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MODULAR-TYPE TRANSFORMATIONS AND INTEGRALS INVOLVING THE RIEMANN Ξ-FUNCTION

ATUL DIXIT

In memory of the great mathematician Hansraj Gupta

Abstract. A survey of various developments in the area of modular-type transformations (along with their generalizations of different types) and integrals involving the Riemann Ξ-function associated to them is given. We discuss their applications in Analytic Number Theory, Special Functions and Asymptotic Analysis.

1. Introduction

The Jacobi theta function $\theta(z) := \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}$ is one of the most important special functions of Mathematics. At the beginning of the last chapter on theta functions in his book [26, p. 314], Rainville remarks *'It seems safe to say that no topic in Mathematics is more replete with beautiful formulas than that on which we now embark'.* In Mathematics theta functions are encountered in Special Functions, Partial Differential Equations, Number Theory, and, in general, in Science in Heat Conduction, Electrical Engineering, Physics etc.

For $z \in \mathbb{H}$ (upper half plane), the famous theta transformation formula is given by [5, p. 12]

$$\theta\left(-1/4z\right) = \sqrt{-2iz}\,\theta(z),$$

or, equivalently,

$$\sum_{n=-\infty}^{\infty} \exp\left(\pi n^2/2iz\right) = \sqrt{-2iz}\sum_{n=-\infty}^{\infty} \exp\left(2\pi i n^2 z\right). \tag{1.1}$$

This implies [5, p. 12]

$$\theta\left(z/(4z+1)\right) = \sqrt{4z+1}\,\theta(z).$$

Along with the obvious fact $\theta(z+1) = \theta(z)$, this implies that for any $\gamma \in \Gamma_0(4)$,

$$\theta^2(\gamma z) = \chi_{-1}(d)(cz+d)\theta^2(z),$$

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where χ_{-1} is the Dirichlet character modulo 4 defined by $\chi_{-1}(n) = \left(\frac{-1}{n}\right) = (-1)^{(n-1)/2}$. Thus $\theta \in M_{1/2}(\Gamma_0(4), \chi_{-1})$, that is, the theta function is a weight 1/2 modular form on $\Gamma_0(4)$ twisted by the Dirichlet character χ_{-1}. Even though Eisenstein, and later Hardy, anticipated the theory of modular forms of half integral weight $k/2$, where k is an odd positive integer, a systematic study of such a theory commenced with a seminal paper by Shimura [30].

Letting $z = i\alpha^2/2$ and $\beta = 1/\alpha$, one can easily write (1.1) in a symmetric form, namely, for $\mathrm{Re}(\alpha^2) > 0$, $\mathrm{Re}(\beta^2) > 0$,

$$\sqrt{\alpha}\left(\frac{1}{2\alpha} - \sum\nolimits_{n=1}^{\infty} e^{-\pi\alpha^2 n^2}\right) = \sqrt{\beta}\left(\frac{1}{2\beta} - \sum\nolimits_{n=1}^{\infty} e^{-\pi\beta^2 n^2}\right). \tag{1.2}$$

Hardy [20] obtained an integral representation for the left-hand side of (1.2), namely for $\mathrm{Re}(\alpha^2) > 0$,

$$\sqrt{\alpha}\left(\frac{1}{2\alpha} - \sum\nolimits_{n=1}^{\infty} e^{-\pi\alpha^2 n^2}\right) = \frac{2}{\pi}\int_0^{\infty} \frac{\Xi(t/2)}{1+t^2}\cos\left(\frac{1}{2}t\log\alpha\right) dt, \tag{1.3}$$

and used (1.2) and (1.3) to prove that infinitely many zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line. Note that the integral in (1.3) is invariant if we replace α by β for $\alpha\beta = 1$. Hence, (1.3) also gives (1.2).

Even though the transformation (1.2) is associated with the modularity of the theta function $\theta(z)$, not all transformations of such type are known to be associated with modular forms. We begin with the following beautiful example from page 220 of Ramanujan's Lost Notebook [28].

Theorem 1.1. *Define $\lambda(x) := \psi(x) + \frac{1}{2x} - \log x$, where $\psi(x)$ is the logarithmic derivative of the gamma function. Let the Riemann ξ-function be defined by*

$$\xi(s) = (1/2)s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s),$$

and let

$$\Xi(t) := \xi(1/2 + it)$$

be the Riemann Ξ-function. If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\lambda(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\lambda(n\beta)\right\}$$
$$= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\, dt, \tag{1.4}$$

where γ denotes Euler's constant.

