KOSHLIAKOV KERNEL AND IDENTITIES INVOLVING THE RIEMANN ZETA FUNCTION

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Abstract. Some integral identities involving the Riemann zeta function and functions reciprocal in a kernel involving the Bessel functions $J_z(x), Y_z(x)$ and $K_z(x)$ are studied. Interesting special cases of these identities are derived, one of which is connected to a well-known transformation due to Ramanujan, and Guinand.

1. INTRODUCTION

In their long memoir [20, p. 158, Equation (2.516)], Hardy and Littlewood obtain, subject to certain assumptions unproved as of yet (for example, the Riemann Hypothesis), an interesting modular-type transformation involving infinite series of Möbius function as suggested to them by some work of Ramanujan. By a modular-type transformation, we mean a transformation of the form $F(\alpha) = F(\beta)$ for $\alpha\beta =$ constant. On pages $159 - 160$, they also give a generalization of the transformation for any pair of functions reciprocal to each other in the Fourier cosine transform as indicated to them by Ramanujan.

Let $\Xi(t)$ be Riemann's Ξ-function defined by [31, p. 16]

$$\Xi(t) := \xi(\tfrac{1}{2} + it), \tag{1.1}$$

where

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{1.2}$$

is the Riemann ξ-function [31, p. 16]. Here $\Gamma(s)$ is the gamma function [1, p. 255] and $\zeta(s)$ is the Riemann zeta function [31, p. 1].

A natural way to obtain similar such modular-type transformations is by evaluating integrals of the type

$$\int_0^\infty f(t)\Xi(t)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt,$$

where $f(t) = \phi(it)\phi(-it)$ for some analytic function ϕ, since they are invariant under $\alpha \to 1/\alpha$, although the aforementioned transformation involving series of Möbius function is not obtainable this way. Ramanujan studied an interesting integral of this type in [29].

Motivated by Ramanujan's generalization, the authors of the present paper, in [12], studied integrals of the above type but with the cosine function replaced by a general function $Z\left(\frac{1}{2} + it\right)$, which is an even function of t, real for real t, and depends on the functions reciprocal in the Fourier cosine transform. Several integral evaluations

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such as the one connected with the general theta transformation formula [10, Equation (4.1)], and those of Hardy [19, Equation (2)] and Ferrar [10, p. 170] were obtained in [12] as special cases by evaluating these general integrals for specific choices of f.

Ramanujan [29] also studied integrals of the form

$$\int_0^\infty f\left(\frac{t}{2},z\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt,$$

where

$$f(t,z)=\phi(it,z)\phi(-it,z), \tag{1.3}$$

with ϕ being analytic in the complex variable z and in the real variable t. With f being of the form just discussed, in the present paper, we study a generalization of the above integral of the form

$$\int_0^\infty f\left(\frac{t}{2},z\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2},\frac{z}{2}\right)\,dt, \tag{1.4}$$

where the function $Z\left(\frac{1}{2}+it,z\right)$ depends on a pair of functions which are reciprocal to each other in the kernel

$$\cos\left(\frac{\pi z}{2}\right)M_z(4\sqrt{x})-\sin\left(\frac{\pi z}{2}\right)J_z(4\sqrt{x}), \tag{1.5}$$

where

$$M_z(x):=\frac{2}{\pi}K_z(x)-Y_z(x).$$

Here $J_z(x)$ and $Y_z(x)$ are Bessel functions of the first and second kinds respectively, and $K_z(x)$ is the modified Bessel function.

We call this kernel as the *Koshliakov kernel* since Koshliakov [23, Equation 8] was the first mathematician to construct a function self-reciprocal in this kernel, namely, he showed that for real z satisfying $-\frac{1}{2}<z<\frac{1}{2}$,

$$\int_0^\infty K_z(t)\left(\cos(\pi z)M_{2z}(2\sqrt{xt})-\sin(\pi z)J_{2z}(2\sqrt{xt})\right)\,dt=K_z(x). \tag{1.6}$$

(It is easy to see that this formula actually holds for complex z with $-\frac{1}{2}<\mathrm{Re}(z)<\frac{1}{2}$.) Dixon and Ferrar [13, Equation (1)] had previously obtained the special case $z=0$ of the above integral evaluation.

The Koshliakov kernel occurs in a variety of places in number theory, for example, in the extended form of the Voronoï summation formula [4, Theorems 6.1, 6.3]. In view of Koshliakov's aforementioned work, the integral transform

$$\int_0^\infty g(t,z)\left(\cos(\pi z)M_{2z}(2\sqrt{xt})-\sin(\pi z)J_{2z}(2\sqrt{xt})\right)\,dt,$$

where $g(t,z)$ is a function analytic in the real variable t and in the complex variable z, is named in [4, p. 70] as the *first Koshliakov transform* of g. It arises naturally when one considers a function corresponding to the functional equation of an even Maass form in conjunction with Ferrar's summation formula; see the work of Lewis and Zagier [24, p. 216–217], for example.

Let the functions φ and ψ be related by

$$\varphi(x,z) = 2\int_0^\infty \psi(t,z)\left(\cos(\pi z)\, M_{2z}(4\sqrt{tx}) - \sin(\pi z)\, J_{2z}(4\sqrt{tx})\right)\, dt, \tag{1.7}$$

and

$$\psi(x,z) = 2\int_0^\infty \varphi(t,z)\left(\cos(\pi z)\, M_{2z}(4\sqrt{tx}) - \sin(\pi z)\, J_{2z}(4\sqrt{tx})\right)\, dt. \tag{1.8}$$

Moreover, define the normalized Mellin transforms $Z_1(s,z)$ and $Z_2(s,z)$ of the functions $\varphi(x,z)$ and $\psi(x,z)$ by

$$\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right) Z_1(s,z) = \int_0^\infty x^{s-1}\varphi(x,z)\, dx, \tag{1.9}$$

$$\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right) Z_2(s,z) = \int_0^\infty x^{s-1}\psi(x,z)\, dx, \tag{1.10}$$

where each equation is valid in a specific vertical strip in the complex s-plane. Set

$$Z(s,z) := Z_1(s,z) + Z_2(s,z) \quad \text{and} \quad \Theta(x,z) := \varphi(x,z) + \psi(x,z), \tag{1.11}$$

so that

$$\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right) Z(s,z) = \int_0^\infty x^{s-1}\Theta(x,z)\, dx \tag{1.12}$$

for values of s in the intersection of the two vertical strips.

In this paper, we evaluate the integrals in (1.4) for two specific choices of $f(t,z)$ satisfying (1.3). The function $Z\left(\frac{1}{2}+it, z\right)$ in these integrals depends on the functions $\varphi(x,z)$ and $\psi(x,z)$ satisfying (1.7) and (1.8), and belonging to the class $\Diamond_{\eta,\omega}$ defined below.

Definition 1.1. *Let $0 < \omega \leq \pi$ and $\eta > 0$. For a fixed z, if $u(s,z)$ is such that*

i) *$u(s,z)$ is analytic of $s = re^{i\theta}$ regular in the angle defined by $r > 0$, $|\theta| < \omega$,*

ii) *$u(s,z)$ satisfies the bounds*

$$u(s,z) = \begin{cases} O_z(|s|^{-\delta}) & \text{if } |s| \leq 1, \\ O_z(|s|^{-\eta-1-|\mathrm{Re}(z)|}) & \text{if } |s| > 1, \end{cases} \tag{1.13}$$

for every positive δ and uniformly in any angle $|\theta| < \omega$, then we say that u belongs to the class $\Diamond_{\eta,\omega}$ and write $u(s,z) \in \Diamond_{\eta,\omega}$.

We are now ready to state our two main theorems.

Theorem 1.2. *Let $\eta > 1/4$ and $0 < \omega \leq \pi$. Suppose that $\varphi, \psi \in \Diamond_{\eta,\omega}$, are reciprocal in the Koshliakov kernel as per (1.7) and (1.8), and that $-1/2 < \mathrm{Re}(z) < 1/2$. Let $Z(s,z)$ and $\Theta(x,z)$ be defined in (1.11). Let $\sigma_{-z}(n) = \sum_{d|n} d^{-z}$. Then,*

$$\frac{32}{\pi}\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right) Z\left(\frac{1+it}{2}, \frac{z}{2}\right)\frac{dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}$$

$$= \sum_{n=1}^\infty \sigma_{-z}(n) n^{z/2}\Theta\left(\pi n, \frac{z}{2}\right) - R(z), \tag{1.14}$$

where

$$R(z) := \pi^{z/2}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)Z\left(1+\frac{z}{2},\frac{z}{2}\right)+\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)Z\left(1-\frac{z}{2},\frac{z}{2}\right). \quad (1.15)$$

With $\alpha\beta = 1$, and the pair $(\varphi(x,z),\psi(x,z)) = (K_z(2\alpha x), \beta K_z(2\beta x))$ which easily satisfies (1.7) and (1.8) (as can be seen from (1.6), we obtain the following result [9, Equation (3.18)]:

Corollary 1.3. *Let* $-1 < \operatorname{Re}(z) < 1$. *Then*

$$-\frac{32}{\pi}\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt$$
$$=\sqrt{\alpha}\left(\alpha^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\alpha^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\alpha\right)\right). \quad (1.16)$$

This result is illustrated below.

