MONOTONICITY RESULTS FOR DIRICHLET L-FUNCTIONS

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Abstract. We present some monotonicity results for Dirichlet L-functions associated to real primitive characters. We show in particular that these Dirichlet L-functions are far from being logarithmically completely monotonic. Also, we show that, unlike in the case of the Riemann zeta function, the problem of comparing the signs of $\frac{d^k}{ds^k}\log L(s,\chi)$ at any two points $s_1, s_2 > 1$ is more subtle.

1. Introduction

A function f is said to be completely monotonic on $[0,\infty)$ if $f \in C[0,\infty)$, $f \in C^\infty(0,\infty)$ and $(-1)^k f^{(k)}(t) \geq 0$ for $t > 0$ and $k = 0, 1, 2\cdots$, i.e., the successive derivatives alternate in sign. The following theorem due to S.N. Bernstein and D. Widder gives a complete characterization of completely monotonic functions [10, p. 95]:

A function $f : [0,\infty) \to [0,\infty)$ is completely monotonic if and only if there exists a non-decreasing bounded function γ such that $f(t) = \int_0^\infty e^{-st} d\gamma(s)$.

Lately, the class of completely monotonic functions have been greatly expanded to include several special functions, for example, functions associated to gamma and psi functions by Chen [9], Guo, Guo and Qi [15] and quotients of K-Bessel functions by Ismail [16]. A conjecture that certain quotients of Jacobi theta functions are completely monotonic was formulated by the first author and Solynin in [12], and slightly corrected later by the present authors in [13]. Certain other classes of such functions were introduced by Alzer and Berg [1], Qi and Chen [22]. Completely monotonic functions have applications in diverse fields such as probability theory [17], physics [4], potential theory [6], combinatorics [3] and numerical and asymptotic analysis [14], to name a few.

A close companion to the class of completely monotonic functions is the class of logarithmically completely monotonic functions. This was first studied, although implicitly, by Alzer and Berg [2]. A function $f : (0,\infty) \to (0,\infty)$ is said to be logarithmically completely monotonic [5] if it is C^∞ and $(-1)^k[\log f(x)]^{(k)} \geq 0$, for $k = 0, 1, 2, 3, \cdots$. Moreover, a function is said to be strictly logarithmically completely monotonic if $(-1)^k[\log f(x)]^{(k)} > 0$. The following is true:

2010 *Mathematics Subject Classification.* Primary 11M06, 11M26.

Keywords and phrases. Dirichlet L-function, complete monotonicity, logarithmically complete monotonicity.

Every logarithmic completely monotonic function is completely monotonic.

The reader is referred to Alzer and Berg [2], Qi and Guo [20], and Qi, Guo and Chen [21] for proofs of this statement.

One goal of this paper is to study the Dirichlet L-functions from the point of view of logarithmically complete monotonicity. For Re $s>1$, the Riemann zeta function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Consider $s>1$. Since $\log\zeta(s)>0$ and

$$(-1)^k\frac{d^k}{ds^k}\log\zeta(s)=(-1)^k\frac{d^{k-1}}{ds^{k-1}}\left(\frac{\zeta'(s)}{\zeta(s)}\right)=\sum_{n=1}^{\infty}\frac{\Lambda(n)(\log n)^{k-1}}{n^s},$$

where $\Lambda(n)\geq 0$ is the von Mangoldt function, $(-1)^k\frac{d^k}{ds^k}\log\zeta(s)>0$ for all $s>1$. This implies that $\zeta(s)$ is a logarithmically completely monotonic function for $s>1$ (in fact, strictly logarithmically completely monotonic). But this approach fails in the case of $L(s,\chi)$ with $s>1$ and χ, a real primitive Dirichlet character modulo q, since

$$(-1)^k\frac{d^k}{ds^k}\log L(s,\chi)=(-1)^k\frac{d^{k-1}}{ds^{k-1}}\left(\frac{L'(s,\chi)}{L(s,\chi)}\right)=\sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)(\log n)^{k-1}}{n^s}$$

may change sign for different values of s as $\chi(n)$ takes the values $-1,0$ or 1. Hence, we need to consider a different approach for studying $L(s,\chi)$ in the context of logarithmically complete monotonicity. This naturally involves studying the zeros of derivatives of $\log L(s,\chi)$.

There have been several studies made on the number of zeros of $\zeta^{(k)}(s)$ and $L^{(k)}(s,\chi)$, one of which dates back to Spieser [23], who showed that the Riemann Hypothesis is equivalent to the fact that $\zeta'(s)$ has no zeros in $0<$ Re $s<1/2$. Spira [24] conjectured that

$$N(T)=N_k(T)+\left[\frac{T\log 2}{2\pi}\right]\pm 1,$$

where $N_k(T)$ denotes the number of zeros of $\zeta^{(k)}(s)$ with positive imaginary parts up to height T, and $N(T)=N_0(T)$. Berndt [7] showed that for any $k\geq 1$, as $T\to\infty$,

$$N_k(T)=\frac{T\log T}{2\pi}-\left(\frac{1+\log 4\pi}{2\pi}\right)T+O(\log T).$$

Levinson and Montgomery [18] proved a quantitative result implying that most of the zeros of $\zeta^{(k)}(s)$ are clustered about the line Re $s=1/2$ and also showed that the Riemann Hypothesis implies that $\zeta^{(k)}(s)$ has at most finitely many non-real zeros in Re $s<1/2$. Their results were further improved by Conrey and Ghosh [8]. Analogues of several of the above-mentioned results for Dirichlet L-functions were given by Yildirim [30]. Our results in this paper are related to the zeros of $\log L(s,\chi)$ and its derivatives.

Throughout the paper, we assume that s is a real number and χ is a real primitive Dirichlet character modulo q. Let $F(s,\chi) := \log L(s,\chi)$, and for $s > 1$, define

$$A_{\chi,k} := \{s : F^{(k)}(s,\chi) = 0\}. \tag{1.1}$$

Then we obtain the following result:

Theorem 1.1. *Let χ be a real primitive character modulo q and $L(s,\chi) \neq 0$ for $0 < s < 1$. Then there exists a constant c_χ such that $[c_\chi,\infty) \cap (\cup_{k=1}^{\infty} A_{\chi,k})$ is dense in $[c_\chi,\infty)$.*

Let us note that Theorem 1.1 shows in particular that $L(s,\chi)$ is not logarithmically completely monotonic on any subinterval of $[c_\chi,\infty)$. A stronger assertion is as follows:
For any subinterval of $[c_\chi,\infty)$, however small it may be, infinitely many derivatives $F^{(k)}(s,\chi)$ change sign in this subinterval.