Note that [1, p. 259, formula 6.3.18] for $|\arg z| < \pi$, as $z \to \infty$,

$$\psi(z) \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots .$$

This implies that $\lambda(x) = O(x^{-2})$, and hence the series $\sum_{n=1}^{\infty}\lambda(n\alpha)$ and $\sum_{n=1}^{\infty}\lambda(n\beta)$ converge.

This formula was first proved in [2] where the authors gave two proofs. Later in [7], [8], it was obtained as a special case of a more general result which we will soon discuss. A yet another proof was given in [6].

A transformation of the form $\mathfrak{F}(z) = \mathfrak{F}(-1/z), z \in \mathbb{H}$, can be equivalently written in the form $F(\alpha) = F(\beta)$, where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and $\alpha\beta = 1$. Indeed, if $\mathrm{Im}(z) > 0$, then letting $\alpha = -iz$ gives $\mathrm{Re}(\alpha) > 0$. Thus, if $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$ and $\alpha\beta = 1$, then $-1/z = i\beta$, so that $\mathrm{Re}(\beta) > 0$. Now let $g(w) = h(e^{2\pi i w})$ so that $g(-1/z) = g(z)$ is equivalent to $h(e^{-2\pi\beta}) = h(e^{-2\pi\alpha})$. Now for $x > 0$, let $F(x) = h(e^{-2\pi x})$, so that $F(\alpha) = F(\beta)$. The process can also be reversed so that the transformation $\mathfrak{F}(z) = \mathfrak{F}(-1/z), z \in \mathbb{H}$, is actually equivalent to $F(\alpha) = F(\beta)$, where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\alpha\beta = 1$.

By a *modular-type transformation*, we mean a relation of the form $F(\alpha) = F(\beta), \alpha\beta = 1$. The word 'modular-type' is used to indicate that there may be some such transformations which cannot be made 'modular' in the sense that they may not be associated to a modular form on $\mathrm{SL}_2(\mathbb{Z})$ or its congruence subgroups. There are umpteen examples of modular-type transformations in Ramanujan's Notebooks [29] as well as in his Lost Notebook [28]. He preferred writing them in the form $F(\alpha) = F(\beta)$ over $\mathfrak{F}(z) = \mathfrak{F}(-1/z)$, such as the one in (1.4), and even though he always considered α, β to be positive real numbers, by analytic continuation, one can almost always extend his identities for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$.

In this survey, we will also discuss more general modular-type transformations of the form $F(z,\alpha) = F(z,\beta), F(w,\alpha) = F(iw,\beta)$, and $F(z,w,\alpha) = F(z,iw,\beta)$, where $\alpha\beta = 1$ and $i = \sqrt{-1}$.

Using the theory of Mellin transforms and residue calculus, or some ad-hoc techniques from special functions, the integrals involving the Riemann Ξ-function such as the ones in (1.3) and (1.4) can be respectively evaluated to one of the two expressions in a modular-type transformation such as the ones in (1.2) and (1.4) and then the corresponding modular-type transformations can be established through the invariance of the integrals upon replacing α by β. For the results obtained through this approach, see [2], [3], [6], [7], [8], [9] and [13]. Alternatively, one might first establish a modular-type transformation and then link it to an integral involving the Riemann Ξ-function. An indispensable part of this latter approach is the theory of reciprocal functions, and of self-reciprocal functions. Since the results obtained through the former approach are already surveyed in [10], we concentrate on the latter in this survey.

2. Modular-type transformations and integrals of $\Xi(t)$ through the theory of reciprocal functions

We first begin with a generalization of integrals of the type $\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)\cos(\frac{1}{2}t\log\alpha)\,dt$. where $f(t)$ is of the form $f(t) = g(it)g(-it)$ with g analytic in t, in which the cosine is replaced by a more general class of functions [14].

