# RAMANUJAN'S PUBLISHED PAPERS ON DEFINITE INTEGRALS

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Ramanujan published six papers on properties and evaluations of definite integrals. The numbers prior to the titles below indicate their numbering in Ramanujan's *Collected Papers* [16].

(1) #7. On the integral $\int_0^x \frac{\tan^{-1} t}{t} dt$ [10]
(2) #11. Some definite integrals [11]
(3) #12. Some Definite Integrals Connected with Gauss's Sums [12]
(4) #22. Some Definite Integrals [13]
(5) #23. Some Definite Integrals [14]
(6) #27. A Class of Definite Integrals [15]

## 1. ON THE INTEGRAL $\int_0^x \frac{\tan^{-1} t}{t} dt$

For the integral in the title, Ramanujan derived several integral and series relations [10]. The commentary on pages 337–338 in [16] points out a few inaccuracies in Ramanujan's paper [10], and also gives references to previous work by earlier authors. Ramanujan's paper was published after he arrived in England, but perhaps most, if not all, of its contents were established prior to his arrival. In Section 17 of Chapter 9 in his first notebook [17], [3, pp. 265–267], Ramanujan lists four of his results from [10]. A preliminary version of [10] was published with Ramanujan's lost notebook [18], [2, pp. 361–365].

## 2. SOME DEFINITE INTEGRALS (#11)

In the paper [11], Ramanujan examines several integrals involving finite or infinite products appearing either in the integrands or integral evaluations. Also, Gamma functions appear in the formulas. (Integrals involving $q$-products appear in Ramanujan's voluminous work on $q$-series; these are *not* considered in [11].)

Several results are particularly attractive. For example [11, p. 54, equation (4)], for $0 < a < b - \frac{1}{2}$,

$$\int_0^\infty \left| \frac{\Gamma(a+ix)}{\Gamma(b+ix)} \right|^2 dx = \frac{1}{2}\sqrt{\pi} \frac{\Gamma(a)\Gamma(a+\frac{1}{2})\Gamma(b-a-\frac{1}{2})}{\Gamma(b-\frac{1}{2})\Gamma(b)\Gamma(b-a)}. \tag{2.1}$$

This result can also be found as Entry 22(ii) in Chapter 13 in Ramanujan's second notebook [17]. See also [4, p. 225], where references to related work can be found. In several integrals, hyperbolic trigonometric functions appear in integrands and/or evaluations. For example, if $\alpha$ and $\beta$ are positive and $\alpha\beta = \pi$, then

$$\sqrt{\alpha} \int_0^\infty \frac{e^{-x^2}}{\cosh \alpha x} dx = \sqrt{\beta} \int_0^\infty \frac{e^{-x^2}}{\cosh \beta x} dx, \tag{2.2}$$

which can be found in Ramanujan's first letter to G. H. Hardy [6, p. 27], and also in Section 21 of Chapter 13 in his second notebook [17], [4, p. 225]. Furthermore, (2.2) appears in a manuscript published with the lost notebook [18], [2, p. 368]. The identity (2.2) was also established by Hardy [7, p. 203]. Ramanujan was fond of symmetric formulas such as (2.2) with conditions of the sort $\alpha\beta = \pi$. Four additional relations in the spirit of (2.2) can be found in the aforementioned manuscript [2, p. 368].

Several evaluations are derived using a formal process involving inverting the order of integration in double integrals.

Certain integrals in this paper have connections with integrals arising in the work of several mathematicians. Many theorems in [11] can be found in Ramanujan's second notebook [17]. In particular, (2.1) is part of Entry 22 of Chapter 13 [4, p. 225]. See also [3, pp. 105–107], [4, pp. 223–225], [5, pp. 291, 318–319]. The commentary in Ramanujan's *Collected Papers* [16, pp. 361–362] amplifies our comments.

## 3. Some Definite Integrals Connected with Gauss's Sums

This paper is primarily devoted to integrals with either $\sin x, \cos x, \sinh x$, or $\cosh x$ in their integrands. A partial manuscript, found on pages 190–191 in Ramanujan's lost notebook [18], [2, pp. 367–372], was intended to be Section 4 of a manuscript that is unknown to us. The manuscript has connections with both the present paper [12] and [11], but perhaps more with [11]. The connections of these integrals with the classical Gauss sum defined by

$$\sum_{k=0}^{n-1} e^{2\pi i k^2/n}$$

are somewhat tenuous. The integrands generally contain functions with a quadratic integration variable, for example, $\cos \pi x^2$, and so have a superficial connection with Gauss sums. This was likely a motivating factor when Ramanujan chose a title for his paper. The evaluation [16, p. 63]

$$\int_0^\infty x^2 \frac{\sin \pi x^2}{\cosh \pi x} dx = \frac{1}{8} - \frac{1}{8\sqrt{2}}$$

is a typical example.

The sums in this paper have a more tangible relation with Gauss sums. For example [16, p. 66], if $a/b$ is a positive rational number, let

$$\phi\left(\frac{a}{b}\right) := \frac{1}{4}\sum_{r=1}^{b}(b-2r)\cos\left(\frac{r^2\pi a}{b}\right) - \frac{b}{4a}\sqrt{\frac{b}{a}}\sum_{r=1}^{a}(a-2r)\sin\left(\frac{1}{4}\pi + \frac{r^2\pi b}{a}\right).$$