Now consider any two points s_1, s_2 with $1 < s_1 < s_2$. In the case of the Riemann zeta function, if we compare the signs of the values of $\frac{d^k}{ds^k}\log\zeta(s)$ at s_1 and s_2 for all values of k, we see that they are always the same. Then a natural question arises - what can we say if we make the same comparison in the case of a Dirichlet L-function? We will see below that the answer is completely different (actually it is as different as it could be). We first define a function ψ_χ for a real primitive Dirichlet character modulo q as follows:

Let $\mathcal{B} := \{g : \mathbb{N} \to \{-1,0,1\}\}$. Define an equivalence relation $\sim$ on $\mathcal{B}$ by $g \sim h$ if and only if $g(n) = h(n)$ for all n large enough. Let $\hat{\mathcal{B}} = \mathcal{B}/\sim$. By abuse of notation, we define $\psi_\chi : (1,\infty) \to \hat{\mathcal{B}}$ to be a function whose image is a sequence given by $\{\operatorname{sgn}(F^{(k)}(s,\chi))\}$, i.e.,

$$\psi_\chi(s)(k) := \operatorname{sgn}(F^{(k)}(s,\chi)). \tag{1.2}$$

With this definition, we answer the above question in the form of the following theorem.

Theorem 1.2. *Let χ be a real primitive character modulo q and let ψ_χ be defined as above. Then there exists a constant C_χ with the following property:*

(a) *The Riemann hypothesis for $L(s,\chi)$ implies that ψ_χ is injective on $[C_\chi,\infty)$.*

(b) *Let ψ_χ be injective on $[C_\chi,\infty)$. Then there exists an effectively computable constant D_χ such that if all the nontrivial zeros ρ of $L(s,\chi)$ up to the height D_χ lie on the critical line Re $s = 1/2$, then the Riemann Hypothesis for $L(s,\chi)$ is true.*

2. Proof of theorem 1.1

First we will compute $F^{(k)}(s,\chi)$ in terms of the zeros of $L(s,\chi)$. The logarithmic derivative of $L(s,\chi)$ satisfies [11, page. 83]

$$F'(s,\chi) = \frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'(s/2+b/2)}{\Gamma(s/2+b/2)} + B(\chi) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{2.1}$$

where $B(\chi)$ is a constant depending on χ,

$$b = \begin{cases} 1 & \text{if } \chi(-1) = -1 \\ 0 & \text{if } \chi(-1) = 1 \end{cases}, \tag{2.2}$$

and $\rho = \beta + i\gamma$ are the non trivial zeros of $L(s, \chi)$. Since $B(\overline{\chi}) = \overline{B(\chi)}$ and χ is real, $B(\chi)$ is given by

$$B(\chi) = -\sum_{\rho} \frac{1}{\rho} = -2\sum_{\gamma>0} \frac{\beta}{\beta^2 + \gamma^2} < \infty,$$

see [11, page. 83]. Note that $B(\chi)$ is negative. The Weierstrass infinite product for $\Gamma(s)$ is [11, p. 73]

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} (1 + s/n)^{-1} e^{s/n}, \tag{2.3}$$

with $s = 0, -1, -2, \ldots$ being its simple poles. The functional equation for $\Gamma(s)$ is

$$\Gamma(s+1) = s\Gamma(s) \tag{2.4}$$

$$\tag{2.5}$$

where as the duplication formula for $\Gamma(s)$ is

$$\Gamma(s)\Gamma(s + 1/2) = 2^{(1-2s)} \pi^{1/2} \Gamma(2s), \tag{2.6}$$

see [11, p. 73]. The following can be easily derived from (2.4), (2.6) and the logarithmic derivative of (2.3):

$$\frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} = -\frac{\gamma}{2} - \frac{1}{s} - \sum_{n=1}^{\infty} \left(\frac{1}{s + 2n} - \frac{1}{2n} \right), \tag{2.7}$$

$$\frac{1}{2} \frac{\Gamma'(s/2 + 1/2)}{\Gamma(s/2 + 1/2)} = -\log(2) - \frac{\gamma}{2} - \sum_{n=0}^{\infty} \left(\frac{1}{s + 2n + 1} - \frac{1}{2n + 1} \right), \tag{2.8}$$

From (2.1), (2.2), (2.7) and (2.8), we have

$$F'(s, \chi) = -\frac{1}{2} \log \frac{q}{\pi} + b \log 2 + \frac{\gamma}{2} + B(\chi) + \frac{1 - b}{s} + \sum_{\rho \neq 0} \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right), \tag{2.9}$$

where ρ runs through all the zeros of $L(s, \chi)$. The successive differentiation of (2.9) gives for $k \geq 2$,

$$\begin{aligned} F^{(k)}(s, \chi) &= (-1)^{k-1} (k-1)! \left(\frac{1 - b}{s^k} + \sum_{\substack{\rho \neq 0 \\ L(\rho, \chi) = 0}} \frac{1}{(s - \rho)^k} \right) \\ &= (-1)^{k-1} (k-1)! \left(\sum_{L(\rho, \chi) = 0} \frac{1}{(s - \rho)^k} \right). \end{aligned} \tag{2.10}$$

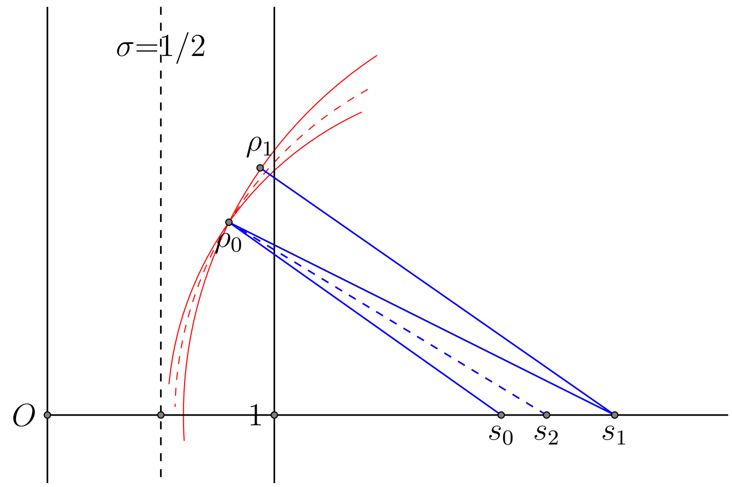

FIGURE 1. Construction for identifying the unique ρ_0 at which $l(s)$ is attained for $s \in (s_0 - \epsilon, s_0 + \epsilon)$.

