THE LAPLACE TRANSFORM OF THE PSI FUNCTION

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ABSTRACT. An expression for the Laplace transform of the Psi function

$$L(a) := \int_0^\infty e^{-at}\psi(t+1)\,dt$$

is derived using two different methods. It is then applied to evaluate the definite integral

$$M(a) = \frac{4}{\pi}\int_0^\infty \frac{x^2\,dx}{x^2 + \ln^2(2e^{-a}\cos x)},$$

for $a > \ln 2$ and resolve a conjecture posed by Olivier Oloa.

1. INTRODUCTION

Let $\psi(x)$ denote the logarithmic derivative of the Gamma function $\Gamma(x)$, i.e.,

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.1}$$

The Psi function has been studied extensively and still continues to receive attention from many mathematicians. Many of its properties are listed in [6, pp. 952–955]. Surprisingly, an explicit formula for the Laplace transform of the Psi function, i.e.,

$$L(a) := \int_0^\infty e^{-at}\psi(t+1)\,dt, \tag{1.2}$$

is absent from the literature. Recently in [5], the nature of the Laplace transform was studied by demonstrating the relationship between $L(a)$ and the Glasser-Manna-Oloa integral

$$M(a) := \frac{4}{\pi}\int_0^\infty \frac{x^2\,dx}{x^2 + \ln^2(2e^{-a}\cos x)}, \tag{1.3}$$

namely that, for $a > \ln 2$,

$$M(a) = L(a) + \frac{\gamma}{a}, \tag{1.4}$$

where γ is the Euler's constant. In [1], T. Amdeberhan, O. Espinosa and V. H. Moll obtained certain analytic expressions for $M(a)$ in the complementary range $0 < a \leq \ln 2$. Our goal here is to derive an explicit expression of $L(a)$ for $a > 0$, in terms of elementary functions and a certain simple infinite series which cannot be evaluated in terms of elementary functions. We give two different proofs, the second of which is shorter. However, it makes use of a formula (Equation 2.31) in [1], viz., Equation (3.1) in our paper, which requires considerable work for its derivation. Also, our first proof

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is self-contained and elementary. We then use our expression for the Laplace transform of the Psi function to evaluate $M(a)$ for $a > \ln 2$ and thereby resolve a conjecture posed by Olivier Oloa in [8]. Our main theorem can be stated as follows:

Theorem 1.1. *Let $a > 0$. If $\psi(x)$ is defined as in (1.1), then*

$$L(a) = \int_0^\infty e^{-at}\psi(t+1)\,dt = \left(\frac{1}{e^a-1} - \frac{1}{a} + 1\right)\ln\left(\frac{2\pi}{a}\right) + 2a\sum_{n=1}^\infty \frac{\ln n}{a^2+4n^2\pi^2} + \frac{1}{4}\left(\psi\left(\frac{ia}{2\pi}\right) + \psi\left(-\frac{ia}{2\pi}\right)\right) - \frac{\gamma+\ln a}{a}. \tag{1.5}$$

2. Laplace transform of Psi function: First Proof

First, from [2, p. 259, formula 6.3.18] for $|\arg z| < \pi$, as $z \to \infty$,

$$\psi(z) \sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots. \tag{2.1}$$

We now state a lemma[1] which will be subsequently used in the proof. The integral involved in the lemma exists because of (2.1). This integral gives the motivation for decomposing $L(a)$ into the two integrals mentioned in Equation (2.9) below. This integral evaluation is implicit in the work of A.P. Guinand on a certain transformation formula involving the Psi function (see the footnote on the last page in [7]), but he neither proves it nor explicitly states it in [7].