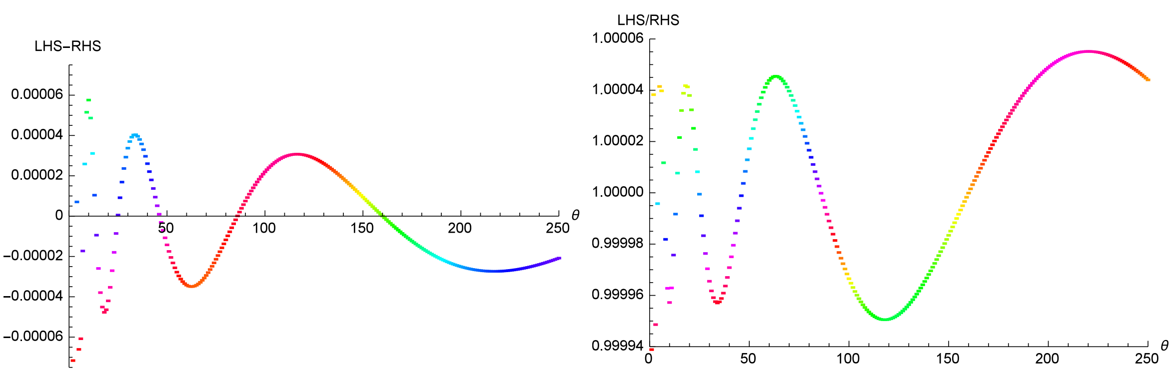

FIGURE 1. Left: Difference between the left and right sides of (1.16);
Right: Quotient $\frac{\text{left side}}{\text{right side}}$ of (1.16);
Here $z = 0$ and the series on the right is truncated to 10 sums.

This identity is connected with the Ramanujan-Guinand formula (See Equation (3.7) below.). Our second result is

Theorem 1.4. *Let* $\eta > 1/4$ *and* $0 < \omega \leq \pi$. *Suppose that* $\varphi, \psi \in \Diamond_{\eta,\omega}$, *are reciprocal in the Koshliakov kernel as per* (1.7) *and* (1.8), *and that* $-1/2 < \operatorname{Re}(z) < 1/2$. *Let* $Z(s,z)$ *and* $\Theta(x,z)$ *be defined in* (1.11). *Then,*

$$\pi^{\frac{z-3}{2}}\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2},\frac{z}{2}\right)\frac{dt}{t^2+(z+1)^2}$$
$$=\pi^{z+1/2}\Gamma\left(\frac{z+3}{2}\right)\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z+1}\int_0^\infty \Theta\left(x,\frac{z}{2}\right)\frac{x^{1+z/2}}{(x^2+\pi^2n^2)^{(z+3)/2}}\,dx - S(z), \quad (1.17)$$

where

$$S(z) := 2^{-1-z}\Gamma(1+z)\zeta(1+z)Z\left(1+\frac{z}{2},\frac{z}{2}\right) + 2^{-z}\Gamma(z)\zeta(z)Z\left(1-\frac{z}{2},\frac{z}{2}\right).$$

As a special case when again $\alpha\beta = 1$, and $(\phi(x,z),\psi(x,z)) = (K_z(2\alpha x), \beta K_z(2\beta x))$, we obtain the following result established in [9, Equation (4.23)].

Corollary 1.5. *Let* $-1 < \mathrm{Re}(z) < 1$ *and define*

$$\lambda(x,z) = \zeta(z+1,x) - \frac{x^{-z}}{z} - \frac{1}{2}x^{-z-1}, \tag{1.18}$$

where $\zeta(z,x)$ *is the Hurwitz zeta function. Then*

$$\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{t^2+(z+1)^2}\,dt$$
$$= \alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^\infty \lambda(n\alpha,z) - \frac{\zeta(z+1)}{2\alpha^{z+1}} - \frac{\zeta(z)}{\alpha z}\right). \tag{1.19}$$

As in the previous case, the graphical illustration of this corollary is given below.

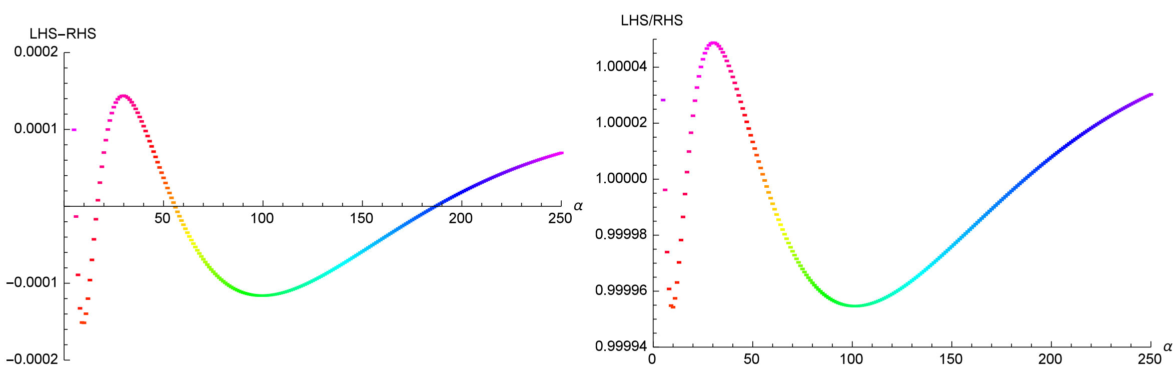

FIGURE 2. Left: Difference between the left and right sides of (1.19);
Right: Quotient $\frac{\text{left side}}{\text{right side}}$ of (1.19);
Here $z = 3/4$ and the series on the right is truncated to 10 sums.

This is related to the modular-type transformation involving infinite series of Hurwitz zeta function. (See Equation (4.9) below.)

Theorem 5.3 from [11] gives a sufficient condition for a function to be equal to its first Koshliakov transform. In the same paper [11, Equations (4.8), (4.17)], there are two new explicit examples of such a function whereas [4, p. 72, Equation (15.18)] contains a further new one. One may be able to obtain further corollaries of our theorems by working with these other examples. However, none of them is as simple as $K_z(x)$, and so we do not pursue this matter here. We note that Theorems 1.2 and 1.4 work for any pair of functions reciprocal in the kernel (1.5), not necessarily self-reciprocal. Dixon and Ferrar [13, Section 5] study, for example, the pair $(\varphi(x),\psi(x)) =$

$\left(e^{-x}, -\frac{2}{\pi}\left(e^{4x}\mathrm{li}(e^{-4x}) + e^{-4x}\mathrm{li}(e^{4x})\right)\right)$, which is reciprocal in (1.5) with $z = 0$. Here $\mathrm{li}(x)$ denotes the logarithmic integral function defined by the Cauchy principal value

$$\mathrm{li}(x) = \lim_{\varepsilon \to 0^+} \left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^{x} \frac{dt}{\log t} \right).$$

Finally, it must be mentioned here that Guinand [17, Theorem 6], [18, Equation (1)] derived the following summation formula involving $\sigma_s(n)$:

$$\sum_{n=1}^{\infty} \sigma_{-z}(n) n^{\frac{z}{2}} f(n) - \zeta(1+z) \int_0^{\infty} x^{\frac{z}{2}} f(x)\, dx - \zeta(1-z) \int_0^{\infty} x^{-\frac{z}{2}} f(x)\, dx$$
$$= \sum_{n=1}^{\infty} \sigma_{-z}(n) n^{\frac{z}{2}} g(n) - \zeta(1+z) \int_0^{\infty} x^{\frac{z}{2}} g(x)\, dx - \zeta(1-z) \int_0^{\infty} x^{-\frac{z}{2}} g(x)\, dx. \quad (1.20)$$

Here $f(x)$ satisfies certain appropriate conditions (see [17] for details) and $g(x)$ is the transform of $f(x)$ with respect to

$$-2\pi \sin\left(\tfrac{1}{2}\pi z\right) J_z(4\pi\sqrt{x}) - \cos\left(\tfrac{1}{2}\pi z\right)\left(2\pi Y_z(4\pi\sqrt{x}) - 4K_z(4\pi\sqrt{x})\right),$$

which, up to a constant factor, is nothing but the Koshliakov kernel in (1.5). Nasim [25] derived a transformation formula involving functions reciprocal in Koshliakov kernel, and which is similar to (1.20).