Let $s > 1/2$ and define

$$l(s) := \min\{|s - \rho| : L(\rho, \chi) = 0\}. \tag{2.11}$$

If the minimum $l(s)$ is attained for a non-trivial zero ρ of $L(s, \chi)$, then since the non-trivial zeros are symmetric with respect to the line $\sigma = 1/2$, we have $\operatorname{Re}\rho \geq 1/2$. Let $\tilde{\rho}_0$ be the non-trivial zero of $L(s, \chi)$ with minimum but positive imaginary part, i.e., $\operatorname{Im}\tilde{\rho}_0 = \min\{\operatorname{Im}\rho > 0 : L(\rho, \chi) = 0, \operatorname{Re}\rho \geq 1/2\}$. Write $\tilde{\rho}_0 = \tilde{\beta}_0 + i\tilde{\gamma}_0$. Then for all $s > \tilde{\gamma}_0^2 + 1/4$, we have $s^2 > (s - 1/2)^2 + \tilde{\gamma}_0^2 \geq |s - \tilde{\rho}_0|^2 \geq (l(s))^2$. Define

$$c_\chi := \operatorname{Inf}\{c > 1 : s > c \Rightarrow |s| > l(s)\}. \tag{2.12}$$

The constant c_χ is defined in this way since we want $l(s)$ to be attained at a non-trivial zero of $L(s, \chi)$, as this will allow us to separate the two terms of the series in (2.10) corresponding to this zero and its conjugate, which together will give a dominating term essential in the proof. Note that if $\tilde{\gamma}_0 \leq \sqrt{3}/2$, $c_\chi = 1$, otherwise $1 \leq c_\chi \leq \tilde{\gamma}_0^2 + 1/4$.

Next we show that for any $s \geq c_\chi$, there is an $s' \in (s - \epsilon, s + \epsilon)$, $\epsilon > 0$, so that $l(s')$ is attained at a unique non-trivial zero ρ' of $L(s, \chi)$ with $\operatorname{Im}\rho' > 0$.

For any real number $s_0 > c_\chi$, consider the interval $(s_0 - \epsilon, s_0 + \epsilon) \subset [c_\chi, \infty)$ for some $\epsilon > 0$. Let

$$A := \{\rho' : \operatorname{Im}\rho' \geq 0 \text{ and } |s_0 - \rho'| = l(s_0), L(\rho', \chi) = 0\}, \tag{2.13}$$

that is, A is comprised of all non-trivial zeros on the circle with center s_0 and radius $l(s_0)$. Clearly A is a finite set since $|A| \leq \mathrm{N}(l(s_0), \chi)$, where $N(T, \chi)$ denotes the number of zeros of $L(s, \chi)$ up to height T. As shown in Figure 1, let $\rho_0 \in A$ with $\operatorname{Re}\rho_0 = \max\{\operatorname{Re}\rho : \rho \in A\}$. Then for any $s \in (s_0, s_0 + \epsilon)$, $|s - \rho_0| < |s - \rho|$, for all $\rho \in A$, $\rho \neq \rho_0$. Fix one such s, say s_1, so that $s_0 < s_1 < s_0 + \epsilon$. Now more than one zeros may lie on the circle with center s_1 and radius $|s_1 - \rho_0|$. If there aren't any (apart from ρ_0), then we have constructed $s' (= s_1)$ that we sought. If there are more than one, we select the one among them, say ρ_1, which has the minimum real part, i.e.,

$\operatorname{Re}\rho_1 = \min\{\operatorname{Re}\rho : |s_1-\rho_0| = |s_1-\rho|, \rho \neq \rho_0, L(\rho,\chi)=0\}$. Note that $\operatorname{Im}\rho_1 > \operatorname{Im}\rho_0$, otherwise it will contradict the fact that the minimum $l(s_0)$ is attained at ρ_0.

For any $s \in (s_0, s_1)$, $|s-\rho_0| < |s-\rho_1|$. Now fix one such s, say $s_2 \in (s_0, s_1)$, and find a ρ_2 so that $\operatorname{Re}\rho_2 = \min\{\operatorname{Re}\rho : |s_2-\rho_0| = |s_2-\rho|, \rho \neq \rho_0, L(\rho,\chi)=0\}$. Since there are only finite may zeros in the rectangle $[0,1]\times[\operatorname{Im}\rho_0, \operatorname{Im}\rho_1]$, repeating the argument allows us to find an $s' \in \mathbb{R}$ and $s_0 < s' < s_1 < s_0+\epsilon$, so that ρ_0 is the only non-trivial zero of $L(s,\chi)$ with $\operatorname{Im}\rho_0 \geq 0$ and $|s'-\rho_0| = \min\{|s'-\rho|, \operatorname{Im}\rho \geq 0 \text{ and } L(\rho,\chi)=0\}$, i.e., the circle with center s' and radius $|s-\rho_0|$ does not contain any zero other than ρ_0 itself. Note that for any $s \in (s_0, s')$, ρ_0 is the only zero at which $l(s)$ is attained.

Next, let $B = \{\rho' : \rho' \neq \rho_0, |s_0-\rho'| < |s_0-\rho|\}$, where ρ, ρ' are zeros of $L(s,\chi)$. Note that B is also a finite set. Arguing in a similar way as above, we can find a $\tilde{\rho} \in B$ and $s'' \in (s_0, s_0+\epsilon)$ so that for all $s \in (s_0, s'')$, $|s-\tilde{\rho}| \leq |s-\rho|$ for $\rho \neq \rho_0$.