Lemma 2.1.

$$\int_0^\infty \left(\psi(t+1) - \frac{1}{2(t+1)} - \ln t\right) dt = \frac{1}{2}\ln 2\pi. \tag{2.2}$$

Proof. Let I denote the integral on the left-hand side. Then,

$$\begin{aligned} I &= \int_0^\infty \frac{d}{dt}\left(\ln\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}}\right) dt \\ &= \lim_{t\to\infty}\ln\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} - \lim_{t\to 0}\ln\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} \\ &= \ln\left(\lim_{t\to\infty}\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}}\right) - \ln\left(\lim_{t\to 0}e^t\Gamma(t+1)\right) - \lim_{t\to 0}t\ln t - \lim_{t\to 0}\frac{1}{2}\ln(t+1) \\ &= \ln\lim_{t\to\infty}\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}}. \end{aligned} \tag{2.3}$$

Now Stirling's formula [6, p. 945, formula 8.327] tells us that,

$$\Gamma(z) \sim z^{z-1/2}e^{-z}\sqrt{2\pi}, \tag{2.4}$$

as $|z| \to \infty$ and $|\arg z| \le \pi - \delta$, where $0 < \delta < \pi$. Hence, employing (2.4), we find that

$$\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} \sim \left(1+\frac{1}{t}\right)^t \frac{\sqrt{2\pi}}{e}, \tag{2.5}$$

[1]The author is indebted to M. L. Glasser for the proof of this lemma. The author's original proof is significantly longer than that given by Glasser.

so that

$$\lim_{t\to\infty} \frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} = \sqrt{2\pi}. \tag{2.6}$$

Thus from (2.3) and (2.6), we conclude that

$$I = \frac{1}{2}\ln 2\pi. \tag{2.7}$$

□

Now we begin with the proof. Without mentioning explicitly, throughout in this paper, we make use of the fact that

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{2.8}$$

Using (2.1), we can decompose the Laplace transform of the Psi function into two integrals

$$\begin{aligned}\int_0^\infty e^{-at}\psi(t+1)\,dt &= \int_0^\infty e^{-at}\left(\psi(t+1) - \frac{1}{2(t+1)} - \ln t\right)dt \\ &\quad + \int_0^\infty e^{-at}\left(\frac{1}{2(t+1)} + \ln t\right)dt \\ &= I_1 + I_2, \text{ say,}\end{aligned} \tag{2.9}$$

where (2.1) ensures that I_1 exists. The integral I_2 is easy to evaluate. Using the definition of the incomplete gamma function

$$\Gamma(0,a) = \int_a^\infty \frac{e^{-t}}{t}\,dt, \tag{2.10}$$

and the fact [6, p. 602, formula 4.331, no. 1] that,

$$\int_0^\infty e^{-ax}\ln x\,dx = -\frac{\gamma+\ln a}{a}, \qquad \text{for } a>0, \tag{2.11}$$

we obtain

$$I_2 = \frac{e^a}{2}\Gamma(0,a) - \frac{\gamma+\ln a}{a}. \tag{2.12}$$

Now let $I_1 = I_1(a)$. Differentiating under the integral sign, we have

$$\begin{aligned}I_1'(a) &= \int_0^\infty -te^{-at}\left(\psi(t+1) - \frac{1}{2(t+1)} - \ln t\right)dt \\ &= \int_0^\infty e^{-at}\left(t\ln t + \frac{t}{2(t+1)} - t\psi(t+1)\right)dt \\ &= \int_0^\infty e^{-at}\left(t\ln t + \frac{1}{2} - \frac{1}{2(t+1)} - t\left(\psi(t) + \frac{1}{t}\right)\right)dt \\ &= -\int_0^\infty \frac{e^{-at}}{2(t+1)}dt - \int_0^\infty te^{-at}\left(\psi(t) + \frac{1}{2t} - \ln t\right)dt \\ &= -\frac{e^a}{2}\Gamma(0,a) - I_3, \text{ say.}\end{aligned} \tag{2.13}$$

Now using a well-known formula [9, p. 248] in a slightly simplified form, we have for Re $z > 0$,

$$\psi(z) + \frac{1}{2z} - \ln z = -\int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) e^{-tz}\, dt. \tag{2.14}$$

