RAMANUJAN'S INGENIOUS METHOD FOR GENERATING MODULAR-TYPE TRANSFORMATION FORMULAS

ATUL DIXIT

Dedicated to Srinivasa Ramanujan on the occasion of his 125th birth anniversary

Abstract. We discuss an ingenious method of Ramanujan for generating modular-type transformations of the form $F(\alpha) = F(\beta)$, $\alpha\beta = 1$, having its origins in one of his published papers and in his Lost Notebook. We also review the developments that have occured since then and list examples of these and more general modular-type transformations.

1. introduction

One of Ramanujan's earlier published papers after his arrival in England, namely [41], is devoted to obtaining new expressions for the Riemann zeta function $\zeta(s)$ and $\Xi(t)$, the Riemann Ξ-function (see (1.3) below), in which he studies two definite integrals containing $\Xi(t)$ under the sign of integration and obtains alternative representations for them. In this paper, Ramanujan gives an elegant method for generating modular-type transformation formulas. By a modular-type transformation, we mean a relation governed by the transformation $z \to -1/z$. By a change of variable, one can recast such a relation into an equivalent one governed by the transformation $\alpha \to 1/\alpha$. Ramanujan's method, given in a slightly different but an equivalent form in [41], is as follows.

Suppose we have an integral or a series involving a positive parameter α, denoted by $F(\alpha)$, and further suppose that $F(\alpha)$ can be represented by an integral of the form

$$F(\alpha) = \int_0^\infty h(t) \cos\left(\frac{1}{2} t \log\alpha\right) dt. \tag{1.1}$$

Now suppose that β is another positive parameter such that $\alpha\beta = 1$. Replacing α by β in (1.1) leaves the integral on the right-hand side invariant, and we thus get the modular-type transformation $F(\alpha) = F(\beta)$. Thus, Ramanujan's clever idea was to obtain an integral representation of the type in (1.1) to obtain the transformation $F(\alpha) = F(\beta)$. We can even start with any h (without knowing F) as long as the integral converges, evaluate it by some means, and then use it to obtain a modular-type transformation.

The following beautiful example illustrating this method can be found on page 220 of Ramanujan's Lost Notebook [42].

2010 *Mathematics Subject Classification.* Primary 11M06, Secondary 33C15.

Keywords and phrases. Ramanujan, Riemann zeta function, Riemann Ξ-function, confluent hypergeometric function, theta transformation, Bessel functions.

Theorem 1.1. *Define*

$$\lambda(x) := \psi(x) + \frac{1}{2x} - \log x, \tag{1.2}$$

where

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{m=0}^{\infty}\left(\frac{1}{m+x} - \frac{1}{m+1}\right)$$

is the logarithmic derivative of the Gamma function and γ is Euler's constant. Let the Riemann ξ-function be defined by

$$\xi(s) := (s-1)\pi^{-\frac{1}{2}s}\Gamma(1+\tfrac{1}{2}s)\zeta(s),$$

and let

$$\Xi(t) := \xi(\tfrac{1}{2}+it) \tag{1.3}$$

be the Riemann Ξ-function. If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\begin{aligned}\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\lambda(n\alpha)\right\} &= \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\lambda(n\beta)\right\}\\ &= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt. \end{aligned} \tag{1.4}$$

Ramanujan notes this identity to be 'curious'. Observe that the only place in the integral on the extreme right of (1.4) where α appears in the cosine term. Thus showing that this integral equals either of the two expressions in the first equality in (1.4) is sufficient for deriving the first equality.

Throughout this paper, α and β denote two positive numbers such that $\alpha\beta = 1$. The region of validity of a modular-type transformation can be easily extended from $\mathbb{R}^+$ to some region in the complex plane (including $\mathbb{R}^+$) by analytic continuation, so we only concentrate on positive values of α (and hence of β).

Traditionally, one may obtain a transformation formula of the type $F(\alpha) = F(\beta)$ via the Poisson summation formula, one version [8, p. 39], [43, p. 194] of which tells us that if f is a non-negative, continuous, decreasing, and Riemann integrable function on $[0,\infty)$ and if $g(y) := \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(xy)\,dx$, then

$$\sqrt{\alpha}\left(\frac{1}{2}f(0) + \sum_{n=1}^{\infty} f\left(\sqrt{2\pi}n\alpha\right)\right) = \sqrt{\beta}\left(\frac{1}{2}g(0) + \sum_{n=1}^{\infty} g\left(\sqrt{2\pi}n\beta\right)\right). \tag{1.5}$$

However, as we shall see in this survey, Ramanujan's above method is much more general and applicable in many different situations. For example, one cannot use (1.5) to prove the first equality in (1.4) because the function λ has a singularity at $x = 0$. Thus, one has to appeal to a more general version of Poisson summation. We note here one such version given by A.P. Guinand [20].

Theorem 1.2. *If $f(x)$ can be represented as a Fourier integral, $f(x)$ tends to zero as $x \to \infty$, and $xf'(x)$ belongs to $L^p(0, \infty)$, for some p, $1 < p \leq 2$, then*

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} f(n) - \int_0^N f(t)\,dt\right) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} g(n) - \int_0^N g(t)\,dt\right),$$

where

$$g(x) = 2\int_0^{\to\infty} f(t)\cos(2\pi xt)\,dt.$$

Guinand [21, 22] rediscovered Ramanujan's transformation formula in (1.4) and mentioned in [21] that it can be proved using Theorem 1.2. However, it does not lead one to the integral involving the Riemann Ξ-function in (1.4). Another advantage of Ramanujan's method is that, using Fourier's integral theorem, one can obtain new expressions for $\zeta(s)$ and $\Xi(t)$ (this is discussed in [41]). For more on the history of Theorem 1.1, the reader is referred to [2], [5]. Its proofs can be found in [2, 5, 10, 11].

