

Hence by absolute convergence, we find that

$$\begin{aligned}
&\int_0^\infty \left(\frac{k}{G(\overline{\chi})^2} \sum_{n=1}^\infty \overline{\chi}(n)\sigma_{-1}(n)e^{-2n\pi^2/kx} \right) x^{-5/2} K_{1/2}(2a\pi^2 k/x)dx \\
&= \frac{\sqrt{k}}{4\sqrt{a}\pi^{5/2}} \int_0^\infty \sum_{n=1}^\infty \frac{\overline{\chi}(n)}{n^2 G(\overline{\chi})} \left(\sum_{m=0}^{k-1} \frac{\overline{\chi}(m)}{G(\overline{\chi})} \left(\frac{e^{um/k}}{e^u - 1} - \frac{1}{u} \right) \right) e^{-aku/n} du \\
&= \frac{\sqrt{k}}{4\sqrt{a}\pi^{5/2}} \sum_{n=1}^\infty \frac{\overline{\chi}(n)}{n^2 G(\overline{\chi})} \left(\sum_{m=0}^{k-1} \frac{\overline{\chi}(m)}{G(\overline{\chi})} \int_0^\infty \left(\frac{e^{um/k}}{e^u - 1} - \frac{1}{u} \right) e^{-aku/n} du \right) \\
&= \frac{\sqrt{k}}{4\sqrt{a}\pi^{5/2}} \sum_{n=1}^\infty \frac{\overline{\chi}(n)}{n^2 G(\overline{\chi})} \sum_{m=0}^{k-1} \frac{\overline{\chi}(m)}{G(\overline{\chi})} \left(\log\left(\frac{ak}{n} \right) + \gamma - \sum_{t=1}^\infty \frac{(ak^2 - mn)}{t(tnk + ak^2 - mn)} \right) \\
&= \frac{-\sqrt{k}}{4\sqrt{a}\pi^{5/2}} \left(\frac{1}{G(\overline{\chi})^2} \sum_{n=1}^\infty \frac{\overline{\chi}(n)}{n^2} \sum_{m=0}^{k-1} \overline{\chi}(m) \sum_{t=1}^\infty \frac{(ak^2 - mn)}{t(tnk + ak^2 - mn)} \right) \\
&= \frac{-\sqrt{k}}{4\sqrt{a}\pi^{5/2} G(\overline{\chi})^2} S(a,k), \qquad (3.36)
\end{aligned}$$

where $S(a,k)$ is defined in (3.20) and we made use of the fact that $\sum_{m=0}^{k-1} \overline{\chi}(m) = 0$ in the penultimate step.

Thus from (3.25), we obtain

$$\frac{1}{ak^{3/2}\pi^{3/2}} \sum_{n=1}^\infty \chi(n)\sigma_{-1}(n)\sqrt{n} K_1(4\pi\sqrt{an}) = \frac{-\sqrt{k}}{4\sqrt{a}\pi^{5/2} G(\overline{\chi})^2} S(a,k), \qquad (3.37)$$

which upon trivial simplification shows the equality of the first and third expressions in (3.19). □

Corollary 3.7. *If $a > 0$ and $\left(\frac{n}{p}\right)$ is the Legendre symbol, where p is a prime with $p \equiv 1 \pmod 4$, then*

$$2\sqrt{a} \sum_{n=1}^\infty \left(\frac{n}{p} \right) \sigma_{-1}(n)\sqrt{n} K_1(4\pi\sqrt{an}) = \frac{p^2 a}{2\pi} \sum_{n=1}^\infty \frac{\left(\frac{n}{p} \right) \sigma_{-1}(n)}{n + p^2 a}. \qquad (3.38)$$

Proof. Set $\chi(n) = \left(\frac{n}{p}\right)$ in (3.19). Then after simplification, the first equality in (3.19) yields (3.38). □

Remark. The middle expression in Theorem 3.6 converges extremely slowly. For example, let $a = 1$ and $\chi(n) = \left(\frac{n}{5}\right)$. The value of the leftmost expression of Theorem 3.6 is $2.51273028 \cdots \times 10^{-6}$, which is correct up to the decimals listed. However, the middle expression

$$\frac{25}{2\pi} \sum_{n=1}^\infty \frac{\left(\frac{n}{5}\right) \sigma_{-1}(n)}{n + 25}, \qquad (3.39)$$

as we indicated above, converges very slowly. To illustrate, the correct value for 49 million terms of (3.39) is $2.59028733\cdots \times 10^{-6}$, while the correct value of (3.39) for 50 million terms is $2.4317244512\cdots \times 10^{-6}$. Thus, the convergence of (3.39) is slow to stabilize, and the partial sums oscillate about the correct value.