Let $\phi(x)$ and $\psi(x)$ be two integrable functions on the real line. The functions

ϕ and ψ are said to be reciprocal in the Fourier cosine transform if

$$\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty \psi(u)\cos(2ux)du \quad \text{and} \quad \psi(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty \phi(u)\cos(2ux)du.$$

Define $Z_1(s)$ and $Z_2(s)$ by

$$\Gamma\left(\frac{s}{2}\right)Z_1(s) := \int_0^\infty x^{s-1}\phi(x)dx, \quad \Gamma\left(\frac{s}{2}\right)Z_2(s) := \int_0^\infty x^{s-1}\psi(x)dx,$$

each valid in a specific vertical strip in the complex s-plane. Note that in case of a non-empty intersection of the two corresponding vertical strips, the Mellin inversion theorem gives

$$\phi(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma\left(\frac{s}{2}\right)Z_1(s)x^{-s}ds, \quad \psi(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma\left(\frac{s}{2}\right)Z_2(s)x^{-s}ds,$$

where $\text{Re}(s) = c$ lies in the intersection. Here and throughout this paper, by $\int_{(c)}$ we mean $\int_{c-i\infty}^{c+i\infty}$. Let

$$\Theta(x) := \phi(x) + \psi(x) \quad \text{and} \quad Z(s) := Z_1(s) + Z_2(s) \tag{2.1}$$

so that

$$\Gamma\left(\frac{s}{2}\right)Z(s) = \int_0^\infty x^{s-1}\Theta(x)dx$$

for values of s in the intersection of the two strips.

Let $0 < \omega \leq \pi$ and $\lambda < \frac{1}{2}$. If $f(z)$ is such that

i) $f(z)$ is analytic with $z = re^{i\theta}$, regular in the angle defined by $r > 0$, $|\theta| < \omega$,

ii) $f(z)$ satisfies the bounds

$$f(z) = \begin{cases} O(|z|^{-\lambda-\varepsilon}) & \text{if } |z| \text{ is small,} \\ O(|z|^{-b-\varepsilon}) & \text{if } |z| \text{ is large,} \end{cases}$$

for every $\varepsilon > 0$ and $b > \lambda$, and uniformly in any angle $\theta < \omega$, then we say that f belongs to the class K and write $f(z) \in K(\omega, \lambda, b)$.

With this set-up, the following result was proved in [14, Theorem 1.2].

Theorem 2.1. *Let $b > 1$ and $\phi, \psi \in K(\omega, 0, b)$ and let Θ and Z be defined in* (2.1). *Then we have*

$$\int_0^\infty \frac{\Xi(t)}{t^2+1/4} Z(1/2+it)dt = (\pi/2)Z(1) - (\pi/2)\sum_{n=1}^\infty \Theta(n\sqrt{\pi}).$$

This not only gives (1.3) as a special case but also the following general theta transformation along with a general integral involving $\Xi(t)$ [14, Corollary 1.2].

For $\alpha\beta = 1$, $\text{Re}(\alpha^2) > 0$, $\text{Re}(\beta^2) > 0$, and $w \in \mathbb{C}$,

$$\sqrt{\alpha}\left((e^{-\frac{w^2}{8}}/2\alpha) - e^{\frac{w^2}{8}}\sum_{n=1}^\infty e^{-\pi\alpha^2n^2}\cos(\sqrt{\pi}\alpha nw)\right)$$
$$= \sqrt{\beta}\left((e^{\frac{w^2}{8}}/2\beta) - e^{-\frac{w^2}{8}}\sum_{n=1}^\infty e^{-\pi\beta^2n^2}\cosh(\sqrt{\pi}\beta nw)\right)$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\Xi(t/2)}{1+t^2}\nabla\left(\alpha, w, (1+it)/2\right)\, dt, \tag{2.2}$$

where

$$\nabla(x, w, s) := \rho(x, w, s) + \rho(x, w, 1-s),$$

$$\rho(x, w, s) := x^{\frac{1}{2}-s}e^{-\frac{w^2}{8}}\,{}_1F_1\left((1-s)/2; 1/2; w^2/4\right),$$

with ${}_1F_1(a; c; z)$ being the confluent hypergeometric function.

Though the first equality in (2.2) is known since Jacobi, the integral involving $\Xi(t)$ in (2.2) was first found in [9]. In fact the first equality in (2.2) was obtained by first evaluating this integral to the expression on far left and then utilizing the fact that the integral is invariant under the simultaneous replacement of α by β and w by iw. This is one among the three examples of the generalized modular-type transformation of the form $F(w, \alpha) = F(iw, \beta)$ studied in [9], the other two being generalizations of some results of Ferrar [18] and Hardy [21].