Thus,

$$\begin{aligned} I_3 &= -\int_0^\infty te^{-at} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) e^{-xt}\, dx\, dt \\ &= -\int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) \int_0^\infty te^{-(x+a)t}\, dt\, dx \\ &= -\int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) \frac{dx}{(x+a)^2}. \end{aligned} \tag{2.15}$$

The inversion of the order of integration above is easily justifiable. Next from [3, p. 191], we have for $x \neq 0$,

$$\sum_{n=1}^\infty \frac{1}{x^2 + 4n^2\pi^2} = \frac{1}{2x}\left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right). \tag{2.16}$$

Hence,

$$\begin{aligned} I_3 &= -\int_0^\infty \frac{2x\, dx}{(x+a)^2} \sum_{n=1}^\infty \frac{1}{x^2 + 4n^2\pi^2} \\ &= -\frac{1}{a} \sum_{n=1}^\infty \int_0^\infty \frac{2ax}{(x+a)^2(x^2 + 4n^2\pi^2)}\, dx \\ &= -\frac{1}{a} \sum_{n=1}^\infty \int_0^\infty \frac{(x^2 + 2ax + a^2) - (x^2 + a^2)}{(x+a)^2(x^2 + 4n^2\pi^2)}\, dx \\ &= -\frac{1}{a} \sum_{n=1}^\infty \left(\int_0^\infty \frac{dx}{x^2 + 4n^2\pi^2} - \int_0^\infty \frac{x^2 + a^2}{(x+a)^2(x^2 + 4n^2\pi^2)}\, dx\right), \end{aligned} \tag{2.17}$$

where the inversion of integration and summation can be easily seen via absolute convergence. Now using Mathematica, we find that

$$\begin{aligned} &\int_0^\infty \frac{x^2 + a^2}{(x+a)^2(x^2 + 4n^2\pi^2)}\, dx \\ &= \frac{a^4 - 8a^2n^2\pi^2 + 32an^3\pi^2 + 16n^4\pi^4 + 8a^3n(1 + \ln(2)) + 32an^3\pi^2 \ln(a/2n\pi) + 8a^3n \ln(n\pi/a)}{4n(a^2 + 4n^2\pi^2)^2} \\ &= \frac{(a^2 + 4n^2\pi^2)^2 - 16a^2n^2\pi^2 + 8a^3n(1 + \ln(2\pi n/a)) + 32an^3\pi^2(1 - \ln(2\pi n/a))}{4n(a^2 + 4n^2\pi^2)^2}. \end{aligned} \tag{2.18}$$

Hence from (2.17) and (2.18), we deduce after some simplification that

$$\begin{aligned} I_3 &= \frac{1}{4a}\sum_{n=1}^{\infty}\frac{32an^3\pi^2(1-\ln(2\pi n/a))+8a^3n(1+\ln(2\pi n/a))-16a^2n^2\pi^2}{n(a^2+4n^2\pi^2)^2} \\ &= S_1+S_2+S_3+S_4-S_5, \end{aligned} \tag{2.19}$$

where

$$S_1 = 8\pi^2\sum_{n=1}^{\infty}\frac{n^2}{(a^2+4n^2\pi^2)^2}, \tag{2.20}$$

$$S_2 = \frac{\ln(2\pi/a)}{4a}\sum_{n=1}^{\infty}\frac{8a^3n-32an^3\pi^2}{n(a^2+4n^2\pi^2)^2}, \tag{2.21}$$

$$S_3 = \frac{1}{4a}\sum_{n=1}^{\infty}\frac{(8a^3n-32an^3\pi^2)\ln n}{n(a^2+4n^2\pi^2)^2}, \tag{2.22}$$

