BEAUTIFUL INTEGRALS

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1. INTRODUCTION

Reading his letters to Hardy, his published papers, and his notebooks, one easily concludes that Ramanujan loved to evaluate definite integrals. In particular, sources for several of Ramanujan's evaluations of integrals are his Quarterly Reports [1], the unorganized pages of his second notebook [10], and his third notebook [2].

Ramanujan wrote three Quarterly Reports for the University of Madras before departing for England. These are thoroughly examined in Berndt's book [1, pp. 295–336]. A separate article in this *Encyclopedia* is devoted to the Quarterly Reports.

In the sequel, references are made to Ramanujan's third notebook, which contains 33 pages [10, 361–393]. The second and third notebooks were published together as Volume 2 of [10], which explains the "large" page numbers below in referencing the third notebook.

2. FUNCTIONAL EQUATIONS

Ramanujan was fond of special integrals or sequences thereof that satisfy functional equations, recurrence relations, or possess a parameter leading to specific elegant evaluations.

Question 783, posed by Ramanujan in the *Journal of the Indian Mathematical Society* [9] and found in Ramanujan's third notebook [10, p. 373], is especially elegant.

Theorem 2.1. ***(Question 783)*** *For $n \geq 0$, put $v = u^n - u^{n-1}$, and define*

$$\varphi(n) := \int_0^1 \frac{\log u}{v} dv. \tag{2.1}$$

Then,

$$\varphi(0) = \frac{\pi^2}{6}, \qquad \varphi(1) = \frac{\pi^2}{12}, \qquad \text{and} \qquad \varphi(2) = \frac{\pi^2}{15}.$$

Furthermore, if $n > 0$,

$$\varphi(n) + \varphi\left(\frac{1}{n}\right) = \frac{\pi^2}{6}. \tag{2.2}$$

Inspired by Theorem 2.1, Berndt and R. J. Evans [4] established the following generalization.

Theorem 2.2. *Let g be a strictly increasing, differentiable function on $[0, \infty)$ with $g(0) = 1$ and $g(\infty) = \infty$. For $n > 0$ and $t \geq 0$, define*

$$v(t) := \frac{g^n(t)}{g(1/t)}.$$

Suppose that

$$\varphi(n) := \int_0^1 \log\, g(t)\frac{dv}{v}$$

converges. Then

$$\varphi(n) + \varphi\left(\frac{1}{n}\right) = 2\varphi(1).$$

Note that if $g(t) = 1 + t$, then Theorem 2.2 reduces to Theorem 2.1. See [4] and [2, pp. 326–329] for more details.

The integral (2.1) is reminiscent of the dilogarithm, defined by

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-w)}{w}dw, \qquad z \in \mathbb{C}, \tag{2.3}$$

where the principal branch of $\log(1-w)$ is chosen. The dilogarithm was studied by Ramanujan in Chapter 9 of his first notebook [10, Entries 5–7], [1, pp. 246–249] and in his third notebook [10, pp. 365], [2, pp. 322–326].

3. Ramanujan's Generalization of Frullani's Theorem

One of the most intriguing theorems in the evaluation of integrals is Frullani's Theorem. (We set $F(\infty) := \lim_{x\to\infty} F(x)$ for any continuous function.)

Theorem 3.1. ***(Frullani)*** *If f is a continuous function on $[0,\infty)$ such that $f(\infty)$ exists, then, for any pair $a, b > 0$,*

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}dx = \{f(0) - f(\infty)\}\log\left(\frac{b}{a}\right). \tag{3.1}$$

If $f(\infty)$ does not exist, but $f(x)/x$ is integrable over $[c,\infty)$ for $c > 0$, then (3.1) *still holds, but with $f(\infty)$ replaced by 0.*

In his second Quarterly Report, Ramanujan offers a beautiful generalization of Frullani's Theorem. A slightly less general version is provided by Ramanujan in the unorganized pages of his second notebook [10, pp. 332, 334], [1, p. 316]. We do not give the hypotheses that are needed for $u(x)$ and $v(x)$; see [1, pp. 299, 313] for these requirements. Set

$$f(x) - f(\infty) = \sum_{k=0}^\infty \frac{u(k)(-x)^k}{k!} \qquad \text{and} \qquad g(x) - g(\infty) = \sum_{k=0}^\infty \frac{v(k)(-x)^k}{k!}.$$

Ramanujan also assumes that the limit below can be taken under the integral sign.