Therefore we can find a closed interval $[c,d] \subset (s_0-\epsilon, s_0+\epsilon)$ so that for all $s \in [c,d]$, we have

$$l(s) = |s-\rho_0| = |s-\bar{\rho}_0| < |s-\rho|, \rho \neq \rho_0, \overline{\rho_0} \tag{2.14}$$

$$|s-\tilde{\rho}| = |s-\overline{\tilde{\rho}}| \leq |s-\rho|, \rho \neq \rho_0, \overline{\rho_0}, \tilde{\rho}, \overline{\tilde{\rho}}. \tag{2.15}$$

Now let $s-\rho_0 = r_s e^{i\theta_s}$ for all $c \leq s \leq d$. Then from (2.10) and the fact that the zeros of $L(s,\chi)$ are symmetric with respect to the real axis, we have

$$\begin{aligned}
F^{(k)}(s,\chi) &= (-1)^{k-1}(k-1)!\left(\frac{1}{(s-\rho_0)^k} + \frac{1}{(s-\bar{\rho}_0)^k} + \sum_{\rho\neq\rho_0,\bar{\rho}_0}\frac{1}{(s-\rho)^k}\right)\\
&= (-1)^{k-1}(k-1)!\left(\frac{2}{r_s^k}\cos(k\theta_s) + \sum_{\rho\neq\rho_0,\bar{\rho}_0}\frac{1}{(s-\rho)^k}\right)\\
&= \frac{(-1)^{k-1}(k-1)!}{r_s^k}\left(2\cos(k\theta_s) + f(s)\right),
\end{aligned} \tag{2.16}$$

where $f(s) := r_s^k \sum_{\rho\neq\rho_0,\bar{\rho}_0}\frac{1}{(s-\rho)^k}$ and $k \geq 2$. Since the series $\sum_{\rho\neq\rho_0,\bar{\rho}_0}\frac{1}{(s-\rho)^k}$ converges absolutely for $k \geq 1$, $f(s)$ is a differentiable function for $s > 1$. Now,

$$\begin{aligned}
|f(s)| \leq 2\sum_{\substack{\rho\neq\rho_0,\bar{\rho}_0\\ \operatorname{Im}\rho\geq 0}}\frac{r_s^k}{|s-\rho|^k} &= 2\sum_{\substack{\rho\neq\rho_0,\bar{\rho}_0,\\ \operatorname{Im}\rho\geq 0}}\frac{|s-\rho_0|^k}{|s-\rho|^k}\\
&= 2|s-\rho_0|^2\sum_{\substack{\rho\neq\rho_0,\bar{\rho}_0,\\ \operatorname{Im}\rho\geq 0}}\frac{1}{|s-\rho|^2}\frac{|s-\rho_0|^{k-2}}{|s-\rho|^{k-2}}\\
&\leq 2|s-\rho_0|^2\sum_{\substack{\rho\neq\rho_0,\bar{\rho}_0,\\ \operatorname{Im}\rho\geq 0}}\frac{1}{|s-\rho|^2}\frac{|s-\rho_0|^{k-2}}{|s-\tilde{\rho}|^{k-2}}\\
&\leq 2|s-\rho_0|^2\sum_{\substack{\rho\neq\rho_0,\bar{\rho}_0,\\ \operatorname{Im}\rho\geq 0}}\frac{1}{|s-\rho|^2}\operatorname{Sup}_{c\leq s\leq d}\left\{\frac{|s-\rho_0|^{k-2}}{|s-\tilde{\rho}|^{k-2}}\right\},
\end{aligned} \tag{2.17}$$

where in the penultimate step we use (2.15). Let $h(s) := \frac{|s-\rho_0|}{|s-\tilde{\rho}|}$. Then $h(s)$ is a continuous function on $[c,d]$ and hence attains its supremum on $[c,d]$. Thus there exists an $x \in [c,d]$ such that

$$\eta := \mathrm{Sup}_{c\le s\le d}\left\{\frac{|s-\rho_0|}{|s-\tilde{\rho}|}\right\} = \frac{|x-\rho_0|}{|x-\tilde{\rho}|}. \tag{2.18}$$

Therefore by (2.14), $\eta < 1$. Combining (2.17) and (2.18), we have

$$|f(s)| \le 2\eta^{k-2}|s-\rho_0|^2 \sum_{\substack{\rho\ne\rho_0,\bar{\rho_0}\\ \mathrm{Im}\,\rho\ge 0}} \frac{1}{|s-\rho|^2} \le 2\eta^{k-2}|d-\rho_0|^2 \sum_{\substack{\rho\ne\rho_0,\bar{\rho_0}\\ \mathrm{Im}\,\rho\ge 0}} \frac{1}{|c-\rho|^2} \le C_{c,d,\chi}\eta^{k-2}. \tag{2.19}$$

Note that the constant term depends only on c, d and χ. Hence for sufficiently large k, we have $|f(s)| < 1$. Let $c-\rho_0 = r_c e^{i\theta_c}$ and $d-\rho_0 = r_d e^{i\theta_d}$. Then $\theta_c > \theta_d$. For k large enough, we can write $2\pi < k(\theta_c - \theta_d)$. Since for $s \in [c,d]$, we have $\theta_d \le \theta_s \le \theta_c$, for a sufficiently large k, $\cos(k\theta_s)$ attends all the values of the interval $[-1,1]$. So from (2.17) and (2.19) we conclude that for each large enough k there will be an s in $[c,d] \subset (s_0-\epsilon, s_0+\epsilon)$ so that $F^{(k)}(s,\chi) = 0$. This shows that $\cup_{k=1}^{\infty} A_{\chi,k}$ has a non-empty intersection with $(s_0-\epsilon, s_0+\epsilon)$ for any $s_0 > c_\chi$. This completes the proof of the theorem.

Remark: Let χ be a real nonprincipal Dirichlet character. If $L(s,\chi)$ has a Siegel zero, call it β, and if every zero of $L(s,\chi)$ has real part $\le \beta$, then for any $s > 1$, (2.10) implies

$$F^{(k)}(s,\chi) = \frac{(-1)^{k-1}(k-1)!}{(s-\beta)^k}\left(1 + \sum_{\substack{\rho\ne\beta\\ L(\rho,\chi)=0}} \left(\frac{s-\beta}{s-\rho}\right)^k\right). \tag{2.20}$$

Arguing as in the proof of Theorem 1.1, we see that there exists an integer M such that for all $k \ge M$, the series in (2.20) is less than 1. This means that for those k, $F^{(k)}(s,\chi)$ maintains the same sign for all $s > 1$. This is why we include the condition that $L(s,\chi) \ne 0$ for $0 < s < 1$ in the hypotheses of Theorem 1.1 .