$$S_4 = 2a^2\sum_{n=1}^{\infty}\frac{1}{(a^2+4n^2\pi^2)^2}, \tag{2.23}$$

and

$$S_5 = 4a\pi^2\sum_{n=1}^{\infty}\frac{n}{(a^2+4n^2\pi^2)^2}. \tag{2.24}$$

Decomposing S_1 into partial fractions,

$$\begin{aligned} S_1 &= 2\sum_{n=1}^{\infty}\frac{1}{a^2+4n^2\pi^2} - 2a^2\sum_{n=1}^{\infty}\frac{1}{(a^2+4n^2\pi^2)^2} \\ &= 2\sum_{n=1}^{\infty}\frac{1}{a^2+4n^2\pi^2} + a\sum_{n=1}^{\infty}\frac{d}{da}\left(\frac{1}{a^2+4n^2\pi^2}\right) \\ &= 2\sum_{n=1}^{\infty}\frac{1}{a^2+4n^2\pi^2} + a\frac{d}{da}\left(\sum_{n=1}^{\infty}\frac{1}{a^2+4n^2\pi^2}\right), \end{aligned} \tag{2.25}$$

and then using (2.16), we find after some simplification that

$$S_1 = \frac{2\coth(a/2)-a/\sinh^2(a/2)}{8a}. \tag{2.26}$$

Similarly, breaking S_2 into partial fractions and then employing (2.16) and (2.26), we find upon simplification that

$$S_2 = \frac{\ln(2\pi/a)}{4a}\left(\frac{a}{\sinh^2(a/2)}-\frac{4}{a}\right). \tag{2.27}$$

The sum S_4 is similarly evaluated as

$$S_4 = \frac{4+a^2+a\sinh(a)-4\cosh(a)}{8a^2\sinh^2(a/2)}. \tag{2.28}$$

Hence from (2.26), (2.27) and (2.28), and using the identities

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$
$$\cosh(2x) = 1 + 2\sinh^2(x),$$

we find upon simplification that

$$S_1 + S_2 + S_4 = \frac{-4 + 2a\coth(a/2) + (-4 + a^2/\sinh^2(a/2))\ln(2\pi/a)}{4a^2}. \tag{2.29}$$

Thus, from (2.19), (2.22), (2.24) and (2.29),

$$\begin{aligned} I_3 = & \frac{-4 + 2a\coth(a/2) + (-4 + a^2/\sinh^2(a/2))\ln(2\pi/a)}{4a^2} \\ & + \sum_{n=1}^{\infty} \frac{2(a^2 + 4n^2\pi^2)\ln n - 4an\pi^2 - 16n^2\pi^2\ln n}{(a^2 + 4n^2\pi^2)^2} \\ = & \frac{-4 + 2a\coth(a/2) + (-4 + a^2/\sinh^2(a/2))\ln(2\pi/a)}{4a^2} \\ & + 2\sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} - 4\pi^2 \sum_{n=1}^{\infty} \frac{an}{(a^2 + 4n^2\pi^2)^2} - 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2\ln n}{(a^2 + 4n^2\pi^2)^2}. \end{aligned} \tag{2.30}$$

Thus from (2.13) and (2.30), we have

$$\begin{aligned} I_1'(a) = & -\frac{e^a}{2}\Gamma(0,a) - \frac{-4 + 2a\coth(a/2) + (-4 + a^2/\sinh^2(a/2))\ln(2\pi/a)}{4a^2} \\ & - 2\sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} + 4\pi^2 \sum_{n=1}^{\infty} \frac{an}{(a^2 + 4n^2\pi^2)^2} + 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2\ln n}{(a^2 + 4n^2\pi^2)^2}. \end{aligned} \tag{2.31}$$

Now using the fact [6, p. 953, formula 8.363, no. 8]

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2} \tag{2.32}$$

to integrate the second to last series in (2.31), we deduce that

$$\begin{aligned} I_1(a) = & -\frac{e^a}{2}\Gamma(0,a) - \frac{1}{2}\ln(a) + \frac{(-2 + a\coth(a/2))\ln(2\pi/a)}{2a} \\ & - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\ln n \tan^{-1}(a/2n\pi)}{n} + \frac{4\pi^2}{16\pi^2}\left(\psi\left(1 + \frac{ia}{2\pi}\right) + \psi\left(1 - \frac{ia}{2\pi}\right)\right) \\ & + 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2 \ln n \left(\tan^{-1}\left(\frac{a}{2n\pi}\right) + \frac{2an\pi}{a^2 + 4n^2\pi^2}\right)}{16n^3\pi^3} + c, \end{aligned} \tag{2.33}$$