In the last section of his paper [27], Ramanujan considered the integral

$$\mathfrak{I}_1(z, x)=\int_0^{\infty} \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos(\frac{t}{2}\log x)dt}{(z+1)^2+t^2}, \tag{2.3}$$

$x > 0$, and obtained alternate integral representations for it in the regions[1] $\mathrm{Re}(s) > 1$, $-1 < \mathrm{Re}(s) < 1$, $-3 < \mathrm{Re}(s) < -1$. In [7, Theorem 1.4], [8, Theorem 1.5], it was shown that this integral generalizes Ramanujan's result (1.4), thereby giving a generalized modular-type transformation of the type $F(z, \alpha) = F(z, \beta), \alpha\beta = 1$. This result is given below.

Theorem 2.2. *Let* $-1 < \mathrm{Re}(z) < 1$. *Let* $\lambda(z, x) = \zeta(z+1, x) - \frac{1}{2}x^{-z-1} + \frac{x^{-z}}{-z}$, *where* $\zeta(z, x)$ *is the Hurwitz zeta function. Let* $\mathfrak{I}_1(z, x)$ *be defined in* (2.3)*. Then for* α, $\beta > 0$, $\alpha\beta = 1$,

$$\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\mathfrak{I}_1(z, \alpha) = \alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty} \lambda(z, n\alpha) - \frac{\zeta(z+1)}{2\alpha^{z+1}} - \frac{\zeta(z)}{\alpha z}\right)$$

$$= \beta^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty} \lambda(z, n\beta) - \frac{\zeta(z+1)}{2\beta^{z+1}} - \frac{\zeta(z)}{\beta z}\right).$$

The integral $\mathfrak{I}_1(z, \alpha)$ involves a product of the Riemann Ξ-function at two different arguments, namely $\Xi(\frac{t+iz}{2})\Xi(\frac{t-iz}{2})$. An integral of a similar type, namely,

$$\mathfrak{I}_2(z, x) := \int_0^{\infty} \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log x\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\, dt \tag{2.4}$$

was studied first in [8]. It is associated to the famous Ramanujan-Guinand formula that will be discussed in the next section.

These examples motivate us, and indeed as will be seen in the next section, it is extremely fruitful to consider a more general integral where the cosine is replaced by a general class of functions. This was done in [15]. We

[1]Each of the representations for $\mathrm{Re}(s) > 1$ and $-3 < \mathrm{Re}(s) < -1$ involves an extra expression which should not be present. See [7, Theorem 1.2] for the corrected version.

provide below the set-up given in [15], albeit with one extra parameter w, for reasons to be clear soon. However, we first note that while the appropriate kernel with respect to which we study the reciprocal functions for studying integrals of the form $\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)Z\left(\frac{1+it}{2}\right)dt$ is the cosine function, the one while studying integrals of the form $\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2}\right)dt$ turns out to be $\cos(\pi z)\,M_{2z}(4\sqrt{tx})-\sin(\pi z)\,J_{2z}(4\sqrt{tx})$, where $M_z(x):=\frac{2}{\pi}K_z(x)-Y_z(x)$, with $J_z(x),Y_z(x)$ being the Bessel functions of the first and second kinds respectively and $K_z(x)$ being the modified Bessel function of the second kind.

Let the functions φ and ψ be related by

$$\varphi(x,z,w)=2\int_0^\infty\psi(t,z,w)\left(\cos(\pi z)\,M_{2z}(4\sqrt{tx})-\sin(\pi z)\,J_{2z}(4\sqrt{tx})\right)dt,$$
$$\psi(x,z,w)=2\int_0^\infty\varphi(t,z,w)\left(\cos(\pi z)\,M_{2z}(4\sqrt{tx})-\sin(\pi z)\,J_{2z}(4\sqrt{tx})\right)dt.$$

Let the normalized Mellin transforms $Z_1(s,z,w)$ and $Z_2(s,z,w)$ of the functions $\varphi(x,z,w)$ and $\psi(x,z,w)$ be defined by

$$\Gamma((s-z)/2)\,\Gamma((s+z)/2)\,Z_1(s,z,w)=\int_0^\infty x^{s-1}\varphi(x,z,w)\,dx,$$
$$\Gamma((s-z)/2)\,\Gamma((s+z)/2)\,Z_2(s,z,w)=\int_0^\infty x^{s-1}\psi(x,z,w)\,dx,$$

where each equation is valid in a specific vertical strip in the complex s-plane. Set

$$Z(s,z,w)=Z_1(s,z,w)+Z_2(s,z,w)\quad\text{and}\quad\Theta(x,z,w)=\varphi(x,z,w)+\psi(x,z,w),\qquad(2.5)$$

so that

$$\Gamma((s-z)/2)\,\Gamma((s+z)/2)\,Z(s,z,w)=\int_0^\infty x^{s-1}\Theta(x,z,w)\,dx$$

for values of s which lie in the intersection of the two vertical strips.