3. Proof of theorem 1.2

Assume that the Riemann hypothesis holds for $L(s,\chi)$. Let $\gamma_0 := \mathrm{Im}\,\rho_0 = \min\{\mathrm{Im}\,\rho \ge 0 : L(\rho,\chi) = 0\}$, where ρ_0, ρ are non-trivial zeros of $L(s,\chi)$. Then $\rho_0 = 1/2 + i\gamma_0$. We show that the function ψ_χ is injective on $[C_\chi, \infty)$, where the constant C_χ will be determined later.

Let $s > c_\chi$, where c_χ is defined in (2.12). Then $l(s) < |s|$ and $l(s) = |s-\rho_0| < |s-\rho|$ for $\rho \neq \rho_0, \bar{\rho_0}$. Let $s - \rho_0 = r_s e^{i\theta_s}$. From (2.16), we have for $k \geq 2$,

$$\begin{aligned}
|f(s)| \leq \sum_{\rho\neq\rho_0,\bar{\rho_0}} \frac{r_s^k}{|s-\rho|^k} &= |s-\rho_0|^2 \sum_{\rho\neq\rho_0,\bar{\rho_0}} \frac{1}{|s-\rho|^2}.\frac{|s-\rho_0|^{k-2}}{|s-\rho|^{k-2}} \\
&\leq |s-\rho_0|^2 \sum_{\rho\neq\rho_0,\bar{\rho_0}} \frac{1}{|s-\rho|^2}.\,\mathrm{Sup}_\rho \left\{\frac{|s-\rho_0|^{k-2}}{|s-\rho|^{k-2}}\right\} \\
&= |s-\rho_0|^2 \eta_s^{k-2} \sum_{\rho\neq\rho_0,\bar{\rho_0}} \frac{1}{|s-\rho|^2} \\
&= O_{s,\chi}(\eta_s^{k-2}). \qquad (3.1)
\end{aligned}$$

Here in the penultimate step,

$$\eta_s = \mathrm{Sup}_\rho \left\{\frac{|s-\rho_0|}{|s-\rho|}\right\} \leq \frac{|s-\rho_0|}{|s-\tilde{\rho}|} < 1,$$

and $\mathrm{Im}\,\rho_0 < \mathrm{Im}\,\tilde{\rho} \leq \mathrm{Im}\,\rho$, resulting from (2.14) and (2.15). Combining (2.16) and (3.1), we obtain

$$F^{(k)}(s,\chi) = \frac{2(-1)^{k-1}(k-1)!}{r_s^k}\left(\cos(k\theta_s) + f(s)\right), \qquad (3.2)$$

where $f(s) = O_{s,\chi}(\eta_s^{k-2})$.

Next, we show that there are infinitely many k for which $\cos(k\theta_s)$, which we view as the main term, dominate the error term. Since $\eta_s < 1$, for a fixed $s > 1$, we can bound the error term in $(-\epsilon, \epsilon)$ for all sufficiently large k and for all $0 < \epsilon < 1$. Write $\cos(k\theta_s) = \cos\left(\pi\frac{k\theta_s}{\pi}\right) = \cos\left(2\pi\frac{k\theta_s}{2\pi}\right)$ and consider the cases when $\frac{\theta_s}{\pi}$ is rational and $\frac{\theta_s}{2\pi}$ is irrational.

If $\frac{\theta_s}{\pi}$ is a rational number, there are infinitely many $k \in \mathbb{N}$ so that $\frac{k\theta_s}{2\pi}$ is an even integer and hence $\cos(k\theta_s) = 1$.

If $\frac{\theta_s}{\pi}$ is a rational number with odd numerator, then there are infinitely many $k \in \mathbb{N}$, namely the odd multiples of the denominator, so that $\frac{k\theta_s}{2\pi}$ is an odd integer and hence $\cos(k\theta_s) = -1$.

Let $\frac{\theta_s}{\pi} = \frac{2m}{n}$ be a rational number with even numerator and odd denominator. Since $(2m, n) = 1$, there exists an integer $l \in [1, n]$ such that $2ml \equiv 1 (\mathrm{mod}\, n)$. For all $k \equiv l (\mathrm{mod}\, n), 2mk \equiv 1 (\mathrm{mod}\, n)$. Therefore for all $k \equiv l \,\mathrm{mod}\, n$, since $2mk$ is even, we have $2mk = (2p+1)n + 1$. Hence there are infinitely many integers k for which $\cos(k\theta_s) = \cos\left(\pi\left(2p+1+\frac{1}{n}\right)\right) = -\cos\left(\frac{\pi}{n}\right)$.

If $\frac{\theta_s}{2\pi}$ is irrational, then we know [28] that the sequence $\left\{\left\{\frac{k\theta_s}{2\pi}\right\}\right\}$ is dense in $[0, 1]$, where $\{x\}$ denotes the fractional part of x. Hence there are infinitely many $k \in \mathbb{N}$ with $\left\{\frac{k\theta_s}{2\pi}\right\}$ close to 1 and hence $\cos(k\theta_s) > 1 - \epsilon$ for any given $\epsilon > 0$. Likewise, there are infinitely many $k \in \mathbb{N}$ with $\left\{\frac{k\theta_s}{2\pi}\right\}$ close to $\frac{1}{2}$ and hence $\cos(k\theta_s) < -1 + \epsilon$.

Fix s_1 and s_2 such that $c_\chi < s_1 < s_2$. Then $l(s_1) = |s_1 - \rho_0|$ and $l(s_2) = |s_2 - \rho_0|$. Let θ_1 and θ_2 be such that $s_1 - \rho_0 = r_1 e^{i\theta_1}$ and $s_2 - \rho_0 = r_2 e^{i\theta_2}$. Note that $0 < \theta_2 < \theta_1 < \pi/2$.

From (3.2), we have

$$F^{(k)}(s_1,\chi) = \frac{2(-1)^{k-1}(k-1)!}{r_1^k}\left(\cos(k\theta_1)+f(s_1)\right), \tag{3.3}$$

$$F^{(k)}(s_2,\chi) = \frac{2(-1)^{k-1}(k-1)!}{r_2^k}\left(\cos(k\theta_2)+f(s_2)\right), \tag{3.4}$$

where $f(s_1) = O_{s_1,\chi}(\eta_{s_1}^{k-2})$ and $f(s_2) = O_{s_2,\chi}(\eta_{s_2}^{k-2})$. Write $\theta_1 = \theta_2 + (\theta_1 - \theta_2)$.