where c is a constant of integration. Thus after simplification, we obtain

$$\begin{aligned} I_1(a) = & -\frac{e^a}{2}\Gamma(0,a) - \frac{1}{2}\ln(a) + \frac{(-2 + a\coth(a/2))\ln(2\pi/a)}{2a} \\ & + \frac{1}{4}\left(\psi\left(1 + \frac{ia}{2\pi}\right) + \psi\left(1 - \frac{ia}{2\pi}\right)\right) + 2a\sum_{n=1}^{\infty} \frac{\ln n}{a^2 + 4n^2\pi^2} + c. \end{aligned} \tag{2.34}$$

Let $a \to 0$ in (2.34). Taking the limit under the integral sign on the left-hand side of (2.34), using Lebesgue's dominated convergence theorem, and then using Lemma 2.1 and the fact that $\psi(1) = -\gamma$,

$$\frac{1}{2}\ln 2\pi = -\frac{1}{2}\lim_{a\to 0}\left(e^a\Gamma(0,a) + \ln(a)\right) + \lim_{a\to 0}\left(\frac{(-2 + a\coth(a/2))\ln(2\pi/a)}{2a}\right) - \frac{\gamma}{2} + c. \tag{2.35}$$

From [6, p. 951, formula 8.359, no. 1], we find that

$$\Gamma(0,x) = -\text{Ei}(-x). \tag{2.36}$$

Also, from [6, p. 935, formula 8.214, no. 1], for $x < 0$,

$$\text{Ei}(x) = \gamma + \ln(-x) + \sum_{k=1}^{\infty}\frac{x^k}{k\cdot k!}. \tag{2.37}$$

From (2.36) and (2.37), we readily get the asymptotic expansion for the incomplete gamma function as

$$\Gamma(0,x) = -\gamma - \ln x + x - \frac{x^2}{4} + \frac{x^3}{18} - \cdots, \tag{2.38}$$

as $x \to 0^+$. Hence,

$$\lim_{a\to 0}\left(e^a\Gamma(0,a) + \ln(a)\right) = -\gamma. \tag{2.39}$$

Next, using the series expansion of $\coth x$ [6, p. 42, formula 1.411, no. 8],

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots, \quad \text{for } 0 < x^2 < \pi^2, \tag{2.40}$$

we deduce that

$$\begin{aligned}
&\lim_{a\to 0}\frac{(-2 + a\coth(a/2))\ln(2\pi/a)}{2a} \\
&= \lim_{a\to 0}\left[\frac{1}{2a}\left(-2 + a\left(\frac{2}{a} + \frac{a}{6} - \frac{a^3}{360} + \cdots\right)\right)\ln\left(\frac{2\pi}{a}\right)\right] \\
&= \lim_{a\to 0}\left[\left(\frac{a}{12} - \frac{a^3}{720} + \cdots\right)\ln 2\pi - \left(\frac{a}{12} - \frac{a^3}{720} + \cdots\right)\ln a\right] \\
&= 0.
\end{aligned} \tag{2.41}$$

Thus from (2.35), (2.39) and (2.41), we obtain

$$c = \frac{1}{2}\ln 2\pi. \tag{2.42}$$

Now substituting (2.42) in (2.34), we have

$$\begin{aligned}
I_1(a) = &-\frac{e^a}{2}\Gamma(0,a) - \frac{1}{2}\ln(a) + \frac{(-2 + a\coth(a/2))\ln(2\pi/a)}{2a} + 2a\sum_{n=1}^{\infty}\frac{\ln n}{a^2 + 4n^2\pi^2} \\
&+ \frac{1}{4}\left(\psi\left(1 + \frac{ia}{2\pi}\right) + \psi\left(1 - \frac{ia}{2\pi}\right)\right) + \frac{1}{2}\ln 2\pi.
\end{aligned} \tag{2.43}$$