4. Character analogues of theorems of Ramanujan, Koshliakov and Guinand for odd, primitive characters

In this section, we consider the case when $\chi(n)$ is a primitive odd character modulo k.

Theorem 4.1. *If α and β are positive numbers such that $\alpha\beta = \pi^2$ and s is any complex number, then*

$$\alpha\sqrt{\alpha}\sum_{n=1}^{\infty}\chi(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2+1}K_{s/2}(2n\alpha/k)$$
$$= -\frac{k\beta\sqrt{\beta}}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2+1}K_{s/2}(2n\beta/k). \tag{4.1}$$

Proof. Invoking (2.10) in the third equality, we find that

$$\begin{aligned}
&\alpha\sqrt{\alpha}\sum_{n=1}^{\infty}\chi(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2+1}K_{s/2}(2n\alpha/k)\\
&=\alpha\sqrt{\alpha}\sum_{n=1}^{\infty}\chi(n)\sum_{d|n}d^{-s}\left(\frac{n}{k}\right)^{s/2+1}K_{s/2}(2n\alpha/k)\\
&=\alpha\sqrt{\alpha}\sum_{d=1}^{\infty}\chi(d)d^{-s/2+1}\sum_{m=1}^{\infty}\chi(m)\left(\frac{m}{k}\right)^{s/2+1}K_{s/2}(2dm\alpha/k)\\
&=\alpha\sqrt{\alpha}k^{-s/2-1}\sum_{d=1}^{\infty}\chi(d)d^{-s/2+1}\\
&\quad\times\left(\frac{i\sqrt{\pi}}{2(\frac{d\alpha}{\pi})^2}\left(\frac{kd\alpha}{\pi^2}\right)^{s/2+2}\Gamma\left(\frac{s+3}{2}\right)\sum_{m=1}^{\infty}\frac{\overline{\chi}(m)m}{G(\overline{\chi})(m^2+(d\alpha/\pi)^2)^{(s+3)/2}}\right)\\
&=\frac{ik\pi\sqrt{\pi}}{2G(\overline{\chi})}\alpha^{(s+3)/2}\Gamma\left(\frac{s+3}{2}\right)\sum_{d=1}^{\infty}\sum_{m=1}^{\infty}\frac{\chi(d)\overline{\chi}(m)md}{(m^2\pi^2+d^2\alpha^2)^{(s+3)/2}}.
\end{aligned}\tag{4.2}$$

By symmetry,

$$\beta\sqrt{\beta}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2+1}K_{s/2}(2n\beta/k)$$
$$=\frac{ik\pi\sqrt{\pi}}{2G(\chi)}\beta^{(s+3)/2}\Gamma\left(\frac{s+3}{2}\right)\sum_{d=1}^{\infty}\sum_{m=1}^{\infty}\frac{\overline{\chi}(d)\chi(m)md}{(m^2\pi^2+d^2\beta^2)^{(s+3)/2}}$$

$$= \frac{i\pi\sqrt{\pi}}{2G(\chi)}\alpha^{(s+3)/2}\Gamma\left(\frac{s+3}{2}\right)\sum_{d=1}^{\infty}\sum_{m=1}^{\infty}\frac{\overline{\chi}(d)\chi(m)md}{(m^2\alpha^2+d^2\pi^2)^{(s+3)/2}}.$$

Thus,

$$\alpha\sqrt{\alpha}\sum_{n=1}^{\infty}\chi(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2+1}K_{s/2}(2n\alpha/k)+\frac{k\beta\sqrt{\beta}}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-s}(n)\left(\frac{n}{k}\right)^{s/2+1}K_{s/2}(2n\beta/k)$$
$$=\frac{i\alpha^{(s+3)/2}\pi^{3/2}k}{2G(\overline{\chi})}\Gamma\left(\frac{s+3}{2}\right)$$
$$\times\left(\sum_{d=1}^{\infty}\sum_{m=1}^{\infty}\frac{\chi(d)\overline{\chi}(m)md}{(m^2\pi^2+d^2\alpha^2)^{(s+3)/2}}-\sum_{d=1}^{\infty}\sum_{m=1}^{\infty}\frac{\overline{\chi}(d)\chi(m)md}{(m^2\alpha^2+d^2\pi^2)^{(s+3)/2}}\right)$$
$$=0,$$

by interchanging the roles of m and d in the second sum, and so the proof is complete. □