Only recently we were able to find that N.S. Koshlyakov (also spelled N.S. Koshliakov) [33, Equation (6)], [34, Equations (21), (27)] had also rediscovered Ramanujan's transformation formula, albeit in a different form involving a function $\Omega(x)$ defined by

$$\Omega(x) := 2\sum_{n=1}^{\infty} d(n)\left(K_0\left(4\pi e^{\frac{i\pi}{4}}\sqrt{nx}\right) + K_0\left(4\pi e^{-\frac{i\pi}{4}}\sqrt{nx}\right)\right),$$

where $K_z(x)$ denotes the modified Bessel function of order z. The function $\Omega(x)$ obeys the identity [33, Equation (4)], [34, Equation (22)] [1]

$$\Omega(x) = -\gamma - \frac{1}{2}\log x - \frac{1}{4\pi x} + \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{d(n)}{x^2+n^2}, \tag{1.6}$$

where $d(n)$ denotes the number of divisors of n. This can be easily proved using the following identity [2] from Ramanujan's Lost Notebook [42, p. 254] (see also [6, Equation (4.1)]):

$$2\sum_{n=1}^{\infty} d(n)K_0(4\pi\sqrt{an}) = \frac{a}{\pi^2}\sum_{n=1}^{\infty}\frac{d(n)\log(a/n)}{a^2-n^2} - \frac{\gamma}{2} - \left(\frac{1}{4} + \frac{1}{4\pi^2 a}\right)\log a - \frac{\log 2\pi}{2\pi^2 a}.$$

To see this, replace a by $ae^{\frac{i\pi}{2}}$ and $ae^{-\frac{i\pi}{2}}$ in the above identity, and then add the resulting two identities. Koshlyakov extensively studied the function $\Omega(x)$ [28], [32] and its generalization [29]. We now state Koshlyakov's version of (1.4).

$$\sqrt{\alpha}\int_0^{\infty} e^{-2\pi\alpha x}\left(\Omega(x) + \frac{1}{4\pi x}\right)dx = \sqrt{\beta}\int_0^{\infty} e^{-2\pi\beta x}\left(\Omega(x) + \frac{1}{4\pi x}\right)dx$$
$$= \frac{1}{4\pi}\int_0^{\infty}\left|\zeta\left(\frac{1}{2}+\frac{it}{2}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{\cosh\frac{1}{2}\pi t}\,dt. \tag{1.7}$$

[1] In both these references, the term $-\frac{1}{2}\log x$ is incorrectly given as $\log x$.

[2] G.Voronoi [45, Equations (5), (6)] was the first mathematician to discover this identity though he did not use the notation $K_0(x)$ but instead, an equivalent integral representation.

To see the equivalence of (1.7) and (1.4), observe first that the integrand on the extreme right of (1.7) is $-\frac{1}{2\pi}$ times the integrand on the extreme right of (1.4). This can be seen from the fact [10, Equation 5.3] that

$$\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)=\frac{32\pi^2}{(1+t^2)\cosh\frac{1}{2}\pi t},$$

and by writing $\zeta\left(\frac{1+it}{2}\right)$ in terms of $\Xi\left(\frac{t}{2}\right)$. Next we show that

$$\int_0^\infty e^{-2\pi\alpha x}\left(\Omega(x)+\frac{1}{4\pi x}\right)\,dx=-\frac{1}{2\pi}\left\{\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{n=1}^\infty\lambda(n\alpha)\right\},\tag{1.8}$$

where $\lambda(x)$ is defined in (1.2). We use for this purpose the representation for $\Omega(x)$ given in (1.6). Note that [19, p. 571, Formula 4.331.1], for $\operatorname{Re}\mu>0$,

$$\int_0^\infty e^{-\mu x}\log x\,dx=-\frac{1}{\mu}(\gamma+\log\mu).$$

This gives

$$\int_0^\infty e^{-2\pi\alpha x}\left(-\gamma-\frac{1}{2}\log x\right)\,dx=-\frac{1}{2\pi}\left(\frac{\gamma-\log 2\pi\alpha}{2\alpha}\right).\tag{1.9}$$

Now

$$\int_0^\infty e^{-2\pi\alpha x}\frac{x}{\pi}\sum_{n=1}^\infty\frac{d(n)}{x^2+n^2}\,dx=\sum_{m=1}^\infty\int_0^\infty\frac{x}{\pi}e^{-2\pi\alpha x}\sum_{k=1}^\infty\frac{1}{x^2+m^2k^2}\,dx,\tag{1.10}$$

where the interchange of the order of summation and integration is justified by absolute convergence. Since [9, p. 191] for $t\neq 0$,

$$2t\sum_{k=1}^\infty\frac{1}{t^2+4k^2\pi^2}=\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2},$$

we can rewrite the integral in (1.10) as

$$\begin{aligned}\int_0^\infty e^{-2\pi\alpha x}\frac{x}{\pi}\sum_{n=1}^\infty\frac{d(n)}{x^2+n^2}\,dx&=\sum_{m=1}^\infty\int_0^\infty e^{-2\pi\alpha x}\left(\frac{1}{m(e^{2\pi x/m}-1)}-\frac{1}{2\pi x}+\frac{1}{2m}\right)\,dx\\&=\sum_{m=1}^\infty\int_0^\infty e^{-2\pi\alpha mt}\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}+\frac{1}{2}\right)\,dt,\end{aligned}\tag{1.11}$$

where in the last step, we made a change of variable $x=mt$. Now using the fact [42, p. 219], [2, p. 286] that

$$\int_0^\infty e^{-2\pi nx}\left(\frac{1}{e^{2\pi x}-1}-\frac{1}{2\pi x}\right)\,dx=\frac{1}{2\pi}\left(\log n-\psi(1+n)\right),$$

we obtain

$$\int_0^\infty e^{-2\pi\alpha mt}\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}+\frac{1}{2}\right)\,dt=\frac{1}{2\pi}\left(\log m\alpha-\psi(1+m\alpha)\right)+\frac{1}{4\pi m\alpha}.$$

This, along with (1.11) and the fact that $\psi(x+1) = \psi(x) + 1/x$, gives

$$\int_0^\infty e^{-2\pi\alpha x} \frac{x}{\pi} \sum_{n=1}^\infty \frac{d(n)}{x^2+n^2}\, dx = -\frac{1}{2\pi} \sum_{m=1}^\infty \left(\psi(m\alpha) + \frac{1}{2m\alpha} - \log m\alpha \right). \tag{1.12}$$

Finally, (1.6), (1.9) and (1.12) give (1.8). This shows the equivalence of (1.4) and (1.7).