From (2.9), (2.12) and (2.43), we have

$$\begin{aligned}\int_0^{\infty} e^{-at}\psi(t+1)\,dt =&\frac{(-2+a\coth(a/2))\ln(2\pi/a)}{2a}+2a\sum_{n=1}^{\infty}\frac{\ln n}{a^2+4n^2\pi^2}\\ &+\frac{1}{4}\left(\psi\left(1+\frac{ia}{2\pi}\right)+\psi\left(1-\frac{ia}{2\pi}\right)\right)+\frac{1}{2}\ln\left(\frac{2\pi}{a}\right)\\ &-\frac{\gamma+\ln a}{a}. \end{aligned} \tag{2.44}$$

Finally (2.44) simplifies to (1.5). This completes the proof of Theorem 1.1.

3. Second Proof

In [1, Cor. 2.2], Amdeberhan, Espinosa and Moll showed that

$$\int_0^{\infty} e^{-at}\ln\Gamma(t)\,dt=-\frac{\gamma+\ln a}{ace^a}+\frac{A(a-c)}{a^2c}-\frac{1}{2a}\Lambda\left(\frac{a}{2\pi}\right)+2\sum_{j=1}^{\infty}\frac{\ln j}{a^2+4\pi^2j^2}, \tag{3.1}$$

where $a>0$, $c=1-e^{-a}$, $A=\ln 2\pi+\gamma$ and

$$\Lambda(z)=\lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{j}{j^2+z^2}-\ln n\right), \tag{3.2}$$

a generalization of Euler's constant. Now

$$\int_0^{\infty} e^{-at}\ln\Gamma(t+1)\,dt=\int_0^{\infty} e^{-at}\ln\Gamma(t)\,dt+\int_0^{\infty} e^{-at}\ln t\,dt. \tag{3.3}$$

and

$$\begin{aligned}\int_0^{\infty} e^{-at}\psi(t+1)\,dt &= \int_0^{\infty}\frac{d}{dt}\left(e^{-at}\ln\Gamma(t+1)\right)\,dt+a\int_0^{\infty} e^{-at}\ln\Gamma(t+1)\,dt\\ =\lim_{t\to\infty}e^{-at}\ln\Gamma(t+1)&-\lim_{t\to 0}e^{-at}\ln\Gamma(t+1)+a\int_0^{\infty} e^{-at}\ln\Gamma(t+1)\,dt. \end{aligned}\tag{3.4}$$

Now using (2.1), we find that

$$\begin{aligned}\lim_{t\to\infty}e^{-at}\ln\Gamma(t+1) &= \lim_{t\to\infty}\frac{\psi(t+1)}{ae^{at}}\\ &=\frac{1}{a}\lim_{t\to\infty}\frac{1}{e^{at}}\left(\ln(t+1)-\frac{1}{2(t+1)}-\frac{1}{12(t+1)^2}+\cdots\right)\\ &=\frac{1}{a}\lim_{t\to\infty}\frac{\ln(t+1)}{e^{at}}\\ &=\frac{1}{a^2}\lim_{t\to\infty}\frac{1}{e^{at}(t+1)}\\ &=0. \end{aligned}\tag{3.5}$$

Also,

$$\lim_{t\to 0}e^{-at}\ln\Gamma(t+1)=0. \tag{3.6}$$

Thus using (3.4), (3.5), (3.6), (3.3), (3.1) and (2.11), we find that

$$\begin{aligned}\int_0^\infty e^{-at}\psi(t+1)\,dt &= a\int_0^\infty e^{-at}\ln\Gamma(t)\,dt + a\int_0^\infty e^{-at}\ln t\,dt\\ &= a\left(-\frac{\gamma+\ln a}{ace^a}+\frac{A(a-c)}{a^2c}-\frac{1}{2a}\Lambda\left(\frac{a}{2\pi}\right)+2\sum_{j=1}^\infty\frac{\ln j}{a^2+4\pi^2j^2}\right)-(\gamma+\ln a)\\ &= -\frac{\gamma+\ln a}{e^a-1}+\frac{A(a-c)}{ac}-\frac{1}{2}\Lambda\left(\frac{a}{2\pi}\right)+2a\sum_{j=1}^\infty\frac{\ln j}{a^2+4\pi^2j^2}-(\gamma+\ln a)\,. \qquad (3.7)\end{aligned}$$