We show that there exist infinitely many integers k such that the terminal rays of $k\theta_1$ and $k\theta_2$ stay away from the y-axis, that $\operatorname{sgn}(\cos k\theta_1) = -\operatorname{sgn}(\cos k\theta_2) \neq 0$, and that $\cos(k\theta_1)$ and $\cos(k\theta_2)$ dominate $f(s_1)$ and $f(s_2)$ in (3.3) and (3.4) respectively. We first determine the signs.

Case 1: If $\frac{\theta_1-\theta_2}{\pi}$ is rational with odd numerator then as we saw before, there are infinitely many positive integers k so that $k\frac{(\theta_1-\theta_2)}{\pi}$ is an odd integer and hence for those $k \in \mathbb{N}$, $\cos(k\theta_1) = \cos(k\theta_2 + \pi) = -\cos(k\theta_2)$.

Case 2: If $\frac{\theta_1-\theta_2}{\pi}$ is rational with even numerator and odd denominator n, there are infinitely many positive integers k so that $k\frac{(\theta_1-\theta_2)}{\pi} = 2p+1+1/n$ for some $p \in \mathbb{N}$ and so $\cos(k\theta_1) = \cos(k\theta_2 + \pi + \pi/n) = -\cos(k\theta_2 + \pi/n)$.

Case 3: If $\frac{(\theta_1-\theta_2)}{2\pi}$ is irrational, there are infinitely many positive integers k so that $\left\{k\frac{(\theta_1-\theta_2)}{2\pi}\right\} \in [1/2, 1/2+\epsilon/2\pi)$, for any given $\epsilon > 0$. So for any δ such that $0 < \delta < \epsilon$, we have $\cos(k\theta_1) = \cos(k\theta_2 + \pi + \delta) = -\cos(k\theta_2 + \delta)$. We can choose ϵ as small as we want and hence $0 < \delta < \epsilon < \pi/n$.

We first show that in Case 2, we have the terminal rays of the angles sufficiently away from the y-axis, with $\cos k\theta_1$ and $\cos k\theta_2$ dominating their corresponding terms $f(s_1)$ and $f(s_2)$. To that end, choose a constant $b_\chi > 1/2$ such that $\tan\left(\frac{\pi}{100}\right) = \frac{\gamma_0}{b_\chi - \frac{1}{2}}$, say. If $s - \rho_0 = r_s e^{i\theta_s}$ and $s > b_\chi$, then $0 < \theta_s < \pi/100$. So if we take $b_\chi < s_1 < s_2$, then $0 < \theta_2 < \theta_1 < \pi/100$. Since $\eta_{s_1}, \eta_{s_2} < 1$ there exists an integer K such that $|f(s_1)|, |f(s_2)| < \theta_2/4$ for all $k > K$. As we saw before, for infinitely many integers $k > K+2$, we have $k\theta_1 = k\theta_2 + \pi + \pi/n$, where n depends on θ_1 and θ_2. We first note that all angles below are considered mod 2π. If $k\theta_2 \in (\pi/2 + \theta_2, \pi)$ then $k\theta_1 \in (-\pi/2 + \theta_2, \pi/2 - \theta_2)$. Thus $\cos(k\theta_1)\cos(k\theta_2) < 0$. Also $|\cos(k\theta_2)| > |\sin(\theta_2)| \geq \theta_2/2 > |f(s_2)|$ and $|\cos(k\theta_1)| = |\cos(k\theta_2 + \pi/n)| > |\sin(\theta_2)| \geq \theta_2/2 > |f(s_1)|$.

Similarly we see that $|\cos(k\theta_1)| > |f(s_1)|$ and $|\cos(k\theta_2)| > |f(s_2)|$ when $k\theta_2 \in (-\pi/2 + \theta_2, 0)$. If $k\theta_2 \in (0, \pi/2 - \theta_2)$ and $k\theta_1 \in (-\pi, -\pi/2 - \theta_2)$ in this case also $|\cos(k\theta_2)| > |\sin(\theta_2)| \geq \theta_2/2 > |f(s_2)|$ and $|\cos(k\theta_1)| = |\cos(k\theta_2 + \pi/n)| > |\sin(\theta_2)| \geq \theta_2/2 > |f(s_1)|$. Now let $k\theta_2 \in (0, \pi/2 + \theta_2)$ and $k\theta_1 \in (-\pi/2 - \theta_2, 0)$. Then since $\pi/n < \theta_1 < \pi/100$, it is easy to check that $(k-2)\theta_2 \in (0, \pi/2 - \theta_2)$ and $(k-2)\theta_1 = k\theta_2 + \pi + \pi/n - 2\theta_1 \in (-\pi, -\pi/2 - \theta_2)$. Hence $|\cos(k\theta_1)| > |f(s_1)|$ and $|\cos(k\theta_2)| > |f(s_2)|$. Similarly we have the same conclusion if $k\theta_2 \in (-\pi, -\pi/2 + \theta_2)$ and $k\theta_1 \in (\pi/2 - \theta_2, \pi)$.

Note that since $k\theta_2 + \pi + \pi/n > k\theta_2 + \pi + \delta$, for the values of θ_1 and θ_2 in Case 3 as well, one can similarly prove that $|\cos(k\theta_2)| > |f(s_2)|$ and $|\cos(k\theta_1)| > |f(s_1)|$. So is the case with the values of θ_1 and θ_2 in Case 1.

Let

$$C_\chi = \max\{c_\chi, b_\chi\}. \tag{3.5}$$

Then for any given real numbers s_1 and s_2 such that $C_\chi < s_1 < s_2$, we have shown that there exist infinitely many integers k such that $\cos(k\theta_1)$ and $\cos(k\theta_2)$ have opposite signs and $|\cos(k\theta_1)| > |f(s_1)|$ and $\cos(k\theta_2) > f(s_2)$. This implies that $F^{(k)}(s_1,\chi)$ and $F^{(k)}(s_2,\chi)$ have opposite signs and that in turn proves that the function ψ_χ is injective in $[C_\chi,\infty)$.