If we now consider a general integral invariant under $\alpha \to \beta$, of which the integral on the extreme right of (1.4) is a special case, we can obtain in general a modular-type transformation $F(\alpha) = F(\beta)$. This is done in Section 2. The representation of the latter integral in terms of another integral, obtained in [41, Equation (22)], is as follows: For $\alpha > 0$,

$$\begin{aligned}\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right) \Gamma\left(\frac{-1-it}{4}\right) \left(\Xi\left(\frac{1}{2}t\right)\right)^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\, dt \\ = \pi^{3/2} \int_0^\infty \left(\frac{1}{e^{x\sqrt{\alpha}}-1} - \frac{1}{x\sqrt{\alpha}}\right)\left(\frac{1}{e^{x/\sqrt{\alpha}}-1} - \frac{1}{x/\sqrt{\alpha}}\right) dx. \end{aligned} \tag{1.13}$$

Compared to Ramanujan's other papers, the paper [41] has certainly not received the attention that it deserves. However, G.H. Hardy definitely understood the importance of this paper so as to include it in the list of Ramanujan's most important papers in an article that he wrote for the Journal of Indian Mathematical Society [26]. Hardy says,

"It is difficult at present to estimate the importance of these results. The unsolved problems concerning the zeros of $\zeta(s)$ or of $\Xi(t)$ are among the most obscure and difficult in the whole range of Pure Mathematics. Any new formulae involving $\zeta(s)$ or $\Xi(t)$ are of very great interest, because of the possibility that they may throw light on some of these outstanding questions. It is, as I have shown in a short note attached to Mr. Ramanujan's paper, certainly possible to apply his formulae in this direction; but the results which can be deduced from them do not at present go beyond those obtained already by Mr. Littlewood and myself in other ways. But I should not be at all surprised if still more important applications were to be made of Mr. Ramanujan's formulae in the future".

The short note which Hardy alludes to in the quoted paragraph is [25]. In it, he says,

"The properties of this integral resemble those of one which Mr. Littlewood and I have used, in a paper to be published shortly in the Acta Mathematica, to prove that [3]

$$\int_{-T}^{T} \left| \zeta\left(\frac{1}{2} + ti\right) \right|^2 dt \sim \frac{2}{\pi} T \log T \tag{1.14}$$

(as $T \to \infty$)" .

The integral in the above quote of Hardy is the one on the left-hand side of (1.13).

In this survey, we discuss the two integrals involving the Riemann Ξ-function that Ramanujan evaluates in his paper [41]. This will lead us to consider two different types of general integrals which yield modular-type transformation formulas.

[3]Note that in (1.14), there should not be π in the denominator on the right-hand side.

The first of the two integrals discussed in Section 2 gives transformation formulas of the type $F(\alpha)=F(\beta)$. The second one discussed in Section 3 gives those of the type $F(z,\alpha)=F(z,\beta)$. There is a third type, first found in [14], that gives transformations of the type $F(z,\alpha)=F(iz,\beta)$, where $i=\sqrt{-1}$. This is discussed in Section 4. Section 5 is devoted to briefly discussing some extensions of these ideas to Dirichlet L-functions and L-functions associated to primitive Hecke forms. In Section 6, we discuss conjectured modular-type transformation formulas which do not seem to result from integrals involving the Riemann Ξ-function or their other analogues. We conclude the paper with some questions and other possible developments mentioned in Section 7.

2. Integral yielding the transformation $F(\alpha)=F(\beta)$

In [41], Ramanujan evaluates

$$\frac{1}{4\pi^{3/2}}\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)dt \tag{2.1}$$

in terms of another integral. The resulting modular-type transformation that is obtained is

$$\alpha^{-\frac{1}{2}}-4\pi\alpha^{-\frac{3}{2}}\int_0^\infty \frac{xe^{-\frac{\pi x^2}{\alpha^2}}}{e^{2\pi x}-1}\,dx=\beta^{-\frac{1}{2}}-4\pi\beta^{-\frac{3}{2}}\int_0^\infty \frac{xe^{-\frac{\pi x^2}{\beta^2}}}{e^{2\pi x}-1}\,dx. \tag{2.2}$$

Now the general integral that gives rise to transformation formulas of the type $F(\alpha)=F(\beta)$ (for some F) can be easily conceived from the above example. Let $f(t)=\phi(it)\phi(-it)$, where $\phi(t)$ is analytic for real values of t. Then the integral

$$\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)dt \tag{2.3}$$

generates the transformation $F(\alpha)=F(\beta)$, provided it converges. (We take $f\left(\frac{t}{2}\right)$ instead of $f(t)$ just to simplify the appearance of the formulas). We also want the evaluation $F(\alpha)$ of the above integral to be "sufficiently interesting". The idea is very simple. Since f and Ξ are even functions of t (of which, the latter follows from the functional equation of $\zeta(s)$), we can convert the integral in (2.3) into an equivalent complex integral over the line Re $s=1/2$. See for example [44, p. 35]. This integral can then be evaluated by appropriately shifting the line of integration, and then using Cauchy's residue theorem and the theory of Mellin transforms. As will be listed at the end, a number of well-known transformations that fall into the category $F(\alpha)=F(\beta)$ can be derived by evaluating the integral in (2.3) for some specific choices of $f(t)$.

In conjunction with these transformation formulas, the works of Hardy, Koshlyakov, and W.L. Ferrar must be mentioned. At the end of his short note [25], Hardy gave the following example without proof:

$$\begin{aligned}\sqrt{\alpha}\int_0^\infty(\psi(x+1)-\log x)\,e^{-\pi\alpha^2x^2}\,dx&=\sqrt{\beta}\int_0^\infty(\psi(x+1)-\log x)\,e^{-\pi\beta^2x^2}\,dx\\&=2\int_0^\infty\frac{\Xi(t/2)}{1+t^2}\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{\cosh\frac{1}{2}\pi t}\,dt.\end{aligned}$$

This compact form of Hardy's identity was in fact given by Koshlyakov [34, Equations (14), (20)], who worked extensively in this area. Koshlyakov is best known for his formula [28] involving the divisor function $d(n)$ and the modified Bessel function $K_0(x)$ given below:

$$\sqrt{\alpha}\left(\frac{\gamma-\log(4\pi\alpha)}{\alpha}-4\sum_{n=1}^{\infty}d(n)K_0(2\pi n\alpha)\right)=\sqrt{\beta}\left(\frac{\gamma-\log(4\pi\beta)}{\beta}-4\sum_{n=1}^{\infty}d(n)K_0(2\pi n\beta)\right)$$
$$=-\frac{32}{\pi}\int_0^{\infty}\frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2\cos\left(\frac{1}{2}t\log\alpha\right)\,dt}{(1+t^2)^2}.$$

However, a lot of his work has not received the recognition it deserves, partly because some of his articles are difficult to obtain. For example, his long manuscript [35] (written while he was in a Siberian labor camp, and under the patronymic name 'N.S. Sergeev') is not known to many. The history of how this manuscript came into light is very interesting [7]. The last chapter of this manuscript contains beautiful generalizations of some of Ramanujan's formulas in [41], in particular, of (2.1). Unfortunately, the second manuscript that he wrote from the labor camp [7], namely, '*Issledovanie nekotorykh voprosov analyticheskoi teorii rational'nogo i kvadratichnogo polya (A study of some questions in the analytic theory of rational and quadratic fields)*', was lost in transit (see [38]).