Ramanujan evaluated $\phi(a/b)$ for seven rational numbers $a/b$ [16, p. 67]. For example,

$$\phi\left(\frac{2}{5}\right) = \frac{8-3\sqrt{5}}{16}.$$

Gauss sums generally obey a reciprocity theorem, and so in continuing the analogy, Ramanujan derived beautiful reciprocity theorems for some of the sums and integrals he investigated. For example, let

$$F(\alpha,\beta) := \sqrt{\alpha}\left\{\frac{1}{2}+\sum_{r=1}^{\infty}\frac{\cos\, r^2\pi\alpha^2}{\cosh\, r\pi\alpha}\right\} - \sqrt{\beta}\sum_{r=1}^{\infty}\frac{\sin\, r^2\pi\beta^2}{\cosh\, r\pi\beta}.$$

Then, for $\alpha,\beta>0$ and $\alpha\beta=1$ [16, p. 64],

$$F(\alpha,\beta)=F(\beta,\alpha)=\sqrt{2\alpha}\left\{\frac{1}{2}+\sum_{n=1}^{\infty}e^{-n^2\pi\alpha}\right\}^2. \tag{3.1}$$

Besides its beauty, identity (3.1) is interesting because, with an appeal to Ramanujan's notation in his theory of theta functions, the sum on the right-hand side is equal to

$$\frac{1}{4}\varphi^2(e^{-\pi\alpha}), \qquad \text{where} \qquad \varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}, \qquad |q|<1.$$

Note that (3.1) implies the standard "modular relation" for $\varphi$, normally given in another notation.

## 4. Some Definite Integrals (#23)

This paper should be read in conjunction with the previous paper, for its content is related to a subset of [14].

Define

$$\phi_w(t) := \int_0^\infty \frac{\cos\,\pi tx}{\cosh\,\pi x}e^{-\pi wx^2}dx \qquad \text{and} \qquad \psi_w(t) := \int_0^\infty \frac{\sin\,\pi tx}{\sinh\,\pi x}e^{-\pi wx^2}dx.$$

In [14], Ramanujan made use of "modular relations" satisfied by $\phi_w(t)$ and $\psi_w(t)$ in order to evaluate them for large classes of values of $t$ and $w$. We give two examples:

$$\int_0^\infty \frac{\sin\,2\pi tx}{\sinh\,\pi x}\cos\,\pi x^2\,dx = \frac{\cosh\,\pi t-\cos\,\pi t^2}{2\sinh\,\pi t}, \tag{4.1}$$
$$\int_0^\infty \frac{\sin\,2\pi tx}{\sinh\,\pi x}\sin\,\pi x^2\,dx = \frac{\sin\,\pi t^2}{2\sinh\,\pi t}.$$

The integral evaluation (4.1) was used by A. K. Mustafy [9] to obtain a new integral representation of the Riemaann zeta function $\zeta(s)$ in terms of an integrand similar to those above.

## 5. A Class of Definite Integrals (#27, #22)

The papers [15] and [12] are perhaps the two most important papers by Ramanujan involving the evaluation of definite integrals in closed form. It should be emphasized that, although we have concentrated on individual examples for illustration in this *Encyclopedia* article, in each of Ramanujan's papers he developed general methods that enabled him to evaluate large classes of integrals. In particular, in this beautiful paper, his methods enable him to treat large classes of definite integrals.

One general integral that can be evaluated [16, pp. 221–222] is:

$$\int_{-\infty}^{\infty} \Gamma(\alpha+x)(\beta-x)e^{inx}dx.$$

Similarly, Ramanujan devised a general approach to evaluating

$$\int_{-\infty}^{\infty} \frac{e^{inx}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+\ell x)\Gamma(\delta-\ell x)}dx,$$

where $n$ and $\ell$ are real numbers, and similar integrals. Again, we forego hypotheses but instead mention only a special case. If $\alpha+\beta+\gamma+\delta=4$, then [15, p. 229]

$$\int_{-\infty}^{\infty} \frac{\cos\{\pi(x+\beta+\gamma)\}}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+2x)\Gamma(\delta-2x)}dx$$
$$=\frac{1}{2\Gamma(\gamma+\delta-1)\Gamma(2\alpha+\delta-2)\Gamma(2\beta+\gamma-2)}.$$

Ramanujan also derived some beautiful and unusual definite integral evaluations involving products of ordinary Bessel functions $J_\nu(x)$. For example, for $\mathrm{Re}(\alpha+\beta) > -1$ [15, p. 225],

$$\int_{-\infty}^{\infty} \frac{J_{\alpha+w}(x)}{x^{\alpha+w}} \frac{J_{\beta-w}(y)}{y^{\beta-w}} dw = \frac{J_{\alpha+\beta}\{\sqrt{(2x^2+2y^2)}\}}{(\frac{1}{2}x^2+\frac{1}{2}y^2)^{(\alpha+\beta)/2}}.$$

Ramanujan concluded his paper with a formula providing the evaluation of a fairly general integral involving the product of four Bessel functions.

It is well-known that Ramanujan was not conversant with the analytic theory of functions of a complex variable. Hardy opined [16, p. xxx], "and he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was." However, in [15] it is clear that Ramanujan knew certain elementary facts about functions of a complex variable, in particular, the care needed in treating branches of "multi-valued" functions. Moreover, on the next-to-last page of Ramanujan's third notebook [17, pp. 391], several integrals from complex analysis are recorded. Next to one of them appear the words, "contour integration." Indeed, this integral can be evaluated by contour integration. So, maybe Ramanujan knew more complex analysis than either Hardy or others have thought.

(Ramanujan's paper [13] is a research announcement for the integrals discussed in [15].)

## 6. Conclusion

We have avoided presenting any of Ramanujan's arguments, but it is hoped that the examples quoted here will inspire many readers to further examine Ramanujan's beautiful ideas in evaluating intriguing integrals. Those desiring more information about Ramanujan's methods in evaluating definite integrals may consult Hardy's book [8, Chapter XI] and R. P. Agarwal's treatise [1]. In particular, see pages 165–171 of [1] for a discussion of Ramanujan's integrals involving $\Gamma(s)$.

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