Monotonicity of quotients of theta functions related to an extremal problem on harmonic measure

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Abstract

We prove that for fixed u and v, $u, v \in [0, 1/2]$, the quotients $\theta_i(u|i\pi t)/\theta_i(v|i\pi t)$, $i = 1, 2, 3, 4$, of the theta functions are monotone on $0 < t < \infty$. The case $v = 0$ had been used by the second author to study a generalization of Gonchar's problem on harmonic measure of radial slits.

Key words: Jacobi theta function, Weierstrass $\wp$-function, monotonicity, harmonic measure

1. Introduction

The classical, but still flourishing theory of theta functions has incredibly many ramifications and applications in modern mathematics. The modest goal of this work is to contribute a small piece to this theory by proving new monotonicity properties of the quotients of theta functions defined for rectangular lattices. In our notation we follow the classical book [4]. So for $z \in \mathbb{C}$ and $q = e^{i\pi\tau}$ with $\Im\tau > 0$, the four theta functions are defined by

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$$\theta_1(z|\tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi z, \tag{1.1}$$

$$\theta_2(z|\tau) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos(2n+1)\pi z, \tag{1.2}$$

$$\theta_3(z|\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2n\pi z, \tag{1.3}$$

$$\theta_4(z|\tau) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2n\pi z. \tag{1.4}$$

For a fixed $|q| < 1$ (that is if $\Im\tau > 0$), each of these series converges for every $z \in \mathbb{C}$ and therefore $\theta_i := \theta_i(z|\tau)$ is an entire function of $z = x + iy$.

In this paper we deal with the quotients

$$S_i(u, v; t) := \frac{\theta_i(u/2|i\pi t)}{\theta_i(v/2|i\pi t)}, \qquad i = 1, 2, 3, 4, \tag{1.5}$$

defined for $u, v \in \mathbb{C}$ and $q = e^{i\pi\tau} = e^{-\pi^2 t}$ with $\Re t > 0$. Our main result is the following monotonicity property.

Theorem 1 *For fixed u and v such that $0 \le u < v < 1$, the functions $S_1(u, v; t)$ and $S_4(u, v; t)$ are positive and strictly increasing on $0 < t < \infty$, while the functions $S_2(u, v; t)$ and $S_3(u, v; t)$ are positive and strictly decreasing on $0 < t < \infty$.*

We remind the reader that each of the functions $\theta_i(x/2|i\pi t)$, $i = 1, 2, 3, 4$, satisfies the heat equation

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}, \tag{1.6}$$

see [4, Section 13.19], which explains their frequent appearance in problems on the heat flow in planar domains.

It is worth mentioning that for $v = 0$ and fixed u, the question about monotonicity of $S_2(u, v; t)$ on $0 < t < \infty$ was explored in the work of the second author [7] where it is related to the steady-state distribution of heat. More precisely, the paper [7] deals with the following problem on harmonic measure originally posed by A. A. Gonchar, see [3].

Let D be a Dirichlet domain on $\mathbb{C}$ and let E be a Borel set on ∂D. The harmonic measure $\omega(z) := \omega(z, E, D)$ of E with respect to D is the Perron solution $\omega(z)$ of the Dirichlet problem in D with boundary values 1 on E and 0 on $\partial D \setminus E$. In particular, $\omega(z)$ is a bounded harmonic function in D.

Let K be a compact subset of the half-open interval $(0, 1]$. For a fixed integer $n \ge 1$, let $\mathcal{E}_n(K) = \{E\}$ denote the family of all compact sets $E \subset \overline{\mathbb{D}}$ having the form $E = \cup_{j=1}^{n}\{e^{i\alpha_j}K\}$, where $\alpha_j \in \mathbb{R}$.

A generalized A. A. Gonchar's problem on harmonic measure is to find all configurations $E^* \in \mathcal{E}_n(K)$ realizing the maximum

$$\max \omega(0, E, \mathbb{D} \setminus E), \qquad E \in \mathcal{E}_n(K). \tag{1.7}$$

To give a physical interpretation to the problem (1.7), we may think that $\mathbb{D}$ represents a round stove with cooling circular boundary, where we keep constant temperature 0. Then the question is how to place n identical heating elements with temperature 1, each of which is of the form $e^{i\alpha}K$, to get the maximal temperature at the center of the stove.