Several of Koshlyakov's papers [30, 34, 35] give examples of transformations of the type $F(\alpha)=F(\beta)$. He seems to be the only mathematician who generalized Ramanujan's method from the first three sections in [41] to obtain new transformation formulas and expressions for the Riemann zeta function [34]. For more details on Koshlyakov's work, see [7, 13].

W.L. Ferrar [18] gave a general method for obtaining solutions of $F(\alpha)=F(\beta)$, and mentioned that it will always work as long as we are working with a Dirichlet series having a functional equation, and then explicitly worked out an example involving the modified Bessel function $K_0(x)$ which is given below:

$$\sqrt{\alpha}\left(\frac{-\gamma+\log 16\pi+2\log\alpha}{\alpha}-2\sum_{n=1}^{\infty}\left(e^{\frac{\pi\alpha^2n^2}{2}}K_0\left(\frac{\pi\alpha^2n^2}{2}\right)-\frac{1}{n\alpha}\right)\right)$$
$$=\sqrt{\beta}\left(\frac{-\gamma+\log 16\pi+2\log\beta}{\beta}-2\sum_{n=1}^{\infty}\left(e^{\frac{\pi\beta^2n^2}{2}}K_0\left(\frac{\pi\beta^2n^2}{2}\right)-\frac{1}{n\beta}\right)\right)$$
$$=4\pi^{-\frac{3}{2}}\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt.$$

His method is similar to the one discussed above (in the lines following (2.3)), i.e., it uses the functional equation of $\zeta(s)$ and the residue theorem. See [10] for more on the formulas of Hardy, Koshlyakov and Ferrar.

3. Integral yielding the transformation $F(z,\alpha)=F(z,\beta)$

To the best of our knowledge, the last two sections in [41], namely Sections 4 and 5, have never been carefully examined. As we have seen, analogues of the results in the first three sections from [41] that we have addressed so far have been found by Hardy, Koshlyakov, Ferrar and possibly others. However, the following integral of Ramanujan from [41] has not attracted the attention of researchers. He considered, for n real, the integral

$$\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos nt}{(z+1)^2+t^2}\,dt, \tag{3.1}$$

which contains a new variable z. He then evaluated this integral, in terms of another integral, in various regions of the complex plane, namely, in Re $z>1$, in $-1<$ Re $z<1$ and in $-3<$ Re $z<-1$, and further said 'and so on', indicating that one can always evaluate it in the strip $-(2m+1)<$ Re $z<-(2m-1)$ for $m\geq 0$. The integral on the left-hand side of (1.13) is a special case, when $z=0$, of the above integral. Note that instead of using Theorem 1.2 to obtain the transformation in (1.4), as was done by Guinand, we can simply use (1.13) to show that the integral on the right-hand side is equal to each of the expressions in the first equality of (1.4) (see [5]). But with Ramanujan's evaluations of (3.1), we can do the same for a general z. This leads to modular-type transformations involving the Hurwitz zeta function, out of which the following one[4] generalizes (1.4) (see [11]).

Theorem 3.1. *Let $-1<$ Re $z<1$. Define $\varphi(z,x)$ by*

$$\varphi(z,x):=\zeta(z+1,x)-\frac{x^{-z}}{z}-\frac{1}{2}x^{-z-1},$$

where $\zeta(z,x)$ denotes the Hurwitz zeta function. Then if α and β are any positive numbers such that $\alpha\beta=1$,

$$\alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^\infty\varphi(z,n\alpha)-\frac{\zeta(z+1)}{2\alpha^{z+1}}-\frac{\zeta(z)}{\alpha z}\right)=\beta^{\frac{z+1}{2}}\left(\sum_{n=1}^\infty\varphi(z,n\beta)-\frac{\zeta(z+1)}{2\beta^{z+1}}-\frac{\zeta(z)}{\beta z}\right)$$
$$=\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^\infty\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt. \tag{3.2}$$

It is now clear that the general integral

$$\int_0^\infty f\left(z,\frac{t}{2}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt, \tag{3.3}$$

where $f(z,t)=\phi(z,it)\phi(z,-it)$, ϕ is analytic in z in some complex domain and analytic in t as a function of a real variable, gives transformation of the type $F(z,\alpha)=F(z,\beta)$. We can again convert the above integral into a complex integral over the line Re $s=\frac{1}{2}$, this time using the fact that $\Xi\left(-t\pm\frac{iz}{2}\right)=\Xi\left(t\mp\frac{iz}{2}\right)$ (which follows from the functional equation of $\zeta(s)$), shifting the line of integration appropriately and then

[4]Ramanujan's identity [41, Equation (20)] would give a modular-type transformation between two integrals, equivalent to (3.2), after letting $n=\frac{1}{2}\log\alpha$. Same is the case with Equations (19) and (21) in [41] which would lead to entries 1 and 2 in Table 2 below. However, there are errors in each of the Equations (19) and (21). See [11] for the corrected versions.

using the residue theorem and theory of Mellin transforms. See [12]. Thus, provided the integral converges, we can evaluate (3.3) for specific choices of f.

Besides the one in (3.1), a new integral of the type in (3.3) is treated in [12]. It gives the Ramanujan-Guinand formula [42, p. 253][5], [23], [12], given below, as a consequence of its invariance under $\alpha \to \beta$.