We now prove part (b) of Theorem 1.2.

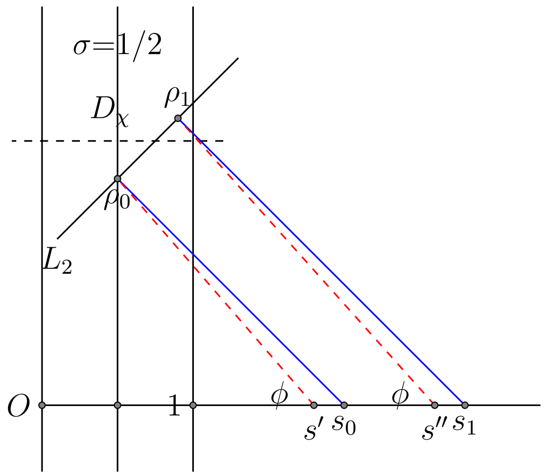

FIGURE 2. Constructing the angle $\phi = 2\pi(a+b\sqrt{2})$.

Let ρ_0 be the lowest zero of $L(s,\chi)$ above the real axis (so ρ_0 is not a real number). Let L_1 be the line passing through ρ_0 and perpendicular to the line which passes through ρ_0 and C_χ, where C_χ is defined in (3.5). Let $(1, D_\chi)$ be the point of intersection of the lines $\sigma = 1$ and L_1. We first show that if there is only one zero ρ_1 with $\operatorname{Im}\rho_1 \geq D_\chi$ off the critical line $\sigma = 1/2$, then this contradicts the injectivity of ψ_χ on $[C_\chi,\infty)$.

Without loss of generality, let $\operatorname{Re}\rho_1 > 1/2$. As shown in Figure 2, let L_2 be the line passing through ρ_0 and ρ_1. Let s_0 and s_1 be the points of intersection of the real axis with the lines perpendicular to L_2 and passing through ρ_0 and ρ_1 respectively. Clearly $s_1 > s_0 > C_\chi$. Note that by our construction, $l(s_0) = |s_0-\rho_0|$ and $l(s_1) = |s_1-\rho_1|$, where $l(s)$ is defined in (2.11), and there exists a θ such that $(s_0-\rho_0) = r_{s_0}e^{i\theta}$ and $(s_1-\rho_1) = r_{s_1}e^{i\theta}$. From the proof of the Theorem 1.1, we know that there exists an $\epsilon > 0$ so that $l(s) = |s-\rho_0|$ for all $s \in (s_0-\epsilon, s_0+\epsilon)$ and $l(s) = |s-\rho_1|$ for all $s \in (s_1-\epsilon, s_1+\epsilon)$. Without loss of generality, we can assume that $s_0+\epsilon < s_1-\epsilon$. Therefore, there exists a $\delta > 0$ such that $\theta_s \in (\theta-\delta, \theta+\delta)$, where $s-\rho_0 = r_s e^{i\theta_s}$ and $l(s) = |s-\rho_0|$ for all $s \in (s_0-\epsilon, s_0+\epsilon)$, and such that $\theta_s \in (\theta-\delta,\theta+\delta)$, where $s-\rho_1 = r_s e^{i\theta_s}$ and $l(s) = |s-\rho_1|$ for all $s \in (s_1-\epsilon, s_1+\epsilon)$.

Since the sequence $\{\{n\sqrt{2}\}\}$ is dense in $[0,1)$, and $\{n\sqrt{2}\} = n\sqrt{2} - \lfloor n\sqrt{2}\rfloor$, there exists an integer a and an integer $b \neq 0$ such that $a + b\sqrt{2} \in (\frac{\theta-\delta}{2\pi}, \frac{\theta+\delta}{2\pi})$. Let $\phi = 2\pi(a+b\sqrt{2})$, $s' \in (s_0-\epsilon, s_0+\epsilon)$ and $s'' \in (s_1-\epsilon, s_1+\epsilon)$ be such that $s'-\rho_0 = r_{s'}e^{i\phi}$

and $s'' - \rho_1 = r_{s''} e^{i\phi}$. Therefore,

$$F^{(k)}(s', \chi) = \frac{2(-1)^{k-1}(k-1)!}{r_{s'}^k} \left(\cos(k\phi) + f(s')\right) \tag{3.6}$$

$$F^{(k)}(s'', \chi) = \frac{2(-1)^{k-1}(k-1)!}{r_{s''}^k} \left(\cos(k\phi) + f(s'')\right), \tag{3.7}$$

where $|f(s')| = O(\eta_{s'}^{k-2})$ and $|f(s'')| = O(\eta_{s''}^{k-2})$. Let $\eta = \min\{\eta_{s'}, \eta_{s''}\}$. Then $|f(s')|, |f(s'')| \leq C_{s',s''}\eta^{k-2}$ for some constant $C_{s',s''}$.

We next show that there exist positive constants $C_{a,b}$ and $K_{a,b}$ so that

$$|4k(a + b\sqrt{2}) + r| > \frac{C_{a,b}}{k}, \tag{3.8}$$

for any integers r and k, with $k > K_{a,b}$. Let $|4k(a + b\sqrt{2}) + r| \leq 1$. Then,

$$|4k(a - b\sqrt{2}) + r| \leq |4k(a + b\sqrt{2}) + r| + 8k|b|\sqrt{2} \leq 1 + 8k|b|\sqrt{2} < \frac{k}{C_{a,b}}. \tag{3.9}$$

Therefore for $k \geq 2$,

$$|4k(a+b\sqrt{2})+r|\frac{k}{C_{a,b}} > |4k(a-b\sqrt{2})+r||4k(a+b\sqrt{2})+r| = |(4ka+r)^2 - 2(4kb)^2| \geq 1, \tag{3.10}$$

since $b \neq 0$. If $|4k(a + b\sqrt{2}) + r| \geq 1$, then of course, there exists a $K_{a,b}$, such that for $k > K_{a,b}$, we have $|4k(a + b\sqrt{2}) + r| > \frac{C_{a,b}}{k}$. Hence in conclusion, for a large positive integer N and for all $k > N$, if we choose m so that $|4k(a + b\sqrt{2}) \pm 1 \pm 4m| < 1$, we have

$$\begin{aligned} |\cos k\phi| = \left|\sin \frac{\pi}{2}(4k(a + b\sqrt{2}) \pm 1 \pm 4m)\right| &\geq \sin\left(\frac{\pi C_{a,b}}{2k}\right) \\ &\geq \frac{\pi C_{a,b}}{4k} \\ &> C_{s',s''}\eta^{k-2}. \end{aligned} \tag{3.11}$$