2. Preliminaries

The Riemann zeta function satisfies the following functional equation [31, p. 22, eqn. (2.6.4)]

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s), \quad (2.1)$$

sometimes also written in the form

$$\xi(s) = \xi(1-s), \quad (2.2)$$

where $\xi(s)$ is defined in (1.2).

For $\mathrm{Re}(s) > \max(1, 1+\mathrm{Re}(a))$, the following Dirichlet series representation is well known [31, p. 8, Equation (1.3.1)]:

$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}, \quad (2.3)$$

Throughout the paper we use R_a to denote the residue of a function being considered at its pole $z = a$.

We also use Parseval's theorem in the form [28, p. 83, Equation (3.1.13)]

$$\frac{1}{2\pi i} \int_{(\sigma)} \mathfrak{G}(s)\mathfrak{H}(s) t^{-s} ds = \int_0^{\infty} g(x) h\left(\frac{t}{x}\right) \frac{dx}{x}, \quad (2.4)$$

where $\mathfrak{G}$ and $\mathfrak{H}$ are Mellin transforms of g and h respectively.

The following lemma will be instrumental in proving our theorems.

Lemma 2.1. *Let $\eta > 0$ and $0 < \omega \leq \pi$. Let $-1/4 < \mathrm{Re}(z) < 1/4$. Suppose that $\varphi, \psi \in \Diamond_{\eta,\omega}$ are Koshliakov reciprocal functions. One has*

(1) $Z(s,z) = Z(1-s,z)$ *for all* s *in* $-\eta - |\mathrm{Re}(z)|/2 < \mathrm{Re}(s) < 1 + \eta + |\mathrm{Re}(z)|/2$.
(2) $Z(\sigma + it, z) \ll_z e^{\left(\frac{\pi}{2} - \omega + \varepsilon\right)|t|}$ *for every* $\varepsilon > 0$.

Proof. Fix $\eta > 0$, $0 < \omega \leq \pi$ and let $\varphi, \psi \in \Diamond_{\eta,\omega}$. We begin by proving the first part of the claim involving the functional equation between Z_1 and Z_2. Set

$$w(s,z) := \Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right). \tag{2.5}$$

Note that

$$\begin{aligned}
&w(1-s,z)Z_1(1-s,z)\\
&= \int_0^\infty x^{-s}\varphi(x,z)\,dx\\
&= 2\int_0^\infty x^{-s}\int_0^\infty \psi(t,z)\left(\cos(\pi z)\,M_{2z}(4\sqrt{tx}) - \sin(\pi z)\,J_{2z}(4\sqrt{tx})\right)\,dt\,dx\\
&= 2\int_0^\infty \psi(t,z)\int_0^\infty x^{-s}\left(\cos(\pi z)\,M_{2z}(4\sqrt{tx}) - \sin(\pi z)\,J_{2z}(4\sqrt{tx})\right)\,dx\,dt\\
&= 2\pi^{2-2s}\int_0^\infty t^{s-1}\psi(t,z)\int_0^\infty u^{-s}\left(\cos(\pi z)\,M_{2z}(4\pi\sqrt{u}) - \sin(\pi z)\,J_{2z}(4\pi\sqrt{u})\right)\,du\,dt\\
&= \frac{2^{2s-1}}{\pi}\Gamma(1-s-z)\,\Gamma(1-s+z)\left(\cos(\pi z) - \cos(\pi s)\right)\int_0^\infty t^{s-1}\psi(t,z)\,dt\\
&= \frac{2^{2s-1}}{\pi}\Gamma(1-s-z)\,\Gamma(1-s+z)\left(\cos(\pi z) - \cos(\pi s)\right)w(s,z)Z_2(s,z),
\end{aligned} \tag{2.6}$$

where in the penultimate step, we used the integral evaluation

$$\begin{aligned}
&\int_0^\infty x^{-s}\left(\cos(\pi z)\,M_{2z}(4\pi\sqrt{x}) - \sin(\pi z)\,J_{2z}(4\pi\sqrt{x})\right)\,dx\\
&= \frac{1}{2^{2-2s}\pi^{3-2s}}\Gamma(1-s-z)\,\Gamma(1-s+z)\left(\cos(\pi z) - \cos(\pi s)\right),
\end{aligned}$$

valid for $\frac{1}{4} < \mathrm{Re}(s) < 1 \pm \mathrm{Re}(z)$. This can in turn be obtained by replacing s by $1-s$, z by $2z$, and letting $y = 1$ in Lemma 5.1 of [11]. Using the duplication formula for the gamma function

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2s-1}}\Gamma(2s), \tag{2.7}$$

and the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)},$$

we see that

$$Z_1(1-s,z) = Z_2(s,z).$$

Similarly, $Z_2(1-s,z) = Z_1(s,z)$, and hence from (1.11), we see that

$$Z(1-s,z) = Z(s,z).$$

The interchange of the order of integration in the third step in (2.6) requires justification. We provide that below using Fubini's theorem. We only show that the double integral

$$\int_0^\infty \psi(t,z)\int_0^\infty x^{-s}\left(\cos(\pi z)\,M_{2z}(4\sqrt{tx})-\sin(\pi z)\,J_{2z}(4\sqrt{tx})\right)\,dx\,dt$$

converges absolutely for $\frac{3}{4} < \mathrm{Re}(s) < 1-|\mathrm{Re}(z)|$. (This necessitates $\mathrm{Re}(z)$ to be between $-1/4$ and $1/4$.) The absolute convergence of the other one can be proved similarly.

Fix $\varepsilon_0 > 0$ such that

$$\frac{3}{4}+\varepsilon_0 \leqslant \mathrm{Re}(s) \leqslant 1-|\mathrm{Re}(z)|-\varepsilon_0.$$

Then

$$|x^{-s}| \leq \begin{cases} x^{\varepsilon_0-1+|\mathrm{Re}(z)|}, & \text{if} \quad 0\leq x\leq 1,\\ x^{-\varepsilon_0-3/4}, & \text{if} \quad x\geq 1. \end{cases} \tag{2.8}$$

The asymptotics of Bessel functions of the first and second kinds [1, p. 360, 364] give

$$J_z(v) \ll_z \begin{cases} v^{\mathrm{Re}(z)}, & \text{if} \quad 0\leq v\leq 1,\\ v^{-1/2}, & \text{if} \quad v>1, \end{cases}$$

$$Y_z(v) \ll_z \begin{cases} 1+|\log v|, & \text{if} \quad z=0, 0\leq v\leq 1,\\ v^{-|\mathrm{Re}(z)|}, & \text{if} \quad z\neq 0, 0\leq v\leq 1,\\ v^{-1/2}, & \text{if} \quad v>1, \end{cases}$$

whereas those for the modified Bessel function [1, p. 375, 378] give

$$K_z(v) \ll_z \begin{cases} 1+|\log v|, & \text{if} \quad z=0, 0\leq v\leq 1,\\ v^{-|\mathrm{Re}(z)|}, & \text{if} \quad z\neq 0, 0\leq v\leq 1,\\ v^{-1/2}e^{-v}, & \text{if} \quad v>1. \end{cases} \tag{2.9}$$

Therefore

$$\left|\cos(\pi z)\,M_{2z}(4\sqrt{tx})-\sin(\pi z)\,J_{2z}(4\sqrt{tx})\right| \ll_z \begin{cases} 1+|\log(tx)|, & \text{if} \quad z=0, 0\leq tx\leq 1,\\ (tx)^{-|\mathrm{Re}(z)|}, & \text{if} \quad z\neq 0, 0\leq tx\leq 1,\\ (tx)^{-1/4}, & \text{if} \quad tx\geq 1. \end{cases} \tag{2.10}$$