Now we show that

$$\Lambda\left(\frac{a}{2\pi}\right)=\lim_{n\to\infty}\left(\sum_{j=1}^n\frac{4\pi^2 j}{a^2+4\pi^2j^2}-\ln n\right)=\frac{-1}{2}\left(\psi\left(\frac{ia}{2\pi}\right)+\psi\left(-\frac{ia}{2\pi}\right)\right). \qquad (3.8)$$

A series representation for the ψ function [6, p. 952, formula 8.632, no. 1] gives

$$\psi(x)=-\gamma-\sum_{j=0}^\infty\left(\frac{1}{x+j}-\frac{1}{j+1}\right). \qquad (3.9)$$

Thus, using (3.9), we find that

$$\begin{aligned}\psi\left(\frac{ia}{2\pi}\right)+\psi\left(-\frac{ia}{2\pi}\right) &= -2\gamma-\sum_{j=0}^\infty\left(\frac{1}{j+ia/2\pi}-\frac{1}{j+1}\right)-\sum_{j=0}^\infty\left(\frac{1}{j-ia/2\pi}-\frac{1}{j+1}\right)\\ &= -2\gamma-2\sum_{j=0}^\infty\left(\frac{j}{j^2+a^2/4\pi^2}-\frac{1}{j+1}\right)\\ &= 2-2\left(\sum_{j=1}^\infty\frac{j}{j^2+a^2/4\pi^2}-\frac{1}{j+1}+\gamma\right)\\ &= 2-2\lim_{n\to\infty}\left(\sum_{j=1}^n\left(\frac{1}{j}-\frac{1}{j+1}\right)+\sum_{j=1}^n\frac{j}{j^2+a^2/4\pi^2}-\ln n\right)\\ &= 2-2\lim_{n\to\infty}\left(1+4\pi^2\sum_{j=1}^n\frac{j}{a^2+4\pi^2j^2}-\ln n\right)\\ &= -2\lim_{n\to\infty}\left(\sum_{j=1}^n\frac{4\pi^2 j}{a^2+4\pi^2j^2}-\ln n\right), \qquad (3.10)\end{aligned}$$

where in the antepenultimate line, we have made use of the limit definition of γ,

$$\gamma=\lim_{n\to\infty}\left(\sum_{j=1}^n\frac{1}{j}-\ln n\right). \qquad (3.11)$$

This proves (3.8). Finally using (3.8), we see that after a routine simplification, (3.7) is equivalent to (1.5). This finishes the second proof.

4. Oloa's conjecture

In [8], Olivier Oloa conjectured the following:

Let $\alpha > 0$ and
$$I(\alpha) = \int_0^{\pi/2} \frac{\theta^2}{\theta^2 + \ln^2(2\alpha\cos\theta)}\, d\theta. \tag{4.1}$$
If $\alpha < 1/2$, then
$$\frac{4}{\pi} I(\alpha) = -\frac{\gamma}{\ln\alpha} - \frac{\gamma}{1-\alpha} - \frac{1}{1-\alpha}\ln(-\ln\alpha) - \frac{\ln\alpha}{1-\alpha}\int_0^1 \alpha^t \ln\Gamma(t)\, dt, \tag{4.2}$$
and if $\alpha \geq 1/2$, then
$$\frac{4}{\pi} I(\alpha) = -\frac{\gamma}{\ln\alpha} - \frac{\gamma}{1-\alpha} - \frac{1}{1-\alpha}\ln\left(\frac{\alpha\ln\alpha}{\alpha-1}\right) + \frac{\ln\alpha}{\alpha-1}\int_0^1 \alpha^t \ln\Gamma(t)\, dt. \tag{4.3}$$