An engineering intuition immediately suggests that the most symmetric configuration $E_n = \cup_{k=1}^{n}\{e^{2\pi ik/n}K\}$ and its rotations around the origin should provide the maximal temperature at the center. But for a general compact set K, a rigorous proof of this guess remains elusive. Figure 1 displays admissible non-symmetric and symmetric configurations for Gonchar's problem with four heating elements.

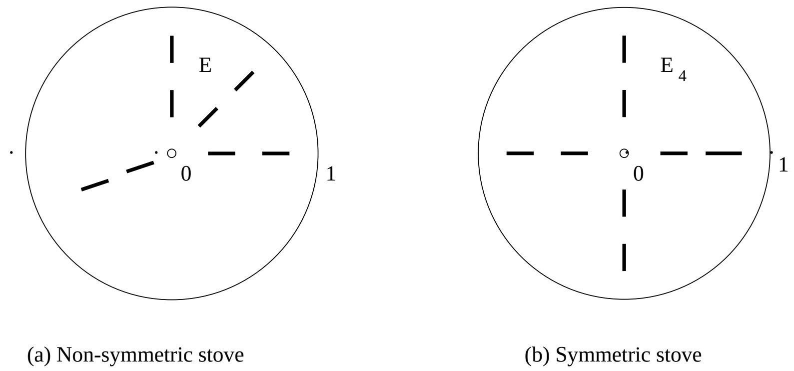

Fig. 1. Admissible configurations for Gonchar's problem with four heaters

Originally in the early 1980's, A. A. Gonchar asked this question for the case when $K = [\rho, 1]$, $0 < \rho < 1$. In [3], V. N. Dubinin solved Gonchar's problem by introducing a new geometric transformation now known as *dissymmetrization*. The case of a general K was studied by A. Baernstein II [2] who proved a more general inequality for the integral means of $\omega(re^{i\theta}, E, \mathbb{D} \setminus E)$ but only for $n = 2$ and $n = 3$.

For the case $K = [r_1, r_2]$, $0 < r_1 < r_2 < 1$, and any $n \geq 1$, Gonchar's problem was studied by the second author in [7], where he first reduced the original problem to the question about the monotonicity of the function $m(r, \omega) + 2\sigma\omega$. Here $\sigma = -(1/2\pi)\log r$ and $m(r, \omega)$ denotes the reduced module of the slit semidisk $T(r, \theta) = \{z : 0 < |z| < 1, 0 < \arg z < \pi\} \setminus \{z : |z| = r, 0 \leq \arg z \leq \theta\}$ considered as a topological triangle with vertices $z_1 = 0$, $z_2 = 1$, and $z_3 = -1$. For the definition and properties of the reduced module of a triangle the reader may consult [8].

For a fixed r in $0 < r < 1$, the angle $\theta = \theta(\omega)$ is an increasing function of the harmonic measure ω. It was shown in [7] that the function $m(r, \omega) + 2\sigma\omega$ can be explicitly expressed as

$$m(r, \omega) + 2\sigma\omega = -\frac{2}{\pi}\log S_2(\omega, 0; \frac{1}{2\pi|\tau|}),$$

where S_2 is defined by (1.5). Therefore for studying monotonicity properties of the function $m(r, \omega) + 2\sigma\omega$, we may work with the quotient of the theta functions (1.5). The proof of monotonicity of $S_2(u, 0; t)$ outlined in Lemma 5 in [7] contains an error since the constant term c_0 is missing in formula (4.20) in [7]. Thus, by proving Theorem 1, this particular case corrects this error.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1 for functions S_1 and S_2 and in Section 3, we prove this theorem for functions S_3 and S_4.

In Section 4, we collect series expansions and inequalities necessary for our proofs in Sections 2 and 3. It is worth mentioning that some proofs in Section 4 rest on certain convolution formulas and inequalities for the divisor function, which occurs frequently in

Number Theory. Some of the relations used in our proofs may be known. To make this paper self-contained we provide the proofs of the relations which we could not locate in major texts.