For example, let $\chi(n)=\chi_4(n)$, where

$$\chi_4(n)=\begin{cases}1, & \text{if } n\equiv 1 \pmod 4\\ -1, & \text{if } n\equiv 3 \pmod 4\\ 0, & \text{if } n \text{ is even,}\end{cases} \tag{4.3}$$

which is an odd primitive character modulo 4. Now let $s=1$ in (4.1). Using (2.1), we deduce the following result since $G(\chi_4)=2i$.

Corollary 4.2. *If α and β are positive numbers such that $\alpha\beta=\pi^2$ and if $\chi_4(n)$ is defined by* (4.3)*, then*

$$\alpha\sum_{n=1}^{\infty}\chi_4(n)\sigma_{-1}(n)ne^{-n\alpha/2}=\beta\sum_{n=1}^{\infty}\chi_4(n)\sigma_{-1}(n)ne^{-n\beta/2}. \tag{4.4}$$

Since $G(\left(\frac{n}{p}\right))=i\sqrt{p}$, when p is a prime such that $p\equiv 3 \pmod 4$, we also have the following result.

Corollary 4.3. *If α and β are positive numbers, $\alpha\beta=\pi^2$ and $\left(\frac{n}{p}\right)$ is the Legendre symbol, where p is a prime such that $p\equiv 3 \pmod 4$, then*

$$\alpha\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{-1}(n)ne^{-2n\alpha/p}=\beta\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{-1}(n)ne^{-2n\beta/p}. \tag{4.5}$$

Next we state and prove a character analogue of Koshliakov's formula for odd characters.

Theorem 4.4. *If α and β are positive numbers, $\alpha\beta=\pi^2$ and s is any complex number, then*

$$\alpha\sqrt{\alpha}\sum_{n=1}^{\infty}\chi(n)d(n)nK_0(2n\alpha/k)=-\frac{k\beta\sqrt{\beta}}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)d(n)nK_0(2n\beta/k). \tag{4.6}$$

Proof. Setting $s = 0$ in (4.1) and noting that $\sigma_0(n) = d(n)$, we arrive at (4.6). $\square$

Theorem 4.5. *If $a > 0$, then*

$$\begin{aligned}
&\int_0^\infty \left(\sum_{m=1}^\infty \frac{\overline{\chi}(m)}{G(\overline{\chi})} e^{-2\pi mx}\right)\left(\sum_{n=1}^\infty \frac{\overline{\chi}(n)}{G(\overline{\chi})} e^{-2\pi na/x}\right)\frac{dx}{x}\\
=&\frac{1}{G(\overline{\chi})^2}\int_0^\infty \left(\frac{\sum_{s=0}^{k-1}\overline{\chi}(s)e^{-2\pi sx}}{1-e^{-2\pi kx}}\right)\left(\frac{\sum_{v=0}^{k-1}\overline{\chi}(v)e^{-2\pi av/x}}{1-e^{-2\pi ak/x}}\right)\frac{dx}{x}\\
=&\frac{2}{G(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)d(n)K_0(4\pi\sqrt{an})\\
=&-\frac{1}{\pi^2}\sum_{n=1}^\infty \frac{\chi(n)d(n)n\log(n/ak^2)}{n^2-a^2k^4}.
\end{aligned} \tag{4.7}$$