Theorem 3.2. *Let $K_\nu(s)$, γ be defined as before and let $\sigma_k(n) = \sum_{d|n} d^k$. Let $-1 <$ Re $z < 1$. Then if α and β are positive numbers such that $\alpha\beta = 1$, we have*

$$\sqrt{\alpha}\left(\alpha^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) + \alpha^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z) - 4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\alpha\right)\right)$$
$$= \sqrt{\beta}\left(\beta^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) + \beta^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z) - 4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\beta\right)\right)$$
$$= -\frac{32}{\pi}\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt.$$

Some more integrals of this type and new modular-type transformations are obtained in [17]. For example, below is one such integral along with the resulting transformation, which generalizes another formula of Koshlyakov involving $d(n)$ and $K_0(x)$ (see Entry 2 in Table 1).

Theorem 3.3. *Let $-1 <$ Re $z < 1$. Let $\gamma, K_\nu(z)$ be defined as before. Define the function $\Phi(x, z)$ by*

$$\Phi(x,z) := x^{\frac{z}{2}}\Gamma(1+z)\Bigg\{\frac{x^{-z}}{-z}\zeta(1-z) + (2\gamma + \log x + \psi(1+z))\zeta(1+z) + \zeta'(1+z) + \sum_{n=1}^{\infty}\sigma_{-z}(n)\left(\frac{n^z}{(n+x)^{z+1}} - \frac{1}{n}\right)\Bigg\}, \tag{3.4}$$

Then for $\alpha, \beta > 0$ such that $\alpha\beta = 1$,

$$\sqrt{\alpha}\int_0^\infty K_{\frac{z}{2}}(2\pi\alpha x)\Phi(x,z)\,dx = \sqrt{\beta}\int_0^\infty K_{\frac{z}{2}}(2\pi\beta x)\Phi(x,z)\,dx$$
$$= \frac{2^{z-2}}{\pi^2}\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Gamma\left(\frac{z+1+it}{4}\right) \times\Gamma\left(\frac{z+1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt. \tag{3.5}$$

[5]The versions of Ramanujan as well as Guinand give only the modular-type transformation. They do not give the integral involving product of two Ξ-functions.

4. Integral yielding the transformation $F(z,\alpha)=F(iz,\beta)$

The general theta transformation formula [40], [46, p. 475], [4, Theorem 1.1] is given by

$$\sqrt{\alpha}\left(\frac{e^{-\frac{z^2}{8}}}{2\alpha}-e^{\frac{z^2}{8}}\sum_{n=1}^{\infty}e^{-\pi\alpha^2n^2}\cos(\sqrt{\pi}\alpha nz)\right)=\sqrt{\beta}\left(\frac{e^{\frac{z^2}{8}}}{2\beta}-e^{-\frac{z^2}{8}}\sum_{n=1}^{\infty}e^{-\pi\beta^2n^2}\cos(i\sqrt{\pi}\beta nz)\right). \tag{4.1}$$

It is of the form $F(z,\alpha)=F(iz,\beta)$. When $z=0$, it gives the famous theta transformation [39, p. 420], [44, Equation (2.16.2)]

$$\sqrt{\alpha}\left(\frac{1}{2\alpha}-\sum_{n=1}^{\infty}e^{-\pi\alpha^2n^2}\right)=\sqrt{\beta}\left(\frac{1}{2\beta}-\sum_{n=1}^{\infty}e^{-\pi\beta^2n^2}\right).$$

Since the latter can be obtained as a consequence of the integral evaluation [24, Equation (2)], [44, p. 36]

$$\int_0^{\infty}\frac{\Xi(t)}{t^2+\frac{1}{4}}\cos xt\,dt=\frac{\pi}{2}\left(e^{\frac{x}{2}}-2e^{-\frac{x}{2}}\sum_{n=1}^{\infty}e^{-\pi n^2e^{-2x}}\right), \tag{4.2}$$

the natural question that comes to our mind is whether there exists an integral generalizing the one in (4.2), and which gives rise to (4.1). Such an integral was found in [14] and is given by

$$\frac{1}{\pi}\int_0^{\infty}\frac{\Xi(t/2)}{1+t^2}\nabla\left(\alpha,z,\frac{1+it}{2}\right)dt,$$

where

$$\nabla(x,z,s):=\rho(x,z,s)+\rho(x,z,1-s),$$

with

$$\rho(x,z,s):=x^{\frac{1}{2}-s}e^{-\frac{z^2}{8}}{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};\frac{z^2}{4}\right),$$

where ${}_1F_1$ denotes Kummer's confluent hypergeometric function. The general form of an integral that generates the transformation $F(z,\alpha)=F(iz,\beta)$ is

$$\int_0^{\infty}f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)\nabla\left(\alpha,z,\frac{1+it}{2}\right)dt, \tag{4.3}$$

where $f(t)$ is of the form $f(t)=\phi(it)\phi(-it)$ and ϕ is analytic in t as a function of a real variable. The reason this integral is invariant under $\alpha\to\beta$ and $z\to iz$ is because $\nabla\left(\beta,iz,\frac{1+it}{2}\right)=\nabla\left(\alpha,z,\frac{1+it}{2}\right)$, which is a simple consequence of Kummer's first transformation for the hypergeometric function [1, p. 191, Equation (4.1.11)]

$${}_1F_1(a;c;z)=e^z{}_1F_1(c-a;c;-z).$$

Note that $\nabla\left(\alpha,0,\frac{1+it}{2}\right)=2\cos\left(\frac{1}{2}t\log\alpha\right)$. Evaluating this integral with $f\left(\frac{t}{2}\right)=\left((1+t^2)\cosh\frac{1}{2}\pi t\right)^{-1}$ and $f\left(\frac{t}{2}\right)=\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)$, respectively, results in one-variable generalizations of the formulas of Hardy and Ferrar [14] mentioned in Section 2 (see Entries 1 and 2 in Table 3). A companion integral

$$\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)\Delta\left(\alpha, z, \frac{1+it}{2}\right)\,dt, \tag{4.4}$$

where

$$\Delta(x,z,s) := \omega(x,z,s) + \omega(x,z,1-s),$$

with

$$\omega(x,z,s) := x^{\frac{1}{2}-s}e^{-\frac{z^2}{8}}\,{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right),$$

also gives the transformation $F(z,\alpha) = F(iz,\beta)$. The same idea based on representing the general integral in terms of a complex integral followed by residue calculus and Mellin transforms can be used to evaluate both the integrals here. See [14].