2. Proof of Theorem 1 for $S_1(u,v;t)$ and $S_2(u,v;t)$

(a) First we prove the monotonicity property of $S_1(u,v;t)$. It follows from (1.1) that for $0 < x < 1$ and $0 < t < \infty$ the function $\theta_1(x/2|i\pi t)$ takes real values only. Since the zeros of $\theta_1(z|\tau)$ are at the points $z = m + n\tau$, $m, n \in \mathbb{Z}$, the latter implies that the quotient $S_1(u,v;t)$ is positive for the considered values of u, v, and t. Since $\theta_1(x/2|i\pi t)$ satisfies the heat equation (1.6), we have

$$\frac{\partial}{\partial t}\log S_1(u,v;t) = \frac{\theta_1''(u/2|i\pi t)}{\theta_1(u/2|i\pi t)} - \frac{\theta_1''(v/2|i\pi t)}{\theta_1(v/2|i\pi t)}. \tag{2.1}$$

Henceforth in this paper the prime will denote differentiation with respect to z in $\theta_i(z|\tau)$. Now equation (2.1) implies that $\frac{\partial}{\partial t}S_1(u,v;t) > 0$ for $0 \le u < v < 1$ and $0 < t < \infty$ if and only if for every fixed $0 < t < \infty$ the function $\theta_1''(x|i\pi t)/\theta_1(x|i\pi t)$ strictly decreases in $0 < x < 1/2$.

For the rest of this proof, we fix t, $0 < t < \infty$. Accordingly, we will abbreviate $\theta_1(z|i\pi t)$ as $\theta_1(z)$.

So, we claim that $\theta_1''(x)/\theta_1(x)$ strictly decreases for $0 < x < 1/2$. The function $\theta_1'(z)/\theta_1(z)$ has real period 1 and imaginary quasi-period τ, see [4, Section 13.19]; i.e.,

$$\frac{\theta_1'(z+1)}{\theta_1(z+1)} = \frac{\theta_1'(z)}{\theta_1(z)} \quad \text{and} \quad \frac{\theta_1'(z+\tau)}{\theta_1(z+\tau)} = -2\pi i + \frac{\theta_1'(z)}{\theta_1(z)}.$$

Differentiating these equations with respect to z, we get

$$\begin{aligned}\frac{\theta_1''(z+1)\theta_1(z+1) - {\theta_1'}^2(z+1)}{\theta_1^2(z+1)} &= \frac{\theta_1''(z+\tau)\theta_1(z+\tau) - {\theta_1'}^2(z+\tau)}{\theta_1^2(z+\tau)} \\ &= \frac{\theta_1''(z)\theta_1(z) - {\theta_1'}^2(z)}{\theta_1^2(z)}.\end{aligned}$$

This shows that $P_1(z) := (\theta_1''(z)\theta_1(z) - (\theta_1')^2(z))/\theta_1^2(z)$ is an elliptic function with periods 1 and τ, which has a single pole of order 2 at $z = 0$ in the period rectangle.

Now it follows from the basic theory of elliptic functions that $P_1(z)$ can be expressed in the form $P_1(z) = a\wp(z) + b$, where $\wp$ is the Weierstrass $\wp$-function with periods 1 and τ and $a, b \in \mathbb{C}$. To find the coefficients a and b, we will use the well-known series expansions at $z = 0$:

$$\frac{\theta_1'(z)}{\theta_1(z)} = \pi\cot(\pi z) + 4\pi\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin(2\pi n z) = \frac{1}{z} + \text{positive powers of } z \tag{2.2}$$

and

$$\wp(z) = \frac{1}{z^2} + \frac{g_2 z^2}{2^2\cdot 5} + \frac{g_3 z^4}{2^2\cdot 7} + \frac{g_2^2 z^6}{2^4\cdot 3\cdot 5^2} + \dots \tag{2.3}$$

Differentiating (2.2), we find

$$\begin{aligned}P_1(z) &= -\pi^2\csc^2(\pi z) + 8\pi^2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos(2\pi n z) \\ &= -\frac{1}{z^2} + c_0 + \text{positive powers of } z,\end{aligned} \tag{2.4}$$

where

$$c_0 = c_0(q) = \lim_{z\to 0}(P_1(z) + z^{-2}) = -\frac{\pi^2}{3} + 8\pi^2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}}. \tag{2.5}$$

Comparing (2.3) and (2.4), we obtain

$$P_1(z) = \frac{d}{dz}\frac{\theta_1'(z)}{\theta_1(z)} = -\wp(z) + c_0. \tag{2.6}$$

Using (2.6), we find

$$\frac{\theta_1''(z)}{\theta_1(z)} = \left(\frac{\theta_1'(z)}{\theta_1(z)}\right)^2 - (\wp(z) - c_0). \tag{2.7}$$