Theorem 3.2. *Let $u(x)$ and $v(x)$ be defined as above, and assume that f and g are continuous functions on $[0,\infty)$. Also assume that $f(0) = g(0)$ and $f(\infty) = g(\infty)$. Therefore, if $a, b > 0$,*

$$\lim_{n\to 0}\int_0^\infty x^{n-1}\left\{f(ax) - g(bx)\right\}dx = \{f(0)-f(\infty)\}\left\{\log\left(\frac{b}{a}\right) + \frac{d}{ds}\left(\log\left(\frac{v(s)}{u(s)}\right)\right)_{s=0}\right\}.$$

Ramanuan's proof depends on his *Master Theorem*, for which a separate article in this *Encyclopedia* has been written. Jim Hafner significantly extended Theorem 3.2. For a proof of Hafner's theorem and several illustrative examples of Theorem 3.2 given by Ramanujan, see [1, pp. 313–321].

Readers may wish to consult an excellent historical account of Frullani's Theorem by A. M. Ostrowski [7].

4. Inversion Formulas

At many places in his published papers, notebooks, and Quarterly Reports, Ramanujan evaluates integrals using inversion formulae, such as Fourier transforms. We offer one of his more unusual inversion formulae [10, p. 332], [2, p. 316–318]. (The hypotheses below are due to Berndt [2, p. 316].)

Theorem 4.1. *Let $F(z)$ be analytic in $R := \{z, Re(z) \geq 0\}$, except possibly at $z = 0$. Assume that $F(z) = O(z^{\alpha})$, as $|z| \to \infty$ in R, for some constant $\alpha < 1$. As $z \to 0$ in R, assume that $F(z) = O(z^{\delta-1})$ for some constant $\delta > 0$. For $Re(a) \geq 0$, define*

$$\varphi(a) := \frac{2}{\pi}\int_0^{\infty} e^{-z^2} F(z/a)dz.$$

Then, for $z > 0$,

$$F(z) = \int_0^{\infty} e^{-u^2}\left\{\varphi\left(\frac{ui}{z}\right) + \varphi\left(-\frac{ui}{z}\right)\right\} du.$$

5. Integrals involving the Riemann Ξ-function

Let the Riemann Ξ- and ξ-functions be defined by

$$\Xi(t) := \xi\left(\tfrac{1}{2} + it\right)$$

and

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

where $s = \sigma + it$, with both σ and t real, and where $\Gamma(s)$ and $\zeta(s)$ denote the Euler Gamma and the Riemann zeta functions, respectively.

Ramanujan derived important integral formulas in which the Riemann Ξ-function appears in the integrands. His two key transformations associated with $\Xi(t)$ are [8]

$$\begin{aligned}\frac{1}{4\pi\sqrt{\pi}}&\int_0^{\infty}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\cos(nt)\,dt\\ &= e^{-n} - 4\pi e^{-3n}\int_0^{\infty}\frac{xe^{-\pi x^2 e^{-4n}}}{e^{2\pi x}-1}\,dx \qquad (5.1)\end{aligned}$$

and

$$\begin{aligned}&\int_0^{\infty}\Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\frac{\cos nt}{(s+1)^2+t^2}\,dt\\ &= \frac{1}{8}(4\pi)^{-\frac{1}{2}(s-3)}\int_0^{\infty}x^s\left(\frac{1}{\exp(xe^n)-1}-\frac{1}{xe^n}\right)\left(\frac{1}{\exp(xe^{-n})-1}-\frac{1}{xe^{-n}}\right)dx, \qquad (5.2)\end{aligned}$$

where $n \in \mathbb{R}$ and $-1 < \text{Re}(s) < 1$.

For more details on these two integral transformations and their applications, see [6] in this *Encyclopedia*. These two examples motivated several new transformations involving similar integrals [5].

6. ONE FINAL EXAMPLE

On page 391 in his third notebook [10], [2, p.329], Ramanujan offers the following beautiful relation.

Theorem 6.1. *For $n > 0$,*

$$\int_0^\infty \frac{\cos(nx)}{x^2+1} \log\, x\, dx + \frac{\pi}{2} \int_0^\infty \frac{\sin(nx)}{x^2+1} dx = 0.$$

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