We now divide the first quadrant of the t,x plane into six different regions whose boundaries are determined by the $t=1$, $x=1$ and the hyperbola $xt=1$.
Let $D=[0,\infty)\times[0,\infty)$. Set

$$F_z(t,x) := x^{-s}\psi(t,z)\left(\cos(\pi z)\,M_{2z}(4\sqrt{tx})-\sin(\pi z)\,J_{2z}(4\sqrt{tx})\right)$$

and

$$I_\psi(s,z) := \int_D |F_z(x,t)|d\lambda,$$

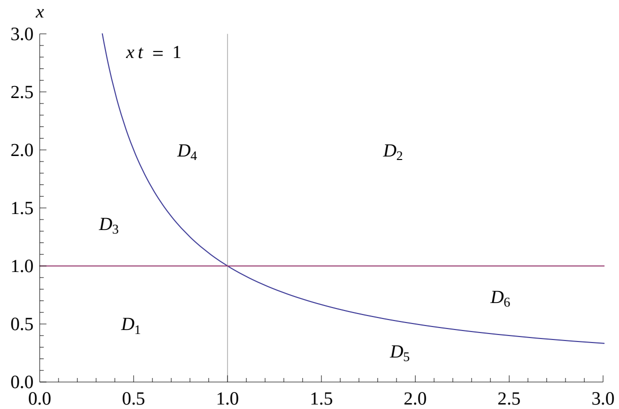

FIGURE 3. Regions from the hyperbola $xt = 1$.

where $d\lambda$ denotes the Lebesgue measure. Let

$$I_\psi(s,z) := I_1(s,z) + I_2(s,z) + \cdots + I_6(s,z), \quad \text{with} \quad I_j(s,z) := \iint_{D_j} |F_z(x,t)| d\lambda,$$

We estimate each I_j separately. Let us assume that $\varepsilon < \varepsilon_0$. Using (1.13), (2.8), and (2.10) in the regions D_1,D_2,$\cdots$,D_6, we have

$$I_1(s,z) \ll_{\varepsilon,z} \iint_{D_1} \frac{1}{x^{1-\varepsilon_0-|\mathrm{Re}(z)|}} \frac{1}{t^\delta} \frac{1}{(xt)^{\varepsilon+|\mathrm{Re}(z)|}}\, dx\, dt = \int_0^1 \frac{dx}{x^{1-\varepsilon_0+\varepsilon}} \int_0^1 \frac{dt}{t^{\delta+\varepsilon+|\mathrm{Re}(z)|}} < \infty,$$

$$I_2(s,z) \ll \iint_{D_2} \frac{1}{x^{3/4+\varepsilon_0}} \frac{1}{t^{1+\eta+|\mathrm{Re}(z)|}} \frac{1}{t^{1/4}x^{1/4}}\, dx\, dt \ll \int_1^\infty \frac{dx}{x^{1+\varepsilon_0}} \int_1^\infty \frac{dt}{t^{5/4+\eta}} < \infty,$$

$$\begin{aligned} I_3(s,z) &\ll \iint_{D_3} \frac{1}{x^{3/4+\varepsilon_0}} \frac{1}{t^\delta} \frac{1}{(xt)^{\varepsilon+|\mathrm{Re}(z)|}}\, dx\, dt \\ &\ll_\varepsilon \iint_{D_3} \frac{1}{x^{3/4+\varepsilon_0+\varepsilon+|\mathrm{Re}(z)|}} \frac{1}{t^{\delta+\varepsilon+|\mathrm{Re}(z)|}}\, dx\, dt \\ &= \int_1^\infty \frac{1}{x^{3/4+\varepsilon_0+\varepsilon+|\mathrm{Re}(z)|}} \int_0^{1/x} \frac{dt}{t^{\delta+\varepsilon+|\mathrm{Re}(z)|}}\, dx \\ &\ll \int_1^\infty \frac{dx}{x^{7/4-\delta+\varepsilon_0}} < \infty, \end{aligned}$$

$$I_4(s,z) \ll \iint_{D_4} \frac{1}{x^{3/4+\varepsilon_0}} \frac{1}{t^\delta} \frac{1}{t^{1/4}x^{1/4}}\, dx\, dt = \int_1^\infty \frac{1}{x^{1+\varepsilon_0}} \int_{1/x}^1 \frac{dt}{t^{\delta+1/4}} dx \ll \int_1^\infty \frac{1}{x^{1+\varepsilon_0}} dx < \infty,$$

$$\begin{aligned} I_5(s,z) &\ll \iint_{D_5} \frac{1}{x^{1-|\mathrm{Re}(z)|-\varepsilon_0}} \frac{1}{t^{1+|\mathrm{Re}(z)|+\eta}} \frac{1}{(xt)^{\varepsilon+|\mathrm{Re}(z)|}}\, dx\, dt \\ &= \iint_{D_5} \frac{1}{x^{1-\varepsilon_0+\varepsilon}} \frac{1}{t^{1+2|\mathrm{Re}(z)|+\eta+\varepsilon}}\, dx\, dt \\ &= \int_1^\infty \frac{1}{t^{1+2|\mathrm{Re}(z)|+\eta+\varepsilon}} \left(\int_0^{1/t} \frac{dx}{x^{1-\varepsilon_0+\varepsilon}} \right) dt \end{aligned}$$

$$\ll_{\varepsilon,\varepsilon_0} \int_1^\infty \frac{dt}{t^{1+2|\mathrm{Re}(z)|+\eta+\varepsilon_0}} < \infty.$$

Finally,

$$\begin{aligned} I_6(s,z) &\ll \iint_{D_6} \frac{1}{x^{1-|\mathrm{Re}(z)|-\varepsilon_0}} \frac{1}{t^{1+|\mathrm{Re}(z)|+\eta}} \frac{1}{t^{1/4}x^{1/4}}\, dx\, dt \\ &= \int_1^\infty \frac{1}{t^{5/4+|\mathrm{Re}(z)|+\eta}} \int_{1/t}^1 \frac{dx}{x^{5/4-|\mathrm{Re}(z)|-\varepsilon_0}} dt \\ &\ll \int_1^\infty \frac{1}{t^{1+2|\mathrm{Re}(z)|+\eta+\varepsilon_0}} dt < \infty. \end{aligned}$$

Hence $I_\psi(s,z) < \infty$, which justifies the interchange of the order of integration in (2.6). Finally, let us look at $Z_2(s,z)$ and the Mellin transform of $\psi(t,z)$. We split up the integral as

$$\int_0^\infty |\psi(t,z)||t^{s-1}|\, dt = \int_0^1 |\psi(t,z)||t^{s-1}|\, dt + \int_1^\infty |\psi(t,z)||t^{s-1}|\, dt.$$

Since $\psi \in \lozenge_{\eta,\omega}$, for the latter integral, we have

$$\int_1^\infty |\psi(t,z)||t^{s-1}|\, dt \ll \int_1^\infty t^{\mathrm{Re}(s)-2-|\mathrm{Re}(z)|/2-\eta}\, dt < \infty,$$

provided that $\mathrm{Re}(s) < 1 + \eta + \frac{|\mathrm{Re}(z)|}{2}$. Similarly, for the first integral, we have

$$\int_0^1 |\psi(t,z)| t^{\mathrm{Re}(s)-1} dt \ll \int_0^1 t^{\mathrm{Re}(s)-1-\delta}\, dt < \infty,$$

provided that $\mathrm{Re}(s) > \delta$. This shows that the function $Z_2(s,z)$ is well-defined and analytic, as a function of s, in the region

$$\delta < \mathrm{Re}(s) < 1 + \eta + \frac{|\mathrm{Re}(z)|}{2}, \tag{2.11}$$

for every $\delta > 0$. Similarly, $Z_1(s,z)$ is well-defined and analytic in s in this region. Thus, by analytic continuation, the equality $Z_1(1-s,z) = Z_2(s,z)$ holds in the vertical strip

$$-\eta - \frac{|\mathrm{Re}(z)|}{2} < \mathrm{Re}(s) < 1 + \eta + \frac{|\mathrm{Re}(z)|}{2}.$$

To prove the second part of the lemma, let us consider the line along any radius vector r and angle θ (we would choose $-\theta$, if $t = \mathrm{Im}(s) < 0$), where $|\theta| < \omega$. Then by Cauchy's theorem we can deform the integral (1.9) to

$$h(\sigma + it, z) Z_1(\sigma + it, z) = \int_0^\infty r^{\sigma+it-1} e^{i\theta(\sigma+it)} \varphi(re^{i\theta}, z)\, dr,$$

where $\theta, t > 0$. Therefore, by splitting the range of integration to $[0,1]$ and $[1,\infty]$ and the fact that Z_1 is analytic in the region defined by (2.11) we see that

$$|h(\sigma + it, z) Z_1(\sigma + it, z)| \le e^{-\theta t} \int_0^\infty r^{\sigma-1} |\varphi(re^{i\theta}, z)|\, dr \ll e^{-|\theta||t|}, \tag{2.12}$$

since $\varphi \in \Diamond_{\eta,\omega}$. By Stirling's formula for $\Gamma(\sigma+it)$ in the vertical strip $p \le \sigma \le q$ [7, p. 224], we have, as $|t| \to \infty$,

$$|\Gamma(s)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|}\left(1+O\left(\frac{1}{|t|}\right)\right), \tag{2.13}$$

as $|t| \to \infty$. Now combining (2.5), (2.12) and (2.13) we get

$$Z_1(1-s,z) = Z_2(s,z) \ll_z e^{(\frac{\pi}{2}-\omega+\varepsilon)|t|},$$

for every $\varepsilon > 0$. This completes the proof the lemma. □

3. Generalization of an integral identity connected with the Ramanujan-Guinand formula

We prove Theorem 1.2 here. The convergence of the series in (1.14) follows from the fact that $\Theta(s,z) \in \Diamond_{\eta,\omega}$.