Differentiating (2.7) once more and taking into account (2.6), we obtain

$$\frac{d}{dz}\frac{\theta_1''(z)}{\theta_1(z)} = -2\frac{\theta_1'(z)}{\theta_1(z)}(\wp(z) - c_0) - \wp'(z). \tag{2.8}$$

Now we will explore some mapping properties of the $\wp$-function. It is well known, see [4, p. 376] that $\wp(z)$ maps the parallelogram R (which is a rectangle in our case) with vertices 0, $\omega = 1/2$, $\omega + \omega' = 1/2 + \tau/2$, and $\omega' = \tau/2$ conformally and one-to-one onto the lower half-plane $\{w : \Im w < 0\}$. In addition, $\wp(z)$ is real and decreases from ∞ to $-\infty$ as z describes the boundary of R in the counter clock-wise direction starting from 0. In particular, using the standard notation $e_1 = \wp(1/2)$, $e_2 = \wp(1/2 + \tau/2)$, and $e_3 = \wp(\tau/2)$ for the images of the vertices of R, we have the inequalities $e_3 < e_2 < e_1$. This monotonicity property and inequality (4.15) of Lemma 1 in Section 4 imply that

$$\wp(x) - c_0 > \wp(1/2) - c_0 = e_1 - c_0 > 0 \quad \text{for } 0 < x < 1/2.$$

This together with (2.8) shows that the inequality $\frac{d}{dz}\frac{\theta_1''(z)}{\theta_1(z)} < 0$ for $z = x$, $0 < x < 1/2$, is equivalent to the following inequality:

$$F_1(x) := 2\frac{\theta_1'(x)}{\theta_1(x)} + \frac{\wp'(x)}{\wp(x) - c_0} > 0 \quad \text{for } 0 < x < 1/2. \tag{2.9}$$

Let us show that F_1 vanishes at $z = 0$ and $z = 1/2$. Since $\wp(z)$ has a stationary point at $z = 1/2$, we have $\wp'(1/2) = 0$. Since $\theta_1'(z + 1/2) = \theta_2'(z)$ is an odd analytic function, we also have $\theta_1'(1/2) = 0$. Therefore, $F_1(1/2) = 0$.

To find $F_1(0)$, we use the series expansions (2.2) and (2.3). From (2.3) one can easily find

$$\frac{\wp'(z)}{\wp(z) - c_0} = -\frac{2}{z} + \text{positive powers of } z. \tag{2.10}$$

Substituting (2.2) and (2.10) into (2.9) and letting $z = x \to 0^+$, we obtain: $F_1(0) = 0$.

Differentiating (2.9), we find

$$F_1'(x) = \frac{-2(\wp(x) - c_0)^3 + \wp''(x)(\wp(x) - c_0) - \wp'^2(x)}{(\wp(x) - c_0)^2}.$$

Using the following well-known differential equations for the $\wp$-function (see [4, p. 332]):

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3 \quad \text{and} \quad \wp''(z) = 6\wp^2(z) - \frac{1}{2}g_2, \tag{2.11}$$

where g_2 and g_3 are the coefficients in (2.3), we can express $F_1'(z)$ as

$$F_1'(z) = \frac{(\frac{g_2}{2} - 6c_0^2)\wp(z) + (g_3 + 2c_0^3 + \frac{g_2 c_0}{2})}{(\wp(z) - c_0)^2}. \tag{2.12}$$

Therefore $F_1'(x) = 0$ for some $0 < x < 1/2$ if and only if

$$\wp(x) = -\frac{g_3 + 2c_0^3 + \frac{g_2 c_0}{2}}{\frac{g_2}{2} - 6c_0^2}. \tag{2.13}$$

Since $\wp(x)$ is monotone in $0 < x < 1/2$, the latter shows that the equation $F_1'(x) = 0$ has at most one solution for $0 < x < 1/2$.

Since $\wp(x) \to +\infty$ as $x \to 0^+$, (2.12) implies

$$F_1'(x) = \frac{\frac{g_2}{2} - 6c_0^2}{\wp(x)} + o(1/\wp(x)) \quad \text{as } x \to 0^+. \tag{2.14}$$

By Lemma 2 of Section 4, we have $\frac{g_2}{2} - 6c_0^2 > 0$. Hence, (2.14) implies that $F_1'(x) > 0$ for all sufficiently small $x > 0$. Since $F_1(0) = 0$ the latter implies that $F_1(x) > 0$ for sufficiently small $x > 0$.