A one-variable generalization of (2.1), and hence of (2.2), is given in [15, Theorem 1.6], which uses (4.4) with $f(t) = \Gamma\left(\frac{-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}-\frac{it}{2}\right)$. It gives the following interesting modular-type transformation involving the error function $\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$ and the imaginary error function $\operatorname{erfi}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{t^2}\,dt$.

$$\begin{aligned}
&\sqrt{\alpha}e^{\frac{z^2}{8}}\left(\operatorname{erf}\left(\frac{z}{2}\right) - 4\int_0^\infty \frac{e^{-\pi\alpha^2x^2}\sin(\sqrt{\pi}\alpha xz)}{e^{2\pi x}-1}\,dx\right)\\
&= \sqrt{\beta}e^{\frac{-z^2}{8}}\left(\operatorname{erfi}\left(\frac{z}{2}\right) - 4\int_0^\infty \frac{e^{-\pi\beta^2x^2}\sinh(\sqrt{\pi}\beta xz)}{e^{2\pi x}-1}\,dx\right)\\
&= \frac{z}{8\pi^2}\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\Delta\left(\alpha, z, \frac{1+it}{2}\right)\,dt.
\end{aligned} \tag{4.5}$$

However, another generalization of (2.1) derived in [14, Theorem 1.5] fails to give a modular-type transformation, as it is not invariant under $\alpha \to \beta$ and $z \to iz$. In view of Hardy's comment [25] that (2.1) has properties similar to those of one that he used to show that there are infinitely many zeros of $\zeta(s)$ on the critical line, it may be interesting to see what information its generalizations, namely (4.3) with $f(t) = \Gamma\left(\frac{-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}-\frac{it}{2}\right)$ and the one on the extreme right of (4.5), can provide.

5. Extensions to Dirichlet L-functions and L-functions associated with primitive Hecke forms

In all of the transformations we have discussed so far, we focused on $\zeta(s)$, $\Xi(t)$ and the functional equation of $\zeta(s)$. However, we can proceed similarly with Dirichlet L-functions and L-functions associated with primitive Hecke forms. In the case of primitive Dirichlet characters, the natural analogue of (3.3) is

$$\int_0^\infty f\left(z, \frac{t}{2}\right)\Xi\left(\frac{t+iz}{2}, \overline{\chi}\right)\Xi\left(\frac{t-iz}{2}, \chi\right)\cos\left(\tfrac{1}{2}t\log\alpha\right)\,dt,$$

where

$$f(z,t)=\frac{\phi(z,it)\phi(z,-it)+\phi(-z,it)\phi(-z,-it)}{2},$$

ϕ is analytic in t as a function of a real variable and analytic in z in some complex domain, and $\Xi(t,\chi)$ is the Ξ-function associated to a primitive Dirichlet characters defined as follows. Let

$$b=\begin{cases}0, & \chi(-1)=1,\\ 1, & \chi(-1)=-1.\end{cases}$$

Then

$$\Xi(t,\chi):=\xi\left(\frac{1}{2}+it,\chi\right),$$

where

$$\xi(s,\chi):=\left(\frac{\pi}{q}\right)^{-(s+b)/2}\Gamma\left(\frac{s+b}{2}\right)L(s,\chi).$$

Here $L(s,\chi)$ is the Dirichlet L-function associated to primitive Dirichlet character χ. Note that simultaneously changing z to $-z$ and χ to $\overline{\chi}$ leaves the integral invariant. It is also invariant if we replace α by β. So it generates transformations of the type

$$F(z,\alpha,\chi)=F(-z,\beta,\overline{\chi})=F(-z,\alpha,\overline{\chi})=F(z,\beta,\chi).$$

The character analogues of (3.2), and of the Ramanujan-Guinand formula are obtained in [13]. The generalizations of some of the formulas of the type $F(\alpha)=F(\beta)$ to rational and number fields have been found by Koshlyakov [36, Equation (30.4)], [37, Equation (34.1)].

6. Conjectured modular-type transformation formulas

While in India, Ramanujan found an interesting identity, a modular-type transformation, involving infinite series of the Möbius function $\mu(n)$ [3, p. 470]. Hardy and Littlewood found a correct version of this identity in [27], actually a conjecture, rephrased below in an equivalent form.

Assume that the series $\sum_\rho \left(\Gamma\left(\frac{1-\rho}{2}\right)/\zeta'(\rho)\right)a^\rho$ *converges, where* ρ *runs through the non-trivial zeros of* $\zeta(s)$ *and* a *denotes a positive real number, and that the non-trivial zeros of* $\zeta(s)$ *are simple. Then*

$$\begin{aligned}\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\pi\alpha^2/n^2}&-\frac{1}{4\sqrt{\pi\alpha}}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}\pi^{\frac{\rho}{2}}\alpha^{\rho}\\ =\sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\pi\beta^2/n^2}&-\frac{1}{4\sqrt{\pi\beta}}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}\pi^{\frac{\rho}{2}}\beta^{\rho}.\end{aligned}\tag{6.1}$$

This conjectured identity is of the type $F(\alpha)=F(\beta)$. An integral generating the above transformation, similar to (2.3), does not exist, for if it did, it would have to necessarily contain $\Xi(t)$ in the denominator (owing to the Dirichlet series associated to $\mu(n)$ being $\sum_{n=1}^{\infty}\mu(n)n^{-s}=1/\zeta(s)$ for Re $s>1$), which is not possible since the integral is then divergent. Recently, a generalization of (6.1) of the form $F(z,\alpha) = F(iz,\beta)$ was obtained in [14]. Further, character analogues of this generalization [15] as well as a generalization for L-functions associated to primitive Hecke forms [16] were recently obtained.

7. Questions and future possible developments

The integral in (2.3) contains one Riemann Ξ-function where as the one in (3.3) contains a product of two Riemann Ξ-functions. Regarding the special case of the latter when $z = 0$ and $f(t) = \Gamma\left(\frac{-1}{4} + \frac{it}{2}\right)\Gamma\left(\frac{-1}{4} - \frac{it}{2}\right)$, we have seen Hardy's remark in (1.14) about it being possibly useful in proving an asymptotic formula for the second moment of $\zeta\left(\frac{1}{2} + it\right)$. It might be interesting to then consider an integral consisting of a product of more than two Riemann Ξ-functions, possibly involving more variables (other than t and z) in order to study higher moments, in the special case when the variables are all 0. Plus, one could possibly get more general modular-type transformations, and via Fourier's integral theorem, new expressions for $\Xi(t)$, and hence for $\zeta(s)$.