Using (1.1) and (2.2), it is easy to see that

$$\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right) = \Xi\left(\frac{-t+iz}{2}\right)\Xi\left(\frac{-t-iz}{2}\right).$$

Along with (1.3) and part (i) of Lemma 2.1, this gives

$$\begin{aligned}
&\int_0^\infty f\left(\frac{t}{2},z\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2},\frac{z}{2}\right)dt\\
&=\frac{1}{2}\int_{-\infty}^\infty f\left(\frac{t}{2},z\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2},\frac{z}{2}\right)dt\\
&=\frac{1}{i}\int_{(\frac{1}{2})}\phi\left(s-\frac{1}{2},z\right)\phi\left(\frac{1}{2}-s,z\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)Z\left(s,\frac{z}{2}\right)ds. \qquad (3.1)
\end{aligned}$$

Now choose

$$\phi(s,z) := \left(\left(s+\frac{1}{2}+\frac{z}{2}\right)\left(s+\frac{1}{2}-\frac{z}{2}\right)\right)^{-1}$$

so that from (3.1),

$$\begin{aligned}
&16\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2},\frac{z}{2}\right)\frac{dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\\
&=\frac{1}{4i}\int_{(\frac{1}{2})}\pi^{-s}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)Z\left(s,\frac{z}{2}\right)ds \qquad (3.2)
\end{aligned}$$

Since $-1/2 < \mathrm{Re}(z) < 1/2$ and $\mathrm{Re}(s) = 1/2$, we have $1/4 < \mathrm{Re}(s-z/2) < 3/4$. In order to use (2.3), with s replaced by $s-z/2$ and a replaced by $-z$, which is therefore valid for $\mathrm{Re}(s) > 1\pm \mathrm{Re}\left(\frac{z}{2}\right)$, we shift the line of integration from $\mathrm{Re}(s) = 1/2$ to $\mathrm{Re}(s) = 5/4$. In doing so, we encounter a pole of order 1 at $s = 1-z/2$ (due to $\zeta(s+z/2)$) and a pole of order 1 at $s = 1+z/2$ (due to $\zeta(s-z/2)$). Using the notation for the residue of a function at a pole, we see that the integral in (3.2) is equal to

$$\frac{1}{4i}\left(\int_{(\frac{5}{4})}\pi^{-s}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)Z\left(s,\frac{z}{2}\right)ds - 2\pi i\left(R_{1+\frac{z}{2}}+R_{1-\frac{z}{2}}\right)\right)$$

$$= \frac{1}{4i}\Bigg(\sum_{n=1}^{\infty} \sigma_{-z}(n) n^{z/2} \int_{(\frac{5}{4})} \Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)(\pi n)^{-s} Z\left(s,\frac{z}{2}\right)\, ds$$
$$- 2\pi i\left(\pi^{-\frac{(1+z)}{2}}\Gamma\left(\frac{1+z}{2}\right)\zeta(1+z)Z\left(1+\frac{z}{2},\frac{z}{2}\right)+\pi^{-\frac{(1-z)}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)Z\left(1-\frac{z}{2},\frac{z}{2}\right)\right)\Bigg)$$
$$= \frac{\pi}{2}\sum_{n=1}^{\infty} \sigma_{-z}(n) n^{z/2}\Theta\left(\pi n,\frac{z}{2}\right)$$
$$- \frac{\pi}{2}\left(\pi^{z/2}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)Z\left(1+\frac{z}{2},\frac{z}{2}\right)+\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)Z\left(1-\frac{z}{2},\frac{z}{2}\right)\right),$$

where in the penultimate step, we used the part (ii) of Lemma 2.1 and (2.13) to see that the integrals along the horizontal segments of the contour $[\frac{1}{2}-iT, \frac{5}{4}-iT, \frac{5}{4}+iT, \frac{1}{2}+iT, \frac{1}{2}-iT]$ tend to zero as $T\to\infty$, then interchanged the order of summation and integration because of absolute convergence, and in the ultimate step we used (2.1) twice.

This completes the proof of Theorem 1.2.

In the following corollary and at few other places in the sequel, we use the notation $\Theta(x)=\Theta(x,0)$ and $Z(s)=Z(s,0)$.

Corollary 3.1. *Let $d(n)$ denote the number of positive divisors of n. Then*

$$\frac{32}{\pi}\int_0^{\infty}\Xi^2\left(\frac{t}{2}\right)Z\left(\frac{1+it}{2}\right)\frac{dt}{(1+t^2)^2} = \sum_{n=1}^{\infty} d(n)\Theta(\pi n) - (Z'(1)+(\gamma-\log 4\pi)Z(1)).$$

Proof. Let $z\to 0$ in Theorem 1.2, and note that the expansion around $z=0$ of $R(z)$, defined in (1.15), is given by

$$R(z) = (\gamma-\log 4\pi)Z(1)+Z'(1)+O(z),$$

where γ is Euler's constant. □

3.1. **Proof of the integral identity connected with the Ramanujan-Guinand formula.** Corollary 1.3 is proved here. As mentioned before its statement, we substitute $\varphi(x,z)=K_z(2\alpha x)$ in (1.8), where $\alpha>0$. The fact that $K_z(x)\in\Diamond_{\eta,\omega}$ for any $\eta>0$ is obvious from (2.9). Let $\beta=1/\alpha$. From (1.6), it is easy to see then that $\psi(x,z)=\beta K_z(2\beta x)$. For $\text{Re}(s)>\pm\,\text{Re}(\nu)$ and $q>0$, we have [26, p. 115, Equation (11.1)]

$$\int_0^{\infty} x^{s-1}K_{\nu}(qx)\,dx = 2^{s-2}q^{-s}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right). \tag{3.3}$$

Using (1.9), (1.10), (3.3), and the fact that $\alpha\beta=1$, we see that

$$Z_1(s,z)=\frac{1}{4}\alpha^{-s}\ \text{ and }\ Z_2(s,z)=\frac{1}{4}\alpha^{s-1} \tag{3.4}$$

so that

$$Z\left(\frac{1+it}{2},\frac{z}{2}\right)=\frac{1}{2\sqrt{\alpha}}\cos\left(\frac{1}{2}t\log\alpha\right) \tag{3.5}$$

and

$$\Theta\left(\pi n, \frac{z}{2}\right) = K_{z/2}(2\pi n\alpha) + \beta K_{z/2}(2\pi n\beta). \tag{3.6}$$

Then Theorem 1.2 gives

$$-\frac{32}{\pi}\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)\, dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)} = \frac{1}{2}\left(\mathfrak{F}(\alpha,z)+\mathfrak{F}(\beta,z)\right),$$

where

$$\mathfrak{F}(\alpha,z) := \sqrt{\alpha}\left(\alpha^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\alpha^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)-4\sum_{n=1}^\infty \sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\alpha\right)\right).$$

Now the Ramanujan-Guinand formula [18], [30, p. 253] gives, for $ab=\pi^2$,

$$\sqrt{a}\sum_{n=1}^\infty \sigma_{-z}(n)n^{z/2}K_{z/2}(2na) - \sqrt{b}\sum_{n=1}^\infty \sigma_{-z}(n)n^{z/2}K_{z/2}(2nb)$$
$$= \frac{1}{4}\Gamma\left(\frac{z}{2}\right)\zeta(z)\{b^{(1-z)/2}-a^{(1-z)/2}\} + \frac{1}{4}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\{a^{(1+z)/2}-a^{(1+z)/2}\}. \tag{3.7}$$

(See [5] for history and other details.)

