ELLIPTIC AND RELATED INTEGRALS

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1. Introduction

The complete elliptic integral of the first kind $K(k)$, $0 < k < 1$, is defined by

$$K(k) := \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}} = \frac{\pi}{2}\, {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right),$$

where the second equality arises from expanding the integrand in a power series and integrating termwise. The number k is called the *modulus*. The function ${}_2F_1$ on the right-hand side is an ordinary hypergeometric function defined by

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (n!)} z^n, \qquad |z| < 1,$$

where

$$(a)_n := a(a+1)(a+2)\cdots(a+n-1).$$

If $\pi/2$ is replaced by another number v, $0 < v < \pi/2$, then the integral is called an incomplete elliptic integral. The integral $K(k)$ is prominent in the theory of the Jacobian elliptic functions $\text{sn}(u)$, $\text{cn}(u)$, and $\text{dn}(u)$. However, Ramanujan developed his own theory of elliptic functions, and these three famous functions play a subsidiary role, at best. Nonetheless, $K(k)$ plays a central role in Ramanujan's theories of theta functions, class invariants, singular moduli, and Eisenstein series; its importance cannot be overemphasized. Separate articles in this *Encyclopedia* are devoted to these topics. Here, we concentrate on $K(k)$ itself and related integrals. Any statement about an elliptic integral of the first kind yields a corresponding statement about an ordinary hypergeometric function, and conversely. Here, the focus is on the integral itself, in particular, transformations and values. At the close of this article, references in this *Encyclopedia* to all of these topics are given.

Elliptic integrals appear at scattered places in Ramanujan's work. However, two locations are prominent. A collection of entries from his notebooks [10] are found in [3], and a set of entries from his lost notebook [11] are gathered together in [1]. Complete references follow in the sequel.

2. Chapter 17, Section 7, Second Notebook

A rich source of identities for elliptic integrals is Section 7 of Chapter 17 Ramanujan's second notebook [10], [3, pp. 104–117]. We begin with the famous *addition theorem* for elliptic integrals. Let

$$u := \int_0^{\alpha} \frac{d\varphi}{\sqrt{1-x^2\sin^2\varphi}}, \quad v := \int_0^{\beta} \frac{d\varphi}{\sqrt{1-x^2\sin^2\varphi}}, \quad w := \int_0^{\gamma} \frac{d\varphi}{\sqrt{1-x^2\sin^2\varphi}}.$$

Ramanujan gave four different conditions for α, β, and γ to ensure the validity of the addition theorem [3, p. 107]

$$u + v = w. \tag{2.1}$$

In particular, if [3, Entry 7(viii) (c)],

$$\cot\,\alpha\,\cot\,\beta = \frac{\cos\,\gamma}{\sin\,\alpha\,\sin\,\beta} + \sqrt{1 - x\sin^2\gamma}, \tag{2.2}$$

then (2.1) holds. The condition (2.2) is equivalent to the condition

$$\frac{\mathrm{cn}(u)\,\mathrm{cn}(v)}{\mathrm{sn}(u)\,\mathrm{sn}(v)} = \frac{\mathrm{cn}(u+v)}{\mathrm{sn}(u)\,\mathrm{sn}(v)} + \mathrm{dn}(u+v).$$

The addition theorem (2.1) is classical, but some of Ramanujan's other identities appear to be new.

In [3, pp. 108–109], the two given proofs of the following result are verifications; Ramanujan must have had a more natural proof.

Entry 2.1. *If* $|x| < 1$*, then*

$$\frac{\pi}{2}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 + x\sin\,\varphi}} = \int_0^{\pi/2} \frac{\cos^{-1}(x\,\sin^2\varphi)d\,\varphi}{\sqrt{1 - x^2\sin^4\varphi}}.$$

The following entry is a beautiful theorem, more recondite than the previous two theorems. It is a wonderful illustration of Ramanujan's ingenuity and quest for beauty.

Entry 2.2. *If* $|x| < 1$*, then*

$$\int_0^{\pi/2}\int_0^{\pi/2} \frac{x\sin\,\varphi\,d\theta\,d\varphi}{\sqrt{(1 - x^2\sin^2\varphi)(1 - x^2\sin^2\theta\,\sin^2\varphi)}}$$

$$= \frac{1}{2}\left(\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \frac{1}{2}(1+x)\sin^2\varphi}}\right)^2 - \frac{1}{2}\left(\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \frac{1}{2}(1-x)\sin^2\varphi}}\right)^2.$$

Despite the fact that Ramanujan's second notebook is a revised edition of the first, there are over 200 claims in the first notebook that cannot be located in the second. In particular, on page 172 in the first notebook [10], two remarkable elliptic integral transformations are recorded [5, pp. 403–404]. One of them is given by

Entry 2.3. *Let* $0 < x < 1$*, and assume for* $0 \le \alpha, \beta \le \pi/2$ *that*

$$\frac{1 + \sin\,\beta}{1 - \sin\,\beta} = \frac{1 + \sin\,\alpha}{1 - \sin\,\alpha}\left(\frac{1 + x\sin\,\alpha}{1 - x\sin\,\alpha}\right)^2.$$

Then,

$$(1 + 2x)\int_0^{\alpha} \frac{d\theta}{\sqrt{1 - x^3\frac{2+x}{1+2x}\sin^2\theta}} = \int_0^{\beta} \frac{d\theta}{\sqrt{1 - x\left(\frac{2+x}{1+2x}\right)^3\sin^2\theta}}.$$

For further formulas for elliptic integrals, consult Whittaker and Watson's text [13, Chapter XX], the tables of P. F. Byrd and M. D. Friedman [7], and M. L. Glasser's tables [8].