We now define a class of functions which will be used in the theorem below. Let $0<\omega\le\pi$ and $\eta>0$. For fixed z and w, let $u(s,z,w)$ be such that

(i) $u(s,z,w)$ is an analytic function of $s=re^{i\theta}$ regular in the angle defined by $r>0$, $|\theta|<\omega$,

(ii) $u(s,z,w)$ satisfies the bounds

$$u(s,z,w)=\begin{cases}O_{z,w}(|s|^{-\delta}) & \text{if } |s|\le 1,\\ O_{z,w}(|s|^{-\eta-1-|\mathrm{Re}(z)|}) & \text{if } |s|>1,\end{cases}$$

for every positive δ and uniformly in any angle $|\theta|<\omega$. Then we say that u belongs to the class $\Diamond_{\eta,\omega}$ and write $u(s,z,w)\in\Diamond_{\eta,\omega}$.

With this set-up, the following result was obtained in [15, Theorem 1.2] (see also [11, Equation (1.18)].

Theorem 2.3. *Let $\eta>1/4$ and $0<\omega\le\pi$. Suppose that $\varphi,\psi\in\Diamond_{\eta,\omega}$, are reciprocal in the Koshliakov kernel, and that $-1/2<\mathrm{Re}(z)<1/2$. Let $Z(s,z,w)$ and $\Theta(x,z,w)$ be defined in* (2.5). *Let $\sigma_{-z}(n)=\sum_{d|n}d^{-z}$. Then,*

$$\frac{32}{\pi}\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2},\frac{z}{2},w\right)\frac{dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}$$
$$=\sum_{n=1}^\infty \sigma_{-z}(n)n^{z/2}\Theta\left(\pi n, z/2, w\right)-R(z,w),$$

where

$$R(z,w)=\pi^{z/2}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)Z\left(1+\frac{z}{2},\frac{z}{2},w\right)+\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)Z\left(1-\frac{z}{2},\frac{z}{2},w\right).$$

This results in the following corollary.

Corollary 2.4. *Let* $-1<\mathrm{Re}(z)<1$*. Let* $\mathfrak{I}_2(z,x)$ *be defined in* (2.4)*. Then*

$$\mathfrak{I}_2(z,\alpha)=-(\pi\sqrt{\alpha}/32)\big(\alpha^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\tfrac{z}{2}\right)\zeta(z)+\alpha^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\tfrac{-z}{2}\right)\zeta(-z)$$
$$-4\sum_{n=1}^\infty \sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\alpha\right)\big). \qquad (2.6)$$

Further integrals of the type $\mathfrak{I}_1(z,x),\mathfrak{I}_2(z,x)$ are studied in [13] and [3, Theorem 15.6]. A companion to Theorem 2.3, which evaluates a generalization of $\mathfrak{I}_1(z,x)$, is also studied in [15, Theorem 1.4].

3. Applications of modular-type transformations and the integrals of $\Xi(t)$ linked to them

Here we discuss three different applications of modular-type transformations and the integrals of $\Xi(t)$ associated to them.

3.1. **Theory of the generalized modified Bessel function** $K_{z,w}(x)$ **and the generalized modular-type transformations** $F(z,w,\alpha)=F(z,iw,\beta)$**, where** $\alpha\beta=1$**.** The theta transformation (1.2) can be simply derived by invoking the Poisson summation formula and the Laplace integral evaluation

$$e^{-\alpha^2x^2}=\frac{2}{\alpha\sqrt{\pi}}\int_0^\infty e^{-u^2/\alpha^2}\cos(2ux)\,du. \qquad (3.1)$$