Invoking (3.7), with $a=\pi\alpha$ and $b=\pi\beta$, it is seen that $\mathfrak{F}(\alpha,z)=\mathfrak{F}(\beta,z)$, with the help of which we deduce that, for $-1/2<\mathrm{Re}(z)<1/2$,

$$-\frac{32}{\pi}\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)\, dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)} = \mathfrak{F}(\alpha,z).$$

Since both sides of the above identity are analytic for $-1<\mathrm{Re}(z)<1$, by the principle of analytic continuation, the result holds for $-1<\mathrm{Re}(z)<1$ as well. This proves Corollary 1.3.

4. Generalization of an integral identity involving infinite series of Hurwitz zeta function

Theorem 1.4 is proved here. To see that the series on the right-hand side of (1.17) is convergent, it suffices to show that

$$\mathfrak{P}(z) := \sum_{n=1}^\infty \sigma_{-z}(n)n^{z+1}\int_0^1 \Theta\left(x,\frac{z}{2}\right)\frac{x^{1+z/2}}{(x^2+\pi^2n^2)^{(z+3)/2}}\,dx$$

converges. Since $\Theta\in\Diamond_{\eta,\omega}$, $\Theta\left(x,\frac{z}{2}\right)\ll x^{-\delta}$ for $0<x<1$ and for every $\delta>0$, so that the inside integral does not blow up as $x\to 0$. Thus,

$$\mathfrak{P}(z) \ll \sum_{n=1}^\infty \frac{\sigma_{-\mathrm{Re}(z)}(n)}{n^2}. \tag{4.1}$$

The series on the right-hand side converges as long as $\mathrm{Re}(z)>-1$ as can be seen from (2.3). Let

$$\phi(s,z) = \frac{1}{s+(z+1)/2}\Gamma\left(\frac{z-1}{4}+\frac{s}{2}\right).$$

Using (1.3) and (3.1), we see that

$$4\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2},\frac{z}{2}\right)\frac{dt}{t^2+(z+1)^2}$$
$$= -\frac{1}{4i}\int_{\left(\frac12\right)}\left(s-\frac{z}{2}\right)\left(s-1+\frac{z}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}-\frac12\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)$$
$$\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)Z\left(s,\frac{z}{2}\right)\pi^{-s}\,ds$$
$$= \frac{1}{i}\Bigg(\int_{\left(\frac54\right)}\Gamma\left(\frac{s}{2}+\frac{z}{4}+\frac12\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)$$
$$\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)Z\left(s,\frac{z}{2}\right)\pi^{-s}\,ds-2\pi i\left(R_{1+\frac{z}{2}}+R_{1-\frac{z}{2}}\right)\Bigg)$$
$$= \frac{1}{i}\Bigg(\sum_{n=1}^\infty\sigma_{-z}(n)n^{z/2}\int_{\left(\frac54\right)}\Gamma\left(\frac{s}{2}+\frac{z}{4}+\frac12\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)$$
$$\times Z\left(s,\frac{z}{2}\right)(\pi n)^{-s}\,ds-2\pi i\left(R_{1+\frac{z}{2}}+R_{1-\frac{z}{2}}\right)\Bigg), \tag{4.2}$$

where in the penultimate step we used the functional equation $\Gamma(w+1)=w\Gamma(w)$ of the Gamma function. Here $R_{1+\frac{z}{2}}$ is the residue of the integrand at the pole of order 1 due to $\zeta\left(s-\frac{z}{2}\right)$, and $R_{1-\frac{z}{2}}$ is the residue of the integrand at the pole of order 1 due to $\zeta\left(s-\frac{z}{2}\right)$. These residues turn out to be

$$R_{1+\frac{z}{2}} = 2^{-z}\pi^{(1-z)/2}\Gamma(1+z)\zeta(1+z)Z\left(1+\frac{z}{2},\frac{z}{2}\right),$$
$$R_{1-\frac{z}{2}} = 2^{1-z}\pi^{(1-z)/2}\Gamma(z)\zeta(z)Z\left(1-\frac{z}{2},\frac{z}{2}\right). \tag{4.3}$$

Note that for $0<\text{Re }u<\text{Re }w$, Euler's beta integral is given by

$$\int_0^\infty\frac{x^{u-1}}{(1+x)^w}\,dx=\frac{\Gamma(u)\Gamma(w-u)}{\Gamma(w)},$$

so that for $-\text{ Re}\left(\frac{z}{2}\right)-1<d=\text{Re}(s)<\text{Re}\left(\frac{z}{2}\right)+2$,

$$\frac{1}{2\pi i}\int_{(d)}\Gamma\left(\frac{s}{2}+\frac{z}{4}+\frac12\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}+1\right)x^{-s}\,ds=2\Gamma\left(\frac{z+3}{2}\right)\frac{x^{(z+2)/2}}{(1+x^2)^{(z+3)/2}}.$$

Along with (1.12) and (2.4), this gives

$$\int_{\left(\frac54\right)}\Gamma\left(\frac{s}{2}+\frac{z}{4}+\frac12\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)Z\left(s,\frac{z}{2}\right)(\pi n)^{-s}\,ds$$
$$=4\pi i\Gamma\left(\frac{z+3}{2}\right)(\pi n)^{(z+2)/2}\int_0^\infty\Theta\left(x,\frac{z}{2}\right)\frac{x^{1+z/2}}{(x^2+\pi^2n^2)^{(z+3)/2}}\,dx. \tag{4.4}$$

From (4.2), (4.3) and (4.4), we obtain (1.17). This completes the proof of Theorem 1.4.

Corollary 4.1. *Let* $d(n)=\sum_{d|n}1$ *as before. Then,*

$$\pi^{-3/2}\int_0^\infty\left|\Gamma\left(\frac{-1+it}{4}\right)\right|^2\Xi^2\left(\frac{t}{2}\right)Z\left(\frac{1+it}{2}\right)\frac{dt}{1+t^2}$$
$$=\frac{\pi}{2}\sum_{n=1}^\infty nd(n)\int_0^\infty\frac{x\Theta(x)}{(x^2+\pi^2n^2)^{3/2}}dx-\frac{1}{2}\left((\gamma-\log(2\pi))Z(1)+Z'(1)\right).$$

Proof. Let $z\to0$ in Theorem 1.4, and note that

$$S(0)=\frac{1}{2}\left((\gamma-\log(2\pi))Z(1)+Z'(1)\right).$$

□

4.1. **Proof of the integral identity involving infinite series of Hurwitz zeta function.** We now prove Corollary 1.5. We again choose the pair $(\phi(x,z),\psi(x,z))=(K_z(2\alpha x),\beta K_z(2\beta x))$ where $\alpha\beta=1$. So from (3.4)-(3.6),

$$\frac{\pi^{\frac{z-3}{2}}}{2\sqrt{\alpha}}\int_0^\infty\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{t^2+(z+1)^2}\,dt$$
$$=\pi^{z+1/2}\Gamma\left(\frac{z+3}{2}\right)\sum_{n=1}^\infty\sigma_{-z}(n)n^{z+1}\int_0^\infty\frac{x^{1+z/2}}{(x^2+\pi^2n^2)^{(z+3)/2}}\left(K_{z/2}(2\alpha x)+\beta K_{z/2}(2\beta x)\right)\,dx$$
$$-\left(2^{-3-z}\Gamma(1+z)\zeta(1+z)\left(\alpha^{-1-\frac{z}{2}}+\alpha^{\frac{z}{2}}\right)+2^{-2-z}\Gamma(z)\zeta(z)\left(\alpha^{-1+\frac{z}{2}}+\alpha^{-\frac{z}{2}}\right)\right).\tag{4.5}$$

For $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$ and $\mathrm{Re}(\nu)>-1$, we have [15, p. 678, formula **6.565.7**]

$$\int_0^\infty x^{1+\nu}(x^2+a^2)^\mu K_\nu(bx)\,dx=2^\nu\Gamma(\nu+1)a^{\nu+\mu+1}b^{-1-\mu}S_{\mu-\nu,\mu+\nu+1}(ab),$$

whereas from the footnote on the first page of [14], we have

$$S_{\mu,\nu}(w)=w^{\mu+1}\int_0^\infty te^{-wt}{}_2F_1\left(\frac{1-\mu+\nu}{2},\frac{1-\mu-\nu}{2};\frac{3}{2};-t^2\right)\,dt.$$

Thus

$$\int_0^\infty x^{1+\nu}(x^2+a^2)^\mu K_\nu(bx)\,dx=a^{2\mu+2}\left(\frac{2}{b}\right)^\nu\Gamma(\nu+1)\int_0^\infty ye^{-aby}{}_2F_1\left(1+\nu,-\mu,\frac{3}{2};-y^2\right)\,dy.\tag{4.6}$$