3. THE LEMNISCATE INTEGRAL

The next two unusual entries are related to elliptic integrals and are found in the unorganized pages of Ramanujan's second notebook [10, pp. 283, 286], [4, p. 255].

Entry 3.1. *Let* $0 \le \theta \le \pi/2$ *and* $0 \le v \le 1$. *Define* μ *to be the constant defined by putting* $v = 1$ *and* $\theta = \pi/2$ *in* (3.1) *below, i.e.,* $\frac{1}{2}\pi\mu = K(1/\sqrt{2})$. *Put*

$$\frac{\theta\mu}{2} = \int_0^v \frac{dt}{\sqrt{1+t^4}} =: G(v). \tag{3.1}$$

Then,

$$2\tan^{-1} v = \theta + \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n\cosh(n\pi)}. \tag{3.2}$$

Despite its unusual character, (3.2) is not too difficult to prove, and follows from the inversion theorem for elliptic integrals.

The integral $G(v)$ has a striking resemblance to the classical lemniscate integral defined next. As above, let $0 \le \theta \le \pi/2$ and $0 \le v \le 1$. Define μ to be the constant defined by putting $v = 1$ and $\theta = \pi/2$ in (3.3) below. Then the lemniscate integral $F(v)$ is defined by

$$\frac{\theta\mu}{\sqrt{2}} = \int_0^v \frac{dt}{\sqrt{1-t^4}} =: F(v) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n v^{4n+1}}{n!(4n+1)}, \tag{3.3}$$

where the right-hand side is a representation for $F(v)$ that arises from expanding the integrand in a binomial series. Ramanujan offers an inversion formula for the lemniscate integral analogous to (3.2). Altogether, Ramanujan states ten inversion formulas, six of them for the lemniscate integral [10, pp. 283, 285, 286]. We offer one of them [4, p. 252]. Proofs for all six are given in [4, 245–260].

Entry 3.2. *Let* θ *and* v *be defined as in* (3.3). *Then,*

$$\begin{aligned}\log\, v + \frac{\pi}{6} - \frac{1}{2}\log 2 + \sum_{n=1}^{\infty} \frac{(\frac{1}{4})_n v^{4n}}{(\frac{3}{4})_n 4n} \\ = \log(\sin\theta) + \frac{\theta^2}{2\pi} - 2\sum_{n=1}^{\infty} \frac{\cos(2n\theta)}{n(e^{2\pi n}-1)}.\end{aligned}$$

If

$$v = \frac{\sqrt{2}\,x}{\sqrt{1+x^4}},$$

then

$$F(v) = \int_0^v \frac{dt}{\sqrt{1-t^4}} = \sqrt{2}\int_0^x \frac{dt}{\sqrt{1+t^4}} = \sqrt{2}G(t),$$

which is an important key step in the historically famous problem of doubling the arc length of the lemniscate.

The lemniscate integral was initially studied by James Bernoulli and Count Giulio Fagnáno. Raymond Ayoub [2] wrote a very informative article emphasizing its history and importance. Carl Ludwig Siegel [12] considered the lemniscate integral so important that he began his development of the theory of elliptic functions with a thorough discussion of it.

4. Incomplete Elliptic Integrals and Theta Functions

On pages 51–53 in his lost notebook [10], Ramanujan states several original, surprising, and unusual integral identities involving elliptic integrals and Ramanujan's theta functions, including

$$f(-q) := (1-q)(1-q^2)\cdots(1-q^n)\cdots, \qquad |q|<1, \tag{4.1}$$

which, except for a factor of $q^{1/24}$, is Dedekind's eta-function $\eta(\tau)$, where $q = e^{2\pi i\tau}, \tau \in \mathbb{H}$. Ramanujan's integrals of theta functions are associated with elliptic integrals of orders 5, 10, 14, or 35. In view of orders 14 and 35, it is surprising that none of order 7 are given. These integral identities were first proved by S. Raghavan and S. S. Rangachari [9] using the theory of modular forms, and later by the author, Heng Huat Chan, and Sen-Shan Huang employing ideas with which Ramanujan would have been familiar. Proofs for all of Ramanujan's identities can be found in Andrews' and the first author's book [1, pp. 327–371]. Certain proofs depend upon transformations of elliptic integrals found in Ramanujan's second notebook and discussed above. Differential equations for products or quotients of theta functions are also featured in some proofs. The first of two examples that we give is associated with modular equations of order 5.