In the similar vein, using a generalization of (3.1), namely

$$e^{-\alpha^2x^2}\cos(wx)=\frac{2e^{-w^2/(4\alpha^2)}}{\alpha\sqrt{\pi}}\int_0^\infty e^{-u^2/\alpha^2}\cosh(wu/\alpha^2)\cos(2ux)\,du \qquad (w\in\mathbb{C}), \qquad (3.2)$$

one gets the general theta transformation in (2.2). Since the inverse Mellin transform of $\Gamma(s)$ is essentially e^{-x^2}, one may want to ask if one can obtain an integral identity similar to (3.1), which renders $K_0(x)$ as a self-reciprocal function in a kernel, since $K_0(x)$ is essentially the inverse Mellin transform of $\Gamma^2(s)$. More generally one may ask the same question for $K_z(x)$. This was already solved by Koshliakov [23, Equation (8)] who obtained the following remarkable identity for $-1/2<z<1/2$ [2],

$$2\int_0^\infty K_z(2t)\left(\cos(\pi z)M_{2z}(4\sqrt{xt})-\sin(\pi z)J_{2z}(4\sqrt{xt})\right)dt=K_z(2x). \qquad (3.3)$$

[2]It is easy to see that this identity actually holds for $-1/2<\mathrm{Re}(z)<1/2$.

For this reason, the kernel $\cos(\pi z)M_{2z}(4\sqrt{xt}) - \sin(\pi z)J_{2z}(4\sqrt{xt})$ is called the *Koshliakov kernel* in [3] and [15].

Now it is natural to ask if there exists a pair of functions *reciprocal* in the Koshliakov kernel, and which gives (3.3) as a special case, similar to how (3.2) subsumes (3.1). This question was answered in [11]. The interesting thing here is, while generalizing (3.1) to (3.2) still involves elementary functions, namely $e^{-\alpha^2x^2}\cos(wx)$ and $e^{-\alpha^2x^2}\cosh(wx)$, generalizing (3.3) involves a *new* special function $K_{z,w}(x)$, which we call the generalized modified Bessel function. It is defined for $z, w \in \mathbb{C}$, $x \in \mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$ and c=Re$(s) > \pm$ Re(z) by an inverse Mellin transform [11], namely,

$$K_{z,w}(x) = \frac{1}{2\pi i}\int_{(c)}\Gamma((s-z)/2)\,\Gamma((s+z)/2)$$
$$ {}_1F_1\bigg((s-z)/2; 1/2; -w^2/4\bigg)\,{}_1F_1\bigg((s+z)/2; 1/2; w^2/4\bigg)2^{s-2}x^{-s}ds. \quad (3.4)$$

Note that if we let $w = 0$, the generalized modified Bessel function reduces to the modified Bessel function $K_z(x)$. It is shown in [11] that $K_{z,w}(x)$ satisfies a rich and a beautiful theory like its special case $K_z(x)$. The generalization of (3.3) is then given in the following theorem [11, Theorem 1.1].

Theorem 3.1. *Let $-\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}$. Let $w \in \mathbb{C}$ and $x > 0$. Let α and β be two positive numbers such that $\alpha\beta = 1$. The functions $e^{-\frac{w^2}{2}}K_{z,iw}(2\alpha x)$ and $\beta\, K_{z,w}(2\beta x)$ form a pair of reciprocal functions in the Koshliakov kernel, that is,*

$$e^{-\frac{w^2}{2}}K_{z,iw}(2\alpha x) = 2\int_0^\infty \beta\, K_{z,w}(2\beta t)\,\Big(\cos(\pi z)M_{2z}(4\sqrt{xt}) - \sin(\pi z)J_{2z}(4\sqrt{xt})\Big)\,dt,$$
$$\beta\, K_{z,w}(2\beta x) = 2\int_0^\infty e^{-\frac{w^2}{2}}K_{z,iw}(2\alpha t)\,\Big(\cos(\pi z)M_{2z}(4\sqrt{xt}) - \sin(\pi z)J_{2z}(4\sqrt{xt})\Big)\,dt.$$

However, we emphasize here that we stumbled upon this interesting generalization of the modified Bessel function while seeking a generalization of a formula of Ramanujan [28, p. 253] rediscovered by Guinand [19]. For $\alpha\beta = \pi^2$, this formula is given by

$$\sqrt{\alpha}\sum_{n=1}^\infty \sigma_{-z}(n)n^{z/2}K_{z/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^\infty \sigma_{-z}(n)n^{z/2}K_{z/2}(2n\beta)$$
$$=\frac{1}{4}\Gamma\left(\frac{z}{2}\right)\zeta(z)\{\beta^{(1-z)/2}-\alpha^{(1-z)/2}\}+\frac{1}{4}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\{\beta^{(1+z)/2}-\alpha^{(1+z)/2}\}. \quad (3.5)$$