Let $a=\pi n$, $b=2\alpha$, $u=-(3+z)/2$ and $v=z/2$ in (4.6) so that

$$\int_0^\infty\frac{x^{1+z/2}K_{z/2}(2\alpha x)}{(x^2+\pi^2n^2)^{(3+z)/2}}\,dx=\frac{(\pi n)^{-1-z}\alpha^{-z/2}}{1+z}\Gamma\left(1+\frac{z}{2}\right)$$
$$\times\int_0^\infty\frac{e^{-2\pi n\alpha y}}{(1+y^2)^{(1+z)/2}}\sin((1+z)\tan^{-1}y)\,dy,$$

since [15, p. 1006, formula **9.121.4**]

$${}_2F_1\left(\frac{1-a}{2},\frac{2-a}{2};\frac{3}{2};\frac{w^2}{u^2}\right)=\frac{(u+w)^a-(u-w)^a}{2awu^{a-1}}.$$

Thus

$$\begin{aligned}
&\pi^{z+1/2}\Gamma\left(\frac{z+3}{2}\right)\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z+1}\int_0^{\infty}\frac{x^{1+z/2}}{(x^2+\pi^2n^2)^{(z+3)/2}}K_{z/2}(2\alpha x)\,dx\\
&=\frac{\alpha^{-z/2}}{2\sqrt{\pi}}\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(1+\frac{z}{2}\right)\sum_{m=1}^{\infty}m^{-z}\int_0^{\infty}\sum_{k=1}^{\infty}e^{-2\pi mk\alpha y}\frac{\sin((1+z)\tan^{-1}y)\,dy}{(1+y^2)^{(1+z)/2}}\\
&=\frac{\alpha^{z/2}}{2\sqrt{\pi}}\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(1+\frac{z}{2}\right)\sum_{m=1}^{\infty}\int_0^{\infty}\frac{\sin\left((1+z)\tan^{-1}\left(\frac{x}{m\alpha}\right)\right)}{(x^2+m^2\alpha^2)^{(1+z)/2}}\frac{dx}{e^{2\pi x}-1}\\
&=\frac{\alpha^{z/2}}{2^{z+2}}\Gamma(z+1)\sum_{m=1}^{\infty}\left(\zeta(z+1,m\alpha)-\frac{(m\alpha)^{-z}}{2}-\frac{(m\alpha)^{-z}}{z}\right), \qquad (4.7)
\end{aligned}$$

where in the last step, we used (2.7), and Hermite's formula for the Hurwitz zeta function [27, p. 609, formula **25.11.29**], namely,

$$\zeta(w,a)=\frac{1}{2}a^{-w}+\frac{a^{1-w}}{w-1}+2\int_0^{\infty}\frac{\sin\left(w\tan^{-1}(t/a)\right)}{(a^2+t^2)^{w/2}}\frac{dt}{e^{2\pi t}-1},$$

which is valid for $w\neq 1$ and $\mathrm{Re}(a)>0$.

Hence from (4.5), (4.7), and the fact that $\alpha\beta=1$, we deduce that

$$\begin{aligned}
&\frac{2^z\pi^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{t^2+(z+1)^2}\,dt\\
&=\frac{1}{2}\left\{\alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\lambda(n\alpha,z)-\frac{\zeta(z+1)}{2\alpha^{z+1}}-\frac{\zeta(z)}{\alpha z}\right)+\beta^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\lambda(n\beta,z)-\frac{\zeta(z+1)}{2\beta^{z+1}}-\frac{\zeta(z)}{\beta z}\right)\right\}, \qquad (4.8)
\end{aligned}$$

where $\lambda(x,z)$ is defined in (1.18). To obtain (1.17), it only remains to show that

$$\alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\lambda(n\alpha,z)-\frac{\zeta(z+1)}{2\alpha^{z+1}}-\frac{\zeta(z)}{\alpha z}\right)=\beta^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\lambda(n\beta,z)-\frac{\zeta(z+1)}{2\beta^{z+1}}-\frac{\zeta(z)}{\beta z}\right). \qquad (4.9)$$

The limiting case $z\to 0$ of this identity appears on page 220 of Ramanujan's Lost Notebook [30], and is discussed in detail in [3].

In [8] as well as in [9], (4.9) was proved as a consequence of Corollary 1.4 and the fact that $\cos\left(\frac{1}{2}t\log\alpha\right)=\cos\left(\frac{1}{2}t\log\beta\right)$ for $\alpha\beta=1$. Hence to avoid circular reasoning, we must obtain a new proof of it which does not make use of the integral involving the Riemann Ξ-function present in this corollary.

The aforementioned limiting case was proved in [3, Section 4] in the manner sought above using Guinand's generalization of the Poisson summation formula [16, Theorem 1]. This requires use of a result of Ramanujan [3, Equation (1.4)] that the function $\psi(x+1)-\log x$ is self-reciprocal in the Fourier cosine transform, i.e.,

$$\int_0^{\infty}\left(\psi(1+x)-\log x\right)\cos(2\pi yx)\,dx=\frac{1}{2}\left(\psi(1+y)-\log y\right).$$

For (4.9), this method, however, does not look feasible as the one-variable generalization of $\psi(x+1)-\log x$ relevant to the problem, namely $x^{-z}/z-\zeta(z+1,x+1)$, is *not* self-reciprocal in the Fourier cosine transform. Hence we prove (4.9) by first obtaining a new proof of the equivalent modular transformation in the first equality in the following result, also proved in [11, Theorem 6.3] using the integral involving the Ξ-function. This new proof is, of course, independent of the use of this integral, and generalizes Koshliakov's proof for the case when $z=0$ [22, p. 248].

Assume $-1<\text{Re } z<1$. *Let* $\Omega(x,z)$ *be defined by*

$$\Omega(x,z):=2\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}\left(e^{\pi iz/4}K_z(4\pi e^{\pi i/4}\sqrt{nx})+e^{-\pi iz/4}K_z(4\pi e^{-\pi i/4}\sqrt{nx})\right),$$

where $\sigma_{-z}(n)=\sum_{d|n}d^{-z}$ *and* $K_\nu(z)$ *denotes the modified Bessel function of order* ν. *Then for* $\alpha,\beta>0,\alpha\beta=1$,

$$\begin{aligned}\alpha^{(z+1)/2}&\int_0^\infty e^{-2\pi\alpha x}x^{z/2}\left(\Omega(x,z)-\frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx\\&=\beta^{(z+1)/2}\int_0^\infty e^{-2\pi\beta x}x^{z/2}\left(\Omega(x,z)-\frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx.\end{aligned}\tag{4.10}$$

Upon proving (4.10), we show that

$$\begin{aligned}&\int_0^\infty e^{-2\pi\alpha x}x^{z/2}\left(\Omega(x,z)-\frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx\\&=\frac{\Gamma(z+1)}{(2\pi)^{z+1}}\sum_{n=1}^{\infty}\left(\lambda(n\alpha,z)-\frac{\zeta(z+1)}{2\alpha^{z+1}}-\frac{\zeta(z)}{\alpha z}\right),\end{aligned}\tag{4.11}$$

thereby proving (4.9). The special case of (4.11) was obtained in [10].

The function $\Omega(x,z)$ has many nice properties. For example, it has a very useful inverse Mellin transform representation [11, Equation (6.6)], valid for $c=\text{Re } s>1\pm\text{Re }\frac{z}{2}$:

$$\Omega(x,z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(1-s+\frac{z}{2})\zeta(1-s-\frac{z}{2})}{2\cos\left(\frac{1}{2}\pi\left(s+\frac{z}{2}\right)\right)}x^{-s}ds.\tag{4.12}$$

It also satisfies the following identity [11, Proposition 6.1] and Re $x>0$:

$$\Omega(x,z)=-\frac{\Gamma(z)\zeta(z)}{(2\pi\sqrt{x})^z}+\frac{x^{z/2-1}}{2\pi}\zeta(z)-\frac{x^{z/2}}{2}\zeta(z+1)+\frac{x^{z/2+1}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{-z}(n)}{n^2+x^2}.\tag{4.13}$$

Lastly, we mention that it plays a vital role in obtaining a very short proof of the extended version of the Voronoï summation formula [4, Theorem 6.1] in the case when the associated function is analytic in a specific region.

We begin with the following lemma, which is interesting in its own right, and shows that the function $\Omega(x,z)-\frac{1}{2\pi}\zeta(z)x^{\frac{z}{2}-1}$ is self-reciprocal when integrated against the Bessel function of the first kind of order z. It is a one variable generalization of a result of Koshliakov [21, Equation (11)].