Proof. Writing $m = kr + s$, $0 \le r < \infty$, $0 \le s \le k-1$, and $n = ku + v$, $0 \le u < \infty$, $0 \le v \le k-1$, we find that, since χ has period k,

$$\begin{aligned}
&\int_0^\infty \left(\sum_{m=1}^\infty \frac{\overline{\chi}(m)}{G(\overline{\chi})} e^{-2\pi mx}\right)\left(\sum_{n=1}^\infty \frac{\overline{\chi}(n)}{G(\overline{\chi})} e^{-2\pi na/x}\right)\frac{dx}{x}\\
=&\frac{1}{G(\overline{\chi})^2}\int_0^\infty \left(\sum_{r=0}^\infty\sum_{s=0}^{k-1}\overline{\chi}(kr+s)e^{-2(kr+s)\pi x}\right)\left(\sum_{u=0}^\infty\sum_{v=0}^{k-1}\overline{\chi}(ku+v)e^{-2(ku+v)\pi a/x}\right)\frac{dx}{x}\\
=&\frac{1}{G(\overline{\chi})^2}\int_0^\infty \left(\sum_{r=0}^\infty e^{-2\pi krx}\sum_{s=0}^{k-1}\overline{\chi}(s)e^{-2\pi sx}\right)\left(\sum_{u=0}^\infty e^{-2\pi aku/x}\sum_{v=0}^{k-1}\overline{\chi}(v)e^{-2\pi av/x}\right)\frac{dx}{x}\\
=&\frac{1}{G(\overline{\chi})^2}\int_0^\infty \left(\frac{\sum_{s=0}^{k-1}\overline{\chi}(s)e^{-2\pi sx}}{1-e^{-2\pi kx}}\right)\left(\frac{\sum_{v=0}^{k-1}\overline{\chi}(v)e^{-2\pi av/x}}{1-e^{-2\pi ak/x}}\right)\frac{dx}{x},
\end{aligned} \tag{4.8}$$

which proves the first equality.

Interchanging the order of summation and integration by absolute convergence, we obtain

$$\begin{aligned}
&\int_0^\infty \left(\sum_{m=1}^\infty \frac{\overline{\chi}(m)}{G(\overline{\chi})} e^{-2\pi mx}\right)\left(\sum_{n=1}^\infty \frac{\overline{\chi}(n)}{G(\overline{\chi})} e^{-2\pi na/x}\right)\frac{dx}{x}\\
=&\frac{1}{G(\overline{\chi})^2}\sum_{m=1}^\infty\sum_{n=1}^\infty \overline{\chi}(mn)\int_0^\infty e^{-2\pi(mx+an/x)}\frac{dx}{x}\\
=&\frac{1}{G(\overline{\chi})^2}\sum_{m=1}^\infty\sum_{n=1}^\infty \overline{\chi}(mn)\int_0^\infty e^{-2\pi(u+amn/u)}\frac{du}{u}\\
=&\frac{2}{G(\overline{\chi})^2}\sum_{m=1}^\infty\sum_{n=1}^\infty \overline{\chi}(mn)K_0(4\pi\sqrt{amn})
\end{aligned}$$

$$= \frac{2}{G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n) d(n) K_0(4\pi\sqrt{an}), \tag{4.9}$$

which proves the second equality.

Now letting $\alpha = \pi a$ and $\beta = \pi/a$ for $a > 0$ in (4.6) and simplifying, we deduce that

$$a \sum_{n=1}^{\infty} \chi(n) d(n) n K_0(2n\pi a/k) = -\frac{k}{a^2 G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n) d(n) n K_0(2n\pi/ak). \tag{4.10}$$

Multiplying both sides by $K_0(2\pi ayk)$ and integrating with respect to a from 0 to ∞, we find that

$$\int_0^{\infty} a \sum_{n=1}^{\infty} \chi(n) d(n) n K_0(2n\pi a/k) K_0(2\pi ayk) da$$
$$= -\frac{k}{G(\overline{\chi})^2} \int_0^{\infty} \frac{1}{a^2} \sum_{n=1}^{\infty} \overline{\chi}(n) d(n) n K_0(2n\pi/ak) K_0(2\pi ayk) da.$$

Interchanging the order of summation and integration and using (2.2) and (2.6), we find that

$$-\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\chi(n) d(n) n \log(n/yk^2)}{n^2 - y^2 k^4} = \frac{2}{G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n) d(n) K_0(4\pi\sqrt{ny}). \tag{4.11}$$

This proves the last equality upon replacing y by a. □

Theorem 4.6. *If $a > 0$, then*

$$-\frac{\sqrt{k}\pi}{G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n) \sigma_{-1/2}(n) e^{-4\pi\sqrt{an}} = \sum_{n=1}^{\infty} \frac{\chi(n)\sigma_{-1/2}(n) n}{(n + ak^2)(\sqrt{n} + k\sqrt{a})}. \tag{4.12}$$