This formula can be written symmetrically in α and β [8, Theorem 1.4], and is, in this latter form, an example of the generalized modular-type transformation of the type $F(z, \alpha) = F(z, \beta)$. As discussed in [4, p. 23], this identity is equivalent to the functional equation of the non-holomorphic Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$. In [8], (3.5) was derived from (2.6) whereas in [15], Theorem 2.3 and (3.5) are used to obtain (2.6).

The elegant generalization of the Ramanujan-Guinand formula, symmetric in α and β, that was established in [11, Theorem 1.5] is now given.

Theorem 3.2. *Let $w \in \mathbb{C}$, $z \in \mathbb{C}\backslash\{-1,1\}$. For $\alpha, \beta > 0$ such that $\alpha\beta = 1$,*

$$\sqrt{\alpha}\Bigg(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{-\frac{w^2}{4}}K_{\frac{z}{2},iw}(2n\pi\alpha)-\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\alpha^{\frac{z}{2}-1}\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right)$$
$$-\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\alpha^{-\frac{z}{2}-1}\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)\Bigg)$$
$$=\sqrt{\beta}\Bigg(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{\frac{w^2}{4}}K_{\frac{z}{2},w}(2n\pi\beta)-\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\beta^{\frac{z}{2}-1}$$
$${}_1F_1\left(\frac{1-z}{2};\frac{1}{2};-\frac{w^2}{4}\right)-\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\beta^{-\frac{z}{2}-1}{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)\Bigg). \quad (3.6)$$

This is an example of a generalized modular-type transformation of the form $F(z, w, \alpha) = F(z, iw, \beta)$, where $\alpha\beta = 1$. Indeed, (3.5) follows at once from (3.6) by letting $w = 0$.

Let $\nabla_2(x, z, w, s)$ be defined by

$$\nabla_2(x, z, w, s) := \rho(x, z, w, s) + \rho(x, z, w, 1-s), \quad (3.7)$$

where

$$\rho(x, z, w, s) := x^{\frac{1}{2}-s}{}_1F_1\left(\frac{1-s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{1-s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right).$$

Using the reciprocal pair $(e^{-w^2/2}K_{z,iw}(2\alpha x), \beta\, K_{z,w}(2\beta x)), \alpha\beta = 1$, in Theorem 2.3 along with (3.6), the integral involving $\Xi(t)$ corresponding to the expressions in (3.6) was obtained [11, Theorem 1.3] as shown below.

Theorem 3.3. *Let $w \in \mathbb{C}$ and $-1 < \mathrm{Re}(z) < 1$. Let $K_{z,w}(x)$ and $\nabla_2(x, z, w, s)$ be defined in (3.4) and (3.7) respectively. If α and β are positive integers satisfying $\alpha\beta = 1$, then*

$$\frac{16}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\nabla_2\left(\alpha,\frac{z}{2},w,\frac{1+it}{2}\right)\,dt}{(t^2+(z+1)^2)\,(t^2+(z-1)^2)}$$
$$=e^{-\frac{w^2}{4}}\sqrt{\alpha}\Bigg\{4\sum\nolimits_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{-\frac{w^2}{4}}K_{\frac{z}{2},iw}(2n\pi\alpha)$$
$$-\Gamma(z/2)\zeta(z)\pi^{-\frac{z}{2}}\alpha^{\frac{z}{2}-1}\,{}_1F_1((1-z)/2;1/2;w^2/4)$$
$$-\Gamma(-z/2)\zeta(-z)\pi^{\frac{z}{2}}\alpha^{-\frac{z}{2}-1}\,{}_1F_1((1+z)/2;1/2;w^2/4)\Bigg\}.$$

3.2. **A far-reaching generalization of Hardy's theorem on infinitude of zeros of $\zeta(s)$ on the critical line.** This sub-section illustrates an application of a modular-type transformation associated with an integral involving $\Xi(t)$, this time the general theta transformation (2.2), in analytic number theory.