Entry 4.1. [1, p. 333], [11, p. 52] *Let* $f(-q)$ *be defined by* (4.1), $\epsilon = (\sqrt{5}+1)/2$, *and*

$$u := u(q) := \frac{q^{1/5}}{1} \, {}_{+} \, \frac{q}{1} \, {}_{+} \, \frac{q^2}{1} \, {}_{+} \, \frac{q^3}{1} \, {}_{+} \cdots, \qquad |q|<1, \tag{4.2}$$

which defines the Rogers–Ramanujan continued fraction. Then,

$$\begin{aligned} 5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt &= \int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}} \\ &= \int_0^{2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right)} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}}. \end{aligned}$$

The next entry, also associated with modular equations of order 5, involves a differential equation involving theta functions.

Lemma 4.2. [1, p. 342], [11, p. 52] *Let*

$$\lambda := \lambda(q) := q\frac{f^6(-q^5)}{f^6(-q)}.$$

Then

$$q\frac{d}{dq}\lambda(q) = \sqrt{q}\, f^2(-q)f^2(-q^5)\sqrt{125\lambda^3+22\lambda^2+\lambda}.$$

Entry 4.3. *Let* u *be defined by* (4.2). *Then there exists a constant* C *such that*

$$u^5+u^{-5} = \frac{1}{2\sqrt{q}}\frac{f^3(-q)}{f^3(-q^5)}\left(C + \int_q^1 \frac{f^8(-t)}{f^4(-t^5)}\frac{dt}{t^{3/2}} + 125\int_0^q \frac{f^8(-t^5)}{f^4(-t)}\sqrt{t}\,dt\right).$$

(The constant C can be determined, but it is different from that claimed by Ramanujan [1, pp. 346–347].)

The next entry is connected with modular equations of order 14.

Entry 4.4. [11, pp. 51–52], [1, p. 359] *Let*

$$v := v(q) := q\left(\frac{f(-q)f(-q^{14})}{f(-q^2)f(-q^7)}\right)^4.$$

Put

$$c = \frac{\sqrt{13+16\sqrt{2}}}{7}.$$

Then

$$\int_0^q f(-t)f(-t^2)f(-t^7)f(-t^{14})dt = \frac{1}{\sqrt{8\sqrt{2}}}\int_{\cos^{-1}\left(c\frac{1+v}{1-v}\right)}^{\cos^{-1} c} \frac{d\varphi}{\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\varphi}}.$$

Our concluding example of Ramanujan's exquisite formulas is an identity linking elliptic integrals and modular equations of degree 35.

Entry 4.5. [11, p. 53], [1, p. 364] *If*

$$v := v(q) := q\frac{f(-q)f(-q^{35})}{f(-q^5)f(-q^7)},$$

then

$$\int_0^q t\, f(-t)f(-t^5)f(-t^7)f(-t^{35})dt = \int_0^v \frac{t\,dt}{\sqrt{(1+t-t^2)(1-5t-9t^3-5t^5-t^6)}}.$$

Related *Encyclopedia* Articles for Further Reading

- Theta Functions
- Class Invariants
- Singular Moduli
- Eisenstein Series, I, II
- The Modular j-Invariant
- Hypergeometric Series

REFERENCES

[1] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook*, Part I, Springer, New York, 2005.

[2] R. Ayoub, *The lemniscate and Fagnáno's contributions to elliptic integrals*, Arch. Hist. Exact Sci. **29** (1984), 131–149.

[3] B. C. Berndt, *Ramanujan's Notebooks*, Part III, Springer–Verlag, New York, 1991.

[4] B. C. Berndt, *Ramanujan's Notebooks*, Part IV, Springer-Verlag, New York, 1994.

[5] B. C. Berndt, *Ramanujan's Notebooks*, Part V, Springer–Verlag, New York, 1998.

[6] B. C. Berndt, H. H. Chan, and S.–S. Huang, *Incomplete elliptic integrals in Ramanujan's lost notebook*, in *q-Series from a Contemporary Perspective*, M. E. H. Ismail and D. Stanton, eds., American Mathematical Society, Providence, RI, 2000, pp. 79–126.

[7] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd. ed., Springer-Verlag, New York, 1971.

[8] M. L. Glasser, *Definite integrals of the complete elliptic integral K*, J. Res. Nat. Bur. Stand. (B), Math. Sci. 80B (1976), 313–323.

[9] S. Raghavan and S. S. Rangachari, *On Ramanujan's elliptic integrals and modular identities* in *Number Theory and Related Topics*, Oxford University Press, Bombay, 1989, pp. 119–149.

[10] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957; second ed., 2012.
[11] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
[12] C. L. Siegel, *Topics in Complex Function Theory*, Vol. 1, Wiley, New York, 1969.
[13] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1966.

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