We now conclude this paper with 3 tables listing some further modular-type transformations of the form $F(\alpha) = F(\beta)$, $F(z,\alpha) = F(z,\beta)$ and $F(z,\alpha) = F(iz,\beta)$. In each table, only one of the two expressions in a modular-type transformation is given along with the specific f in the general form of the integral generating this transformation. See (2.3), (3.3), (4.3) and (4.4). Apart from the definitions

$$X(x) := \frac{\pi^2}{6} + \gamma^2 - 2\gamma_1 + \sum_{n=1}^{\infty} d(n)\left(\frac{1}{x+n} - \frac{1}{n}\right),$$

$$\Lambda(z,x) := \zeta(z+1,x) - \frac{x^{-z}}{z} - \frac{1}{2}x^{-z-1} - \frac{(z+1)x^{-z-2}}{12},$$

the notations employed are either previously given or are standard enough so as to not have them stated here again.

Acknowledgements

The author is funded in part by the grant NSF-DMS 1112656 of Professor Victor H. Moll of Tulane University and sincerely thanks him for the support. The author also thanks Bruce C. Berndt for encouraging him to write this survey, and Arindam Roy for his help with the tables.

Table 1. Formulas of the type $F(\alpha) = F(\beta) = \int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)\, dt$.

No.	$f\left(\frac{t}{2}\right)$	$F(\alpha)$	Mathematician
1.	$\frac{1}{64\pi^5}\left\lvert\Gamma^2\left(\frac{-1+it}{4}\right)\right\rvert^2 \times\Xi\left(\frac{t}{2}\right)\cosh\frac{1}{2}\pi t$	$\sqrt{\alpha^3}\int_0^\infty xJ_0(2\pi\alpha x)\left(\Omega(x) + \frac{1}{4\pi x}\right)\, dx$	Koshlyakov [31][1]
2.	$\frac{8}{(1+t^2)^2\cosh\frac{1}{2}\pi t}\Xi\left(\frac{t}{2}\right)$	$\sqrt{\alpha}\int_0^\infty K_0(2\pi\alpha x) \times\left(X(x) + 2\gamma\log x + \frac{1}{2}\log^2 x\right)\, dx$	Koshlyakov [34]

[1]The transformation formula in this identity is correct. See [32] for a proof. However, the integral involving the Riemann Ξ-function appears to be incorrect.

TABLE 2. Formulas of the type
$F(z,\alpha)=F(z,\beta)=\int_0^{\infty} f\left(z,\frac{t}{2}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt.$

No.	$f\left(z,\frac{t}{2}\right)$	$F(z,\alpha)$	Valid for	Mathematician
1.	$\frac{8(4\pi)^{\frac{z-3}{2}}((z+1)^2+t^2)^{-1}}{\Gamma(z+1)} \times\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)$	$\alpha^{-\frac{(z+1)}{2}}\sum_{k=1}^{\infty}\zeta\left(z+1,1+\frac{k}{\alpha}\right)$	$\mathrm{Re}\, z>1$	Ramanujan [41], Dixit [11]
2.	$\frac{8(4\pi)^{\frac{z-3}{2}}((z+1)^2+t^2)^{-1}}{\Gamma(z+1)} \times\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)$	$\alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\Lambda(z,n\alpha) -\frac{\zeta(z)}{\alpha z}+\frac{\zeta(z+1)}{2}+\frac{(z+1)\zeta(z+2)}{12\alpha^{z+2}}\right)$	$-3<\mathrm{Re}\, z<-1$	Ramanujan [41], Dixit [12]

TABLE 3. Formulas of the type
$F(z,\alpha)=F(iz,\beta)=\int_0^{\infty} f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)g\left(\alpha,z,\frac{1+it}{2}\right)\,dt; g=\nabla$ or Δ.

No.	$f\left(\frac{t}{2}\right)$	g	$F(z,\alpha)$	Mathematician
1.	$\frac{1}{(1+t^2)\cosh\frac{1}{2}\pi t}$	∇	$\sqrt{\alpha}e^{\frac{z^2}{8}}\int_0^{\infty}(\psi(x+1)-\log x) \times e^{-\pi\alpha^2x^2}\cos\left(\sqrt{\pi}\alpha xz\right)\,dx$	Dixit [14]
2.	$\frac{-1}{2\sqrt{\pi}(1+t^2)}\left\lvert\Gamma\left(\frac{1+it}{4}\right)\right\rvert^2$	∇	$\sqrt{\alpha}e^{\frac{z^2}{8}}\int_0^{\infty}e^{-\frac{\alpha^2t^2}{4\pi}}\cos\left(\frac{\alpha tz}{2\sqrt{\pi}}\right) \times\left(\sum_{n=1}^{\infty}K_0(nt)-\frac{\pi}{2t}\right)\,dt$	Dixit [14]

REFERENCES

[1] G.E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, 71, Cambridge University Press, Cambridge, 1999.

[2] G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook*, Part IV, Springer, New York, 2013.

[3] B.C. Berndt, *Ramanujan's Notebooks, Part V*, Springer-Verlag, New York, 1998.

[4] B.C. Berndt, C. Gugg, S. Kongsiriwong and J. Thiel, *A proof of the general theta transformation formula*, in *Ramanujan Rediscovered: Proceedings of a Conference on Elliptic Functions, Partitions, and q-Series in memory of K. Venkatachaliengar: Bangalore,* $1-5$ *June,* 2009, pp. 53–62.

[5] B.C. Berndt and A. Dixit, *A transformation formula involving the Gamma and Riemann zeta functions in Ramanujan's Lost Notebook*, *The legacy of Alladi Ramakrishnan in the mathematical sciences*, K. Alladi, J. Klauder, C. R. Rao, Eds, Springer, New York, 2010, pp. 199–210.

[6] B.C. Berndt, Y. Lee, and J. Sohn, *The formulas of Koshliakov and Guinand in Ramanujan's lost notebook*, Surveys in Number Theory, Series: Developments in Mathematics, vol. 17, K. Alladi, ed., Springer, New York, 2008, pp. 21–42.

[7] N.N. Bogolyubov, L.D. Faddeev, A.Yu. Ishlinskii, V.N. Koshlyakov and Yu.A. Mitropol'skii, *Nikolai Sergeevich Koshlyakov (on the centenary of his birth)*, Uspekhi Mat. Nauk **45** (1990), No. 4, 173–176; English transl. in Russian Math. Surveys **45** (1990), No. 4, 197–202.