Proof. Let $s = 1/2$ in (4.1) and set $\alpha = x$ and $\beta = \pi^2/x$. Then

$$x\sqrt{x} \sum_{n=1}^{\infty} \chi(n) \sigma_{-1/2}(n) \left(\frac{n}{k}\right)^{5/4} K_{1/4}(2nx/k)$$
$$= -\frac{k\pi^3}{x^{3/2} G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n) \sigma_{-1/2}(n) \left(\frac{n}{k}\right)^{5/4} K_{1/4}(2n\pi^2/xk). \tag{4.13}$$

Now multiplying both sides by $K_{1/4}(2akx)$, integrating with respect to x from 0 to ∞, and interchanging the order of summation and integration by absolute convergence, we find that

$$k^{-5/4} \sum_{n=1}^{\infty} \chi(n) \sigma_{-1/2}(n) n^{5/4} \int_0^{\infty} x K_{1/4}(2nx/k) K_{1/4}(2axk) dx$$
$$= -\frac{k^{-1/4}\pi^3}{G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n) \sigma_{-1/2}(n) n^{5/4} \int_0^{\infty} \frac{1}{x^2} K_{1/4}(2n\pi^2/xk) K_{1/4}(2akx) dx. \tag{4.14}$$

Now using (2.1), (2.2) and (2.5) and simplifying, we conclude that

$$-\frac{\sqrt{k}\pi}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1/2}(n)e^{-4\pi\sqrt{an}}=\sum_{n=1}^{\infty}\frac{\chi(n)\sigma_{-1/2}(n)n}{(n+ak^2)(\sqrt{n}+k\sqrt{a})}.$$

□

Remark. The series on the right-hand side of (4.12) converges very slowly. Due to the presence of the expression $\sigma_{-1/2}(n)$, it is very difficult to compute. Hence, even for a simple case like $a=1$ and $\chi(n)=\left(\frac{n}{3}\right)$, we are unable to numerically verify (4.12). However, if we termwise differentiate (4.12) with respect to a, say three times, then, in particular, the series on the right-hand side converges more rapidly, so that numerical verifications are more feasible.

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References

[1] T.M. Apostol, *Introduction to Analytic Number Theory*, 5th ed., Springer-Verlag, New York, 1998.
[2] B.C. Berndt, *Character analogues of the Poisson and Euler-Maclaurin summation formulas with applications*, J. Number Theory **7** (1975), 413–445.
[3] B.C. Berndt, *Periodic Bernoulli numbers, summmation formulas and applications*, in *Theory and Applications of Special Functions*, R.A. Askey, ed., Academic Press, New York, 1975, pp. 143–189.
[4] B.C. Berndt, R.J. Evans, K.S. Williams, *Gauss and Jacobi Sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley, New York, 1998.
[5] B.C. Berndt and M.I. Knopp, *Hecke's Theory of Modular Forms and Dirichlet Series*, World Scientific Publishing Co., Singapore, 2008.
[6] B.C. Berndt, Y. Lee, and J. Sohn, *The formulas of Koshliakov and Guinand in Ramanujan's lost notebook*, in *Surveys in Number Theory*, Developments in Mathematics, vol. 17, K. Alladi, ed., Springer, New York, 2008, pp. 21–42.
[7] B.C. Berndt and L. Schoenfeld, *Periodic analogues of the Euler-Maclaurin and Poisson summation formulas with applications to number theory*, Acta Arith. **28** (1975), 23–68.
[8] A.P. Guinand, *Some rapidly convergent series for the Riemann ξ-function*, Quart. J. Math. (Oxford) **6** (1955), 156–160.
[9] I.S. Gradshteyn and I.M. Ryzhik, eds., *Table of Integrals, Series, and Products*, 5th ed., Academic Press, San Diego, 1994.
[10] N.S. Koshliakov, *On Voronoï's sum-formula*, Mess. Math. **58** (1929), 30–32.
[11] A. Ostrowski, *On some generalizations of the Cauchy-Frullani integral*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 612–616.
[12] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
[13] G.N. Watson, *Some self-reciprocal functions*, Quart. J. Math. (Oxford) **2** (1931), 298–309.
[14] G.N. Watson, *Theory of Bessel Functions*, 2nd ed., University Press, Cambridge, 1966.

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