Finally, since $F_1(x)$ has at most one critical point on $0 < x < 1/2$ and $F_1(1/2) = 0$, it follows that $F_1(x) > 0$ for $0 < x < 1/2$. Therefore the function $\theta_1''(x)/\theta_1(x)$ strictly decreases on $0 < x < 1/2$. Hence, the logarithmic derivative in (2.1) is positive and therefore the function $S_1(u, v; t)$ strictly increases on $0 < t < \infty$. □

(b) The monotonicity property of the function $S_2(u, v; t)$ easily follows from the monotonicity property of $S_1(u, v; t)$. Indeed, since $\theta_2(z|\tau) = \theta_1(1/2 - z|\tau)$ we have

$$S_2(u, v; t) = S_1(1 - u, 1 - v; t) = S_1^{-1}(1 - v, 1 - u; t),$$

which is positive and decreasing by part **(a)** since $1 - v < 1 - u$. □

Remark 1. Since $F_1(0) = F_1(1/2) = 0$, it follows from our proof above that the equation $F_1'(x) = 0$ has precisely one solution on $0 < x < 1/2$. This together with (2.12) and (2.13) implies the inequality

$$\frac{-(g_3 + 2c_0^3 + \frac{g_2 c_0}{2})}{\frac{g_2}{2} - 6c_0^2} > e_1, \tag{2.15}$$

where $e_1 = \wp(1/2)$, which will be used in the proof of the monotonicity property of function $S_4(u, v; t)$ in Section 3.

3. Proof of Theorem 1 for $S_3(u, v; t)$ and $S_4(u, v; t)$

(a) The first part of this proof follows the same lines as in Section 2. First we work with $S_4(u, v; t)$. One can easily see that $S_4(u, v; t)$ is real and positive for $0 \le u < v < 1$ and $0 < t < \infty$. Since $\theta_4(x/2|i\pi t)$ satisfies the heat equation (1.6), we have

$$\frac{\partial}{\partial t} \log S_4(u, v; t) = \frac{\theta_4''(u/2|i\pi t)}{\theta_4(u/2|i\pi t)} - \frac{\theta_4''(v/2|i\pi t)}{\theta_4(v/2|i\pi t)}. \tag{3.1}$$

As in Section 2, our goal now is to prove that for a fixed $0 < t < \infty$, the function $\theta_4''(x)/\theta_4(x) := \theta_4''(x|i\pi t)/\theta_4(x|i\pi t)$ strictly decreases on $0 < x < 1/2$.

Differentiating the well-known periodicity relations

$$\frac{\theta_4'(z+1)}{\theta_4(z+1)} = \frac{\theta_4'(z)}{\theta_4(z)} \quad \text{and} \quad \frac{\theta_4'(z+\tau)}{\theta_4(z+\tau)} = -2\pi i + \frac{\theta_4'(z)}{\theta_4(z)},$$

we obtain

$$\begin{aligned}\frac{\theta_4''(z+1)\theta_4(z+1) - {\theta_4'}^2(z+1)}{\theta_4^2(z+1)} &= \frac{\theta_4''(z+\tau)\theta_4(z+\tau) - {\theta_4'}^2(z+\tau)}{\theta_4^2(z+\tau)} \\ &= \frac{\theta_4''(z)\theta_4(z) - {\theta_4'}^2(z)}{\theta_4^2(z)}.\end{aligned}$$

Therefore the function

$$P_4(z) := \frac{\theta_4''(z)\theta_4(z) - {\theta_4'}^2(z)}{\theta_4^2(z)} \tag{3.2}$$

is an elliptic function with periods 1 and τ having a single pole of order 2 at $z = \tau/2$ in the period rectangle. Taking the logarithm of both sides of the identity $\theta_1(z + \tau/2) = iq^{-1/4}e^{-i\pi z}\theta_4(z)$ and then differentiating twice, we obtain

$$P_4(z) := \frac{\theta_1''(z+\tau/2)\theta_1(z+\tau/2) - {\theta_1'}^2(z+\tau/2)}{\theta_1^2(z+\tau/2)} = -(\wp(z+\tau/2) - c_0), \tag{3.3}$$

where the second equality follows from equation (2.6).