Lemma 4.2. *Let $J_\nu(w)$ denote the Bessel function of the first kind of order ν. Let $-1 < \text{Re } z < 1$. For $\text{Re}(x) > 0$, we have*

$$\int_0^\infty J_z(4\pi\sqrt{xy}) \left(\Omega(y,z) - \frac{1}{2\pi}\zeta(z)y^{\frac{z}{2}-1}\right) dy = \frac{1}{2\pi}\left(\Omega(x,z) - \frac{1}{2\pi}\zeta(z)x^{\frac{z}{2}-1}\right).$$

Proof. For $-\text{ Re}\left(\frac{z}{2}\right) < \text{Re } s < \frac{3}{4}$, we have [11, p. 225]

$$\int_0^\infty x^{s-1}J_z(4\pi\sqrt{xy})dx = 2^{-2s}\pi^{-2s}y^{-s}\frac{\Gamma\left(s+\frac{z}{2}\right)}{\Gamma\left(1-s+\frac{z}{2}\right)},$$

and from (4.12) and an application of the residue theorem, we find that for $\text{Re } s < 1 \pm \text{Re}\left(\frac{z}{2}\right)$,

$$\int_0^\infty y^{s-1}\left(\Omega(y,z) - \frac{1}{2\pi}\zeta(z)y^{\frac{z}{2}-1}\right) = \frac{\zeta(1-s+\frac{z}{2})\zeta(1-s-\frac{z}{2})}{2\cos\left(\frac{1}{2}\pi\left(s+\frac{z}{2}\right)\right)}. \tag{4.14}$$

Hence using Parseval's formula [28, p. 83, Equation (3.1.11)], we see that for $\pm$ Re $\left(\frac{z}{2}\right) < c = \text{Re } s < \min\left(\frac{3}{4}, 1 \pm \text{Re}\left(\frac{z}{2}\right)\right)$,

$$\begin{aligned}
&\int_0^\infty J_z(4\pi\sqrt{xy}) \left(\Omega(y,z) - \frac{1}{2\pi}\zeta(z)y^{\frac{z}{2}-1}\right) dy \\
&= \frac{1}{2\pi i}\int_{(c)} \frac{(2\pi)^{-2s}x^{-s}\Gamma\left(s+\frac{z}{2}\right)}{\Gamma\left(1-s+\frac{z}{2}\right)} \frac{\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)}{2\sin\left(\frac{\pi}{2}\left(s-\frac{z}{2}\right)\right)}\, ds \\
&= \frac{1}{2\pi i}\int_{(c)} \frac{\zeta(1-s+\frac{z}{2})\zeta(1-s-\frac{z}{2})}{4\pi\cos\left(\frac{1}{2}\pi\left(s+\frac{z}{2}\right)\right)} x^{-s} ds,
\end{aligned}$$

where in the last step we used the functional equation for $\zeta(s)$ twice. The result now follows from (4.14). ☐

We now prove the modular transformation 4.10. Using Lemma 4.2, we see that

$$\begin{aligned}
&\alpha^{(z+1)/2}\int_0^\infty e^{-2\pi\alpha x}x^{z/2}\left(\Omega(x,z) - \frac{1}{2\pi}\zeta(z)x^{z/2-1}\right) dx \\
&= 2\pi\alpha^{(z+1)/2}\int_0^\infty e^{-2\pi\alpha x}x^{z/2}\int_0^\infty J_z(4\pi\sqrt{xy})\left(\Omega(y,z) - \frac{1}{2\pi}\zeta(z)y^{\frac{z}{2}-1}\right) dy\, dx \\
&= 2\pi\alpha^{(z+1)/2}\int_0^\infty \left(\Omega(y,z) - \frac{1}{2\pi}\zeta(z)y^{\frac{z}{2}-1}\right)\int_0^\infty e^{-2\pi\alpha x}x^{z/2}J_z(4\pi\sqrt{xy})\, dx\, dy, \quad (4.15)
\end{aligned}$$

where the interchange of the order of integration follows from Fubini's theorem.

Now use the formula [15, p. 709, formula **6.643.1**]

$$\int_0^\infty x^{\mu-\frac{1}{2}}e^{-ax}J_{2\nu}(2b\sqrt{x})\, dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{b\Gamma(2\nu+1)}e^{-\frac{b^2}{2a}}a^{-\mu}M_{\mu,\nu}\left(\frac{b^2}{a}\right),$$

which is valid for Re $\left(\mu+\nu+\frac{1}{2}\right) > 0$, with $b = 2\pi\sqrt{y}$, $\nu = z/2, \mu = (z+1)/2, a = 2\pi\alpha$, and then the definition [15, p. 1024] of the Whittaker function

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-z/2}{}_1F_1\left(\mu-k+\frac{1}{2}; 2\mu+1; z\right), \tag{4.16}$$

to deduce that

$$\int_0^\infty e^{-2\pi\alpha x}x^{z/2}J_z(4\pi\sqrt{xy})\,dx=\frac{e^{-2\pi y/\alpha}y^{z/2}}{2\pi\alpha^{z+1}}. \tag{4.17}$$

Finally we obtain (4.10) from (4.15), (4.17) and the fact that $\alpha\beta=1$.

It only remains to now prove (4.11). We first prove it for $0<\mathrm{Re}\ z<1$ and then extend it by analytic continuation. To that end, we use (4.13) to evaluate the integral on the left-hand side of (4.11). Note that

$$\int_0^\infty e^{-2\pi\alpha x}x^{z/2}\left(-\frac{\Gamma(z)\zeta(z)x^{-z/2}}{(2\pi)^z}-\frac{x^{z/2}}{2}\zeta(z+1)\right)dx=-\frac{\Gamma(z)\zeta(z)}{\alpha(2\pi)^{z+1}}-\frac{\Gamma(z+1)\zeta(z+1)}{2(2\pi\alpha)^{z+1}}. \tag{4.18}$$

Also,

$$\int_0^\infty e^{-2\pi\alpha x}\frac{x^{z+1}}{\pi}\sum_{n=1}^\infty\frac{\sigma_{-z}(n)}{n^2+x^2}\,dx=\sum_{m=1}^\infty m^{-z}\int_0^\infty\frac{x^{z+1}}{\pi}e^{-2\pi\alpha x}\sum_{k=1}^\infty\frac{1}{x^2+m^2k^2}\,dx,$$

where the interchange of the order of summation and integration is justified by absolute convergence. It is well known [6, p. 191] that for $t\neq 0$,

$$2t\sum_{k=1}^\infty\frac{1}{t^2+4k^2\pi^2}=\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}.$$

So

$$\begin{aligned}\int_0^\infty e^{-2\pi\alpha x}\frac{x^{z+1}}{\pi}\sum_{n=1}^\infty\frac{\sigma_{-z}(n)}{x^2+n^2}\,dx&=\sum_{m=1}^\infty m^{-z}\int_0^\infty e^{-2\pi\alpha x}x^z\left(\frac{1}{m(e^{2\pi x/m}-1)}-\frac{1}{2\pi x}+\frac{1}{2m}\right)dx\\&=\sum_{m=1}^\infty\int_0^\infty e^{-2\pi\alpha mt}t^z\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}+\frac{1}{2}\right)dt\\&=\frac{\Gamma(z+1)}{(2\pi)^{z+1}}\sum_{m=1}^\infty\left\{\zeta(z+1,m\alpha)-\frac{(m\alpha)^{-z}}{z}-\frac{(\alpha m)^{-z-1}}{2}\right\}\\&=\frac{\Gamma(z+1)}{(2\pi)^{z+1}}\sum_{m=1}^\infty\lambda(m\alpha,z),\end{aligned} \tag{4.19}$$

where in the penultimate step, we used the following result [27, p. 609, formula (25.11.27)], valid for $\mathrm{Re}(w)>-1, w\neq 1$, $\mathrm{Re}(a)>0$:

$$\zeta(w,a)=\frac{1}{2}a^{-w}+\frac{a^{1-w}}{w-1}+\frac{1}{\Gamma(w)}\int_0^\infty x^{w-1}e^{-ax}\left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}\right)dx$$

with $w=z+1, a=\alpha m$ amd $x=2\pi t$.

Finally (4.18) and (4.19) give (4.11). This completes the proof of the modular transformation in (4.9) for $0<\mathrm{Re}\ z<1$. As explained in [8, p. 1162], the result follows for $-1<\mathrm{Re}\ z<1$ by analytic continuation.

This proves (4.9), and hence along with (4.8), completes the proof of Corollary 1.5.

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