[8] J.M. Borwein and P.B. Borwein, *Pi and the AGM*, A Study in Analytic Number Theory and Computational Complexity, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley-Interscience, John Wiley & Sons Inc., New York, 1987.

[9] J.B. Conway, *Functions of One Complex Variable*, 2nd ed., Springer, New York, 1978.

[10] A. Dixit, *Series transformations and integrals involving the Riemann Ξ-function*, J. Math. Anal. Appl. **368** (2010), 358–373.
[11] A. Dixit, *Analogues of a transformation formula of Ramanujan*, Int. J. Number Theory **7**, No. 5 (2011), 1151-1172.
[12] A. Dixit, *Transformation formulas associated with integrals involving the Riemann Ξ-function*, Monatsh. Math. **164**, No. 2 (2011), 133–156.
[13] A. Dixit, *Character analogues of Ramanujan-type integrals involving the Riemann Ξ-function*, Pacific J. Math. **255**, No. 2 (2012), 317–348.
[14] A. Dixit, *Analogues of the general theta transformation formula*, Proc. Roy. Soc. Edinburgh, Sect. A, **143** (2013), 371–399.
[15] A. Dixit, A. Roy and A. Zaharescu, *Analogues of the general theta transformation formula, II*, submitted for publication.
[16] A. Dixit, A. Roy and A. Zaharescu, *Ramanujan-Hardy-Littlewood-Riesz phenomena for Hecke forms*, submitted for publication.
[17] A. Dixit and V.H. Moll, *Self-reciprocal functions, powers of the Riemann zeta function and modular-type transformations*, in preparation.
[18] W. L. Ferrar, *Some solutions of the equation $F(t) = F(t^{-1})$*, J. London Math. Soc. **11** (1936), 99–103.
[19] I. S. Gradshteyn and I. M. Ryzhik, eds., *Table of Integrals, Series, and Products*, 7th ed., Academic Press, San Diego, 2007.
[20] A.P. Guinand, *On Poisson's summation formula*, Ann. Math. (2) **42** (1941), 591–603.
[21] A.P. Guinand, *Some formulae for the Riemann zeta-function*, J. London Math. Soc. **22** (1947), 14–18.
[22] A.P. Guinand, *A note on the logarithmic derivative of the Gamma function*, Edinburgh Math. Notes **38** (1952), 1–4.
[23] A.P. Guinand, *Some rapidly convergent series for the Riemann ξ-function*, Quart. J. Math. (Oxford) **6** (1955), 156–160.
[24] G.H. Hardy, *Sur les zéros de la fonction $\zeta(s)$ de Riemann*, Comp. Rend. Acad. Sci. **158**, 1012–1014.
[25] G.H. Hardy, *Note by Mr. G.H. Hardy on the preceding paper*, Quart. J. Math. **46** (1915), 260–261.
[26] G.H. Hardy, *Mr. S. Ramanujan's mathematical work in England*, J. Indian Math. Soc. **9** (1917), 30–45; (also a Report to the University of Madras, 1916, privately printed).
[27] G.H. Hardy and J.E. Littlewood, *Contributions to the Theory of the Riemann Zeta-Function and the Theory of the Distribution of Primes*, Acta Math., **41** (1916), 119–196.
[28] N.S. Koshliakov, *On Voronoi's sum-formula*, Mess. Math. **58** (1929), 30–32.
[29] N.S. Koshliakov, *Über eine Verallgemeinerung der Ramanujyan'schen Identitäten*, Izvestiia Akademii nauk Soiuza Sovetskikh Sotsialisticheskikh Respublik. VII Seriia, Otdelenie matematicheskikh i estestvennykh nauk (1931), 1089–1102.
[30] N.S. Koshlyakov, *On a general summation formula and its applications* (in Russian), Comp. Rend. (Doklady) Acad. Sci. URSS **4** (1934), 187–191.
[31] N.S. Koshliakov, *Some integral representations of the square of Riemann's function $\Xi(t)$*, C. R. (Dokl.) Acad. Sci. URSS **2** (1934), 401–405.
[32] N.S. Koshliakov, *On an extension of some formulae of Ramanujan*, Proc. London Math. Soc., II Ser. **41** (1936), 26–32.
[33] N.S. Koshliakov, *Note on some infinite integrals*, C. R. (Dokl.) Acad. Sci. URSS **2** (1936), 247–250.
[34] N.S. Koshliakov, *On a transformation of definite integrals and its application to the theory of Riemann's function $\zeta(s)$*, Comp. Rend. (Doklady) Acad. Sci. URSS **15** (1937), 3–8.
[35] N.S. Sergeev, *A study of a class of transcendental functions defined by the generalized Riemann equation* (in Russian), Izdat. Akad. Nauk SSSR, Moscow, 1949.
[36] N.S. Koshlyakov, *Investigation of some questions of the analytic theory of a rational and quadratic field*, II (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **18** No. 3, 213–260 (1954).

[37] N.S. Koshlyakov, *Investigation of some questions of the analytic theory of a rational and quadratic field*, III (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **18** No. 4, 307–326 (1954).
[38] http://www.ihst.ru/projects/sohist/repress/academy/koshliakov.htm
[39] S.D. Poisson, *Suite du Mémoire sur les intégrales définies et sur la sommation des séries*, Paris Jour. de l'École Polytechnique, **12**, No. 19 (1823), 404–509.
[40] S.D. Poisson, *Sur le calcul numérique des Intégrales définies*, Mém. de l'Acad. des Sci. **6** (1827), 571–602.
[41] S. Ramanujan, *New expressions for Riemann's functions* $\xi(s)$ *and* $\Xi(t)$, Quart. J. Math. **46** (1915), 253–260.
[42] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
[43] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill International Edition, 1987.
[44] E.C. Titchmarsh, *The Theory of the Riemann Zeta Function*, Clarendon Press, Oxford, 1986.
[45] G. Voronoi, *Sur une fonction transcendante et ses appliations à la sommation de quelques séries*, Annales de l'École Normale, Ser. 3, **21** (1904), 207–267.
[46] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, Cambridge, 1966.

Department of Mathematics, Tulane University, New Orleans, LA 70118, USA
E-mail address: adixit@tulane.edu