From (3.2) and (3.3), we find

$$\frac{\theta_4''(z)}{\theta_4(z)} = \left(\frac{\theta_4'(z)}{\theta_4(z)}\right)^2 - (\wp(z+\tau/2) - c_0).$$

Differentiating this once more and using (3.3), we obtain

$$\frac{d}{dz}\frac{\theta_4''(z)}{\theta_4(z)} = -2\frac{\theta_4'(z)}{\theta_4(z)}(\wp(z+\tau/2) - c_0) - \wp'(z+\tau/2). \tag{3.4}$$

Now we consider the function

$$F_4(z) := 2\frac{\theta_4'(z)}{\theta_4(z)} + \frac{\wp'(z+\tau/2)}{\wp(z+\tau/2) - c_0}. \tag{3.5}$$

Equations (3.4) and (3.5) are counterparts of the equations (2.8) and (2.9) in Section 2. But now, as it follows from inequalities (4.4) of Lemma 1 in Section 4, the function $\wp(x + \tau/2) - c_0$ (which increases from $e_3 - c_0$ to $e_2 - c_0$ when x runs from 0 to 1/2) has a zero at some point s, $0 < s < 1/2$. In addition, we have $\wp'(x + \tau/2) > 0$ for $0 < x < 1/2$. This inequality follows, for instance, from the mapping properties of the $\wp$-function, which we discussed in the previous section. The latter shows that

$$\left.\frac{d}{dz}\frac{\theta_4''(x)}{\theta_4(x)}\right|_{z=s} = -\wp'(s+\tau/2) < 0.$$

Thus, to prove that the derivative in (3.4) is negative for $0 < x < 1/2$, we have to show that

$$F_4(x) < 0 \quad \text{for } 0 < x < s \quad \text{and} \quad F_4(x) > 0 \quad \text{for } s < x < 1/2. \tag{3.6}$$

Differentiating (3.5) and using differential equations (2.11), we find for $x \neq s$,

$$F_4'(x) = \frac{(\frac{g_2}{2} - 6c_0^2)\wp(x+\tau/2) + (g_3 + 2c_0^3 + \frac{g_2 c_0}{2})}{(\wp(x+\tau/2) - c_0)^2}. \tag{3.7}$$

We note that $\wp(x + \tau/2)$ strictly increases from $e_3 = \wp(\tau/2)$ to $e_2 = \wp(1/2 + \tau/2)$ when x runs from 0 to 1/2. Since $e_2 < e_1$ and $\frac{g_2}{2} - 6c_0^2 > 0$ by inequality (4.15) of Lemma 2 in Section 4, the latter observation together with (3.7) and inequality (2.15) of Remark 1 implies that

$$F_4'(x) < 0 \quad \text{for all } 0 < x < 1/2,\ x \neq s. \tag{3.8}$$

Now we find the values $F_4(0)$ and $F_4(1/2)$. Using the Fourier expansion

$$\frac{\theta_4'(z)}{\theta_4(z)} = 4\pi\sum_{m=1}^{\infty}\frac{q^m}{1-q^{2m}}\sin 2\pi m z, \tag{3.9}$$

we have

$$\frac{\theta_4'(0)}{\theta_4(0)} = \frac{\theta_4'(1/2)}{\theta_4(1/2)} = 0. \tag{3.10}$$

Since $z = \tau/2$ and $z = 1/2 + \tau/2$ are stationary points of the Weierstrass $\wp$-function, we have

$$\wp'(\tau/2) = \wp'(1/2 + \tau/2) = 0. \tag{3.11}$$

In addition, by Lemma 1 we have

$$\wp(\tau/2) - c_0 < 0 < \wp(1/2 + \tau/2) - c_0. \tag{3.12}$$

Substituting (3.10) and (3.11) into (3.5) and taking into account (3.12), we find

$$F_4(0) = F_4(1/2) = 0,$$

which together with (3.8) proves (3.6).