The second part of the conjecture, namely (4.3), was proved in [1]. Here, using the expression for the Laplace transform of the Psi function that we have obtained, we are able to prove (4.2), namely the first part of the conjecture as well. First, we explicitly evaluate $M(a)$ when $a > \ln 2$, which is then subsequently used to prove (4.2). To that end, from (1.4) and (1.5), we readily obtain
$$\begin{aligned} M(a) &= \left(\frac{1}{e^a - 1} - \frac{1}{a} + 1\right)\ln\left(\frac{2\pi}{a}\right) + 2a\sum_{n=1}^{\infty}\frac{\ln n}{a^2 + 4n^2\pi^2} \\ &\quad + \frac{1}{4}\left(\psi\left(\frac{ia}{2\pi}\right) + \psi\left(-\frac{ia}{2\pi}\right)\right) - \frac{\ln a}{a}. \end{aligned} \tag{4.4}$$
In [1], the following evaluation for $M(a)$ was given for $0 < a \leq \ln 2$:
$$M(a) = \frac{\gamma}{a} + \frac{a + \ln(1-e^{-a}) - \gamma - \ln a}{1-e^{-a}} + \frac{a}{1-e^{-a}}\int_0^1 e^{-at}\ln\Gamma(t)\, dt. \tag{4.5}$$
Also Lemma 2.3 in [1] states that for $a > 0$, $c = 1 - e^{-a}$ and $A = \ln 2\pi + \gamma$, we have
$$\int_0^1 e^{-at}\ln\Gamma(t)\, dt = \frac{A(a-c)}{a^2} - \frac{c}{2a}\Lambda\left(\frac{a}{2\pi}\right) + 2c\sum_{j=1}^{\infty}\frac{\ln j}{a^2 + 4\pi^2 j^2}, \tag{4.6}$$
where a, c and A are defined as in (3.1) and $\Lambda(z)$ is defined in (3.2).

Thus from (4.5) and (4.6), $M(a)$ can be evaluated for $0 < a \leq \ln 2$, which combined with (4.4) implies that the integral $M(a)$ can be evaluated for all positive values of a.

Next we proceed to prove (4.2). First of all, letting $\alpha = e^{-a}$ in (4.1), we see that (4.2) is equivalent to showing
$$M(a) = \frac{\gamma}{a} - \frac{\gamma + \ln a}{1-e^{-a}} + \frac{a}{1-e^{-a}}\int_0^1 e^{-at}\ln\Gamma(t)\, dt. \tag{4.7}$$

From (4.4) and (4.7), it suffices to show that

$$\frac{1}{4}\left(\psi\left(\frac{ia}{2\pi}\right)+\psi\left(-\frac{ia}{2\pi}\right)\right)+2a\sum_{n=1}^{\infty}\frac{\ln n}{a^2+4n^2\pi^2}+\left(\frac{1}{e^a-1}-\frac{1}{a}+1\right)\ln 2\pi$$
$$=\frac{\gamma}{a}-\frac{\gamma}{1-e^{-a}}+\frac{a}{1-e^{-a}}\int_0^1 e^{-at}\ln\Gamma(t)\,dt. \tag{4.8}$$

Simplifying (4.8) leads to

$$\int_0^1 e^{-at}\ln\Gamma(t)\,dt=\frac{1-e^{-a}}{a}\left[\frac{1}{4}\left(\psi\left(\frac{ia}{2\pi}\right)+\psi\left(-\frac{ia}{2\pi}\right)\right)+2a\sum_{n=1}^{\infty}\frac{\ln n}{a^2+4n^2\pi^2}\right.$$
$$\left.-\frac{\gamma}{a}+\frac{\gamma}{1-e^{-a}}+\frac{a}{1-e^{-a}}+\left(\frac{1}{e^a-1}-\frac{1}{a}+1\right)\ln 2\pi\right]. \tag{4.9}$$

Finally using (3.8), after a routine simplification, we find that (4.9) is equivalent to (4.6). This finishes the proof of (4.2).

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