Therefore for the above mentioned s' and s'' such that $s' \neq s''$, and for all $k > N$, $F^{(k)}(s', \chi)$ and $F^{(k)}(s'', \chi)$ have the same sign. This contradicts the injectivity of ψ_χ on $[C_\chi, \infty)$. Now if there is more than one zero ρ with $\operatorname{Im}\rho \geq D_\chi$ off the critical line, then we can choose the zero ρ_1 with the following properties:

i) The angle between the positive x-axis and the line L passing through the zeros ρ_0 and ρ_1 is smaller than the angle between the positive x-axis and the line passing through the zeros ρ_0 and $\rho \neq \rho_1$ and,

ii) $\operatorname{Im}\rho_1 = \min\{\operatorname{Im}\rho \geq D_\chi : \rho \text{ lies on the line } L\}$.

Then we can proceed similarly as above and again get a contradiction. Hence, all the zeros above the line $t = D_\chi$ lie on the critical line $\sigma = 1/2$. This completes the proof.

Acknowledgements

The first author is funded in part by the grant NSF-DMS 1112656 of Professor Victor H. Moll of Tulane University and sincerely thanks him for the support.

References

[1] H. Alzer and C. Berg, *Some classes of completely monotonic functions*, Annales Acad. Sci. Fenn. Math. **27** (2) (2002), 445–460.

[2] H. Alzer and C. Berg, *Some Classes of Completely Monotonic Functions, II*, Ramanujan J. **11**, No. 2 (2006), 225–248.

[3] K. Ball, *Completely monotonic rational functions and Hall's marriage theorem*, J. Combin. Theory Ser. B **61**, No. 1 (1994), 118–124.

[4] C. Berg, *A Pick function related to an inequality for the entropy function*, J. Inequal. Pure Appl. Math. **2**, Issue 2, Article 26 (2001).

[5] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1**, No. 4 (2004), 433–439.

[6] C. Berg and G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Ergebnisse der Math. **87**, Springer, Berlin, 1975.

[7] B.C. Berndt, *The numbers of zeros for* $\zeta^k(s)$, J. London Math. Soc. (2) **2** (1970), 577–580.

[8] J.B. Conrey and A. Ghosh, *Zeros of derivatives of the Riemann Zeta-function near the critical line*, Analytic Number Theory, (ed. Berndt et. al.), Progr. Math. **85**, Birkhäuser Boston, 1990.

[9] C.-P. Chen, *Complete monotonicity and logarithmically complete monotonicity properties for the gamma and psi functions*, J. Math. Anal. Appl. **336**, No. 2 (2007), 812–822.

[10] W. Cheney and W. Light, *A course in Approximation Theory*, Graduate Text in Mathematics, 101, American Mathematical Society, Providence, RI, 2009.

[11] H. Davenport, *Multiplicative Number Theory*, 3rd ed., Springer–Verlag, New York, 2000.

[12] A. Dixit and A.Yu. Solynin, *Monotonicity of quotients of theta functions related to an extremal problem on harmonic measure*, J. Math. Anal. Appl. **336**, No. 2 (2007), 1042–1053.

[13] A. Dixit, A. Roy and A. Zaharescu, *Convexity of quotients of theta functions*, J. Math. Anal. Appl. **386**, No. 1 (2012), 319–331.

[14] C.L. Frenzen, *Error bounds for asymptotic expansions of the ratio of two gamma functions*, SIAM J. Math. Anal. **18** (1987), 890–896.

[15] B.-N. Guo, S. Guo and F. Qi, *Complete monotonicity of some functions involving polygamma functions*, J. Comput. Appl. Math., **233**, No. 9 (2010), 2149–2160.

[16] M.E.H. Ismail, *Integral representations and complete monotonicity of various quotients of Bessel functions*, Can. J. Math. **29**, No. 6 (1977), 1198–1207.

[17] C.H. Kimberling, *A probabilistic interpretation of complete monotonicity*, Aequat. Math. **10** (1974), 152–164.

[18] N. Levinson and H.L. Montgomery, *Zeros of derivatives of the Riemann Zeta-function*, Acta Math. **133** (1974), 49-65.

[19] K.S. Miller and S.G. Samko, *Completely monotonic functions*, Integr. Transf. and Spec. Funct. **12**, No. 4 (2001), 389–402.

[20] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the Gamma and Digamma functions*, RGMIA, Res. Rep. Coll. 7, No. 1 (2004), Art. 6. Available online at http://www.rgmia.org//papers/v7n1/minus-one.pdf

[21] F. Qi, B.-N. Guo and C.-P. Chen, *Some completely monotonic functions involving the Gamma and polygamma functions*, J. Aust. Math. Soc. **80** (2006), 81–88.

[22] F. Qi and S.-X. Chen, *Complete monotonicity of the logarithmic mean*, Math. Inequal. Appl. **10**, No. 4 (2007), 799–804.

[23] A. Speiser, *Geometrisches zur Riemannschen zetafunktion*, Math. Ann. **110** (1934), 514–521.

[24] R. Spira, *Zero-free regions of* $\zeta^{(k)}(s)$, J. London Math. Soc. **40** (1965), 677–682.

[25] R. Spira, *Another zero-free region for* $\zeta^{(k)}(s)$, Proc. Amer. Math. Soc. **35** (1972), 489–497.

[26] R. Spira, *Calculation of Dirichlet L-functions*, Math. Comp. **23** (1969), 489–497.

[27] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed. Oxford, 1986.

[28] W. Hermann, *Über die Gleichverteilung von Zahlen mod. Eins.*, Math. Ann. **77**, No. 3 (1916), 313-352.

[29] C.Y. Yildirim, *A note on $\zeta''(s)$ and $\zeta'''(s)$* , Proc. Amer. Math. Soc. **124**, No. 8, (1996), 2311–2314.

[30] C.Y. Yildirim, *Zeros of derivatives of Dirichlet L-functions*, Turkish J. Math. **20**, No. 4 (1996), 521-534.

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