Therefore the derivative in (3.4) is negative for $0 < x < 1/2$. Hence, the function $\theta_4''(x)/\theta_4(x)$ strictly decreases on $0 < x < 1/2$. Thus, the logarithmic derivative in (3.1) is positive and therefore the proof of the monotonicity property of $S_4(u, v; t)$ is complete. $\square$

(b) To prove the monotonicity property of $S_3(u, v; t)$, we write

$$S_3(u, v; t) = S_4(1 - u, 1 - v; t) = S_4^{-1}(1 - v, 1 - u; t),$$

which is positive and decreasing by part **(a)**. $\square$

4. Some series expansions and inequalities

Lemma 1 *Let $e_1 = e_1(q) = \wp(1/2)$, $e_2 = e_2(q) = \wp(1/2 + \tau/2)$, $e_3 = e_3(q) = \wp(\tau/2)$ and let $c_0 = c_0(q)$ be defined by (2.5). Then for $|q| < 1$,*

$$e_1 - c_0 = \pi^2 + 8\pi^2 \sum_{k=1}^{\infty} \frac{q^{2k}}{(1 + q^{2k})^2}, \tag{4.1}$$

$$e_2 - c_0 = 8\pi^2 \sum_{k=0}^{\infty} \frac{q^{2k+1}}{(1 + q^{2k+1})^2}, \tag{4.2}$$

$$e_3 - c_0 = -8\pi^2 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^{2k}}. \tag{4.3}$$

In particular, for $0 < q < 1$, we have

$$e_3(q) < c_0(q) < e_2(q). \tag{4.4}$$

Proof. **(a)** By (2.4) and (2.6), we have

$$e_1 - c_0 = -P_1(1/2) = \pi^2 - 8\pi^2 \sum_{k=1}^{\infty} \frac{(-1)^k kq^{2k}}{1 - q^{2k}}. \tag{4.5}$$

For $|q| < 1$, the series under consideration converge absolutely, so after some algebra we find

$$\sum_{k=1}^{\infty} \frac{(-1)^k kq^{2k}}{1 - q^{2k}} = \sum_{k=1}^{\infty} (-1)^k kq^{2k} \left(\sum_{m=0}^{\infty} q^{2mk} \right) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} (-1)^k kq^{(2m+2)k} = \tag{4.6}$$
$$= \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} (-1)^k kq^{(2m+2)k} = -\sum_{m=1}^{\infty} \frac{q^{2m}}{(1 + q^{2m})^2},$$

where on the last step we used the binomial expansion for $(1 - x)^{-2}$.

Combining equations (4.5) and (4.6), we obtain (4.1).

(b) By (2.6), we have

$$e_2 - c_0 = \wp(1/2 + \tau/2) - c_0 = -\frac{d}{dz}\frac{\theta_1'(z)}{\theta_1(z)}\bigg|_{z=1/2+\tau/2} = -\frac{d}{dz}\frac{\theta_4'(z)}{\theta_4(z)}\bigg|_{z=1/2}. \tag{4.7}$$

Differentiating (3.9), we find

$$\frac{d}{dz}\frac{\theta_4'(z)}{\theta_4(z)}\bigg|_{z=1/2} = 8\pi^2 \sum_{k=1}^{\infty} \frac{(-1)^k k q^k}{1-q^{2k}}. \tag{4.8}$$

Next, we calculate as in part **(a)**:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^k k q^k}{1-q^{2k}} &= \sum_{k=1}^{\infty} (-1)^k k q^k \left(\sum_{m=0}^{\infty} q^{2mk}\right) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} (-1)^k k q^{(2m+1)k} = \\ &= \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} (-1)^k k q^{(2m+1)k} = -\sum_{m=0}^{\infty} \frac{q^{2m+1}}{(1+q^{2m+1})^2}.\end{aligned} \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we obtain (4.2), which is clearly positive for $0 < q < 1$.

(c) Calculating as in parts **(a)** and **(b)**, we obtain

$$c_0 - e_3 = -\wp(\tau/2) + c_0 = \frac{d}{dz}\frac{\theta_4'(z)}{\theta_4(z)}\bigg|_{z=0}. \tag{4.10}$$

Differentiating (3.9), we find

$$\frac{d}{dz}\frac{\theta_4'(z)}{\theta_4(z)}\bigg|_{z=0} = 8\pi^2 \sum_{k=1}^{\infty} \frac{k q^k}{1-q^{2k}}. \tag{4.11}$$

Combining (4.10) and (4.11), we obtain (4.3), which is negative for $0 < q < 1$. Therefore all assertions of the lemma, including inequalities (4.4), are proved. □

It follows from (4.4) that for a fixed $\tau = i\pi t$, $t > 0$, the equation $\wp(x + \tau/2) = c_0$ has precisely one solution, say s, on $0 < x < 1/2$. The latter equation contains a complex variable τ. Using the addition formula for the $\wp$-function, one can find the following equivalent equation

$$\wp(x) = \frac{2c_0 e_3 + e_1 e_2 + \pi^4 \theta_2^4(0)\theta_3^4(0)}{2(c_0 - e_3)}, \tag{4.12}$$

which is more convenient since it includes only real variables.

In the proof of Lemma 2 below we will use the *divisor function* $\sigma_x(n)$ that is important in Number Theory. For $x \geq 1$, this function is defined by

$$\sigma_x(n) = \sum_{d|n} d^x,$$

where the sum is taken over all divisors d of n, see [1, p. 38]. For $x = 1$, we write $\sigma(n) := \sigma_x(n)$. The divisor function satisfies a variety of nice identities. In particular, the following Liouville's convolution formula is important for us, see [5, equation 3.10]:

$$\sum_{m=1}^{k-1} \sigma(m)\sigma(k-m) = \frac{5}{12}\sigma_3(k) + \frac{1-6k}{12}\sigma(k). \tag{4.13}$$

Lemma 2 *Let $g_2 = g_2(q)$ be the invariant in the expansion (2.3) of the Weierstrass $\wp$-function and let $c_0 = c_0(q)$ be defined by (2.5). Then for $|q| < 1$,*

$$\frac{1}{2}g_2 - 6c_0^2 = 192\pi^2 \sum_{k=1}^{\infty} k\sigma(k)q^{2k}. \tag{4.14}$$

In particular, for $0 < q < 1$, we have

$$\frac{1}{2}g_2(q) - 6c_0(q)^2 > 0. \tag{4.15}$$

Proof. The invariant $g_2 = g_2(q)$ of the Weierstrass $\wp$-function can be represented as

$$g_2 = \frac{4\pi^4}{3} + 320\pi^4 \sum_{k=1}^{\infty} \sigma_3(k)q^{2k}, \tag{4.16}$$

see [1, p. 20].

Next we show that $c_0^2 = c_0^2(q)$ and therefore the function $g_2(q)/2 - 6c_0^2(q)$ itself also can be represented as a power series in q with the coefficients expressed in terms of the divisor function.

We want to emphasize here that for $|q| < 1$, all series under consideration converge absolutely which justifies our algebraic calculations.

For $|q| < 1$, we have

$$\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} = \sum_{n=1}^{\infty} \left(n \sum_{k=1}^{\infty} q^{2(k+1)n} \right) = \sum_{n=1}^{\infty} \sigma(n)q^{2n}. \tag{4.17}$$

Squaring both sides of (4.17), we obtain

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} \right)^2 &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma(n)\sigma(m)q^{2(m+n)} = \sum_{k=2}^{\infty} \left(\sum_{m=1}^{k-1} \sigma(m)\sigma(k-m) \right) q^{2k} \\ &= \frac{1}{12} \sum_{k=2}^{\infty} (5\sigma_3(k) + (1-6k)\sigma(k)q^{2k}, \end{aligned} \tag{4.18}$$

where in the last step we used Liouville's formula (4.13).

Now, combining formulas (2.5), (4.16), (4.17), and (4.18) and using Liouville's formula (4.13), we obtain (4.14):

$$\begin{aligned} \frac{1}{2}g_2 - 6c_0^2 = {} & 160\pi^4 \sum_{k=1}^{\infty} \sigma_3(k)q^{2k} + 32\pi^4 \left(\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} - 12 \left(\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} \right)^2 \right) \\ = {} & \pi^4 \left(160 \sum_{k=1}^{\infty} \sigma_3(k)q^{2k} + 32 \sum_{k=1}^{\infty} \sigma(k)q^{2k} - 32 \sum_{k=1}^{\infty} (5\sigma_3(k) + (1-6k)\sigma(k))\, q^{2k} \right) \\ = {} & 192\pi^4 \sum_{k=1}^{\infty} k\sigma(k)q^{2k}, \end{aligned}$$

which is clearly positive for $0 < q < 1$. Thus, (4.15) is also proved. □

Remark 2. Numerical evidence suggests that the odd and even partial derivatives in t of the quotients $S_i(u, v; t)$ have alternating signs. In modern literature such functions are called *completely monotonic.* It would be interesting to prove that for fixed u and v, $0 \le u < v < 1$, the quotients $S_i(u, v; t)$, $i = 1, 2, 3, 4$, are completely monotonic on $t > 0$.

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