As mentioned in the introduction, Hardy [20] proved in 1914 that infinitely many zeros of $\zeta(s)$ lie on the critical line using (1.2) and (1.3). Let

$$\eta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \quad \text{and} \quad \rho(t) := \eta(1/2 + it).$$

In [14], we generalized Hardy's result by showing that infinitely many zeros of an infinite series whose summands involve the completed zeta function $\rho(t)$ on bounded vertical shifts lie on the critical line too. The precise theorem is now given.

Theorem 3.4. *Let* $\{c_j\}$ *be a sequence of non-zero real numbers so that* $\sum_{j=1}^{\infty}|c_j| < \infty$. *Let* $\{\lambda_j\}$ *be a bounded sequence of distinct real numbers that attains its bounds. Then the function* $F(s) = \sum_{j=1}^{\infty} c_j\eta(s+i\lambda_j)$ *has infinitely many zeros on the critical line* $\mathrm{Re}(s) = 1/2$.

The above theorem also uses (1.2) and (1.3). Hardy's result is simply its special case when all but one c_j's are zero and the remaining non-zero c_j is 1.

Now a natural question arises - can one generalize the above theorem where one uses the general theta transformation (2.2) rather than (1.2) and (1.3)? Indeed, this can be done. It led to the following result that appeared in [12, Theorem 2].

Theorem 3.5. *Let* $\{c_j\}$ *be a sequence of non-zero real numbers so that* $\sum_{j=1}^{\infty}|c_j| < \infty$. *Let* $\{\lambda_j\}$ *be a bounded sequence of distinct real numbers such that it attains its bounds. Let* $\mathfrak{D}$ *denote the region* $|\mathrm{Re}(w) - \mathrm{Im}(w)| < \sqrt{\frac{\pi}{2}} - \sqrt{\frac{2}{\pi}}\,\mathrm{Re}(w)\mathrm{Im}(w)$ *in the* w*-complex plane. Then for any* $w \in \mathfrak{D}$, *the function*

$$F_w(s)=\sum_{j=1}^{\infty} c_j\eta(s+i\lambda_j)\left\{{}_1F_1\left(\frac{1-(s+i\lambda_j)}{2};\frac{1}{2};\frac{w^2}{4}\right)+{}_1F_1\left(\frac{1-(\bar{s}-i\lambda_j)}{2};\frac{1}{2};\frac{\bar{w}^2}{4}\right)\right\}$$

has infinitely many zeros on the critical line $\mathrm{Re}(s) = 1/2$.

3.3. **Asymptotic expansion of an integral involving** $\Xi(t)$**.** The advantage of having an alternate representation for an expression, that is, an identity, is that it may give more information about the expression thereby enhancing our understanding of it. This sub-section bears a testimony to an instance of such a phenomenon.

In [13, Theorem 6.3], the integral $\mathfrak{I}_1(z,x)$, defined in (2.3), was expressed as a Laplace transform:

Theorem 3.6. *Assume* $-1 < \mathrm{Re}(z) < 1$. *Define* $\Omega(x,z)$ *by*

$$\Omega(x,z) = 2\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}\left(e^{\pi iz/4}K_z(4\pi e^{\pi i/4}\sqrt{nx}) + e^{-\pi iz/4}K_z(4\pi e^{-\pi i/4}\sqrt{nx})\right),$$

where $\sigma_{-z}(n) = \sum_{d|n} d^{-z}$. *Then for* $\alpha, \beta > 0, \alpha\beta = 1$,

$$\begin{aligned}\frac{1}{2\pi^{(z+5)/2}}\mathfrak{I}_1(z,\alpha) &= \alpha^{(z+1)/2}\int_0^{\infty} e^{-2\pi\alpha x}x^{z/2}\left(\Omega(x,z) - \frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx\\ &= \beta^{(z+1)/2}\int_0^{\infty} e^{-2\pi\beta x}x^{z/2}\left(\Omega(x,z) - \frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx.\end{aligned}$$

Applying Watson's lemma to the first expression for $\mathfrak{I}_1(z,\alpha)$ involving α led us to its following asymptotic expansion [17, Theorem 1.10]:

Theorem 3.7. *Fix* z *such that* $-1 < \mathrm{Re}\ z < 1$. *As* $\alpha \to \infty$,

$$\frac{1}{\pi^{(z+3)/2}}\mathfrak{I}_1(z,\alpha) \sim -\frac{\Gamma(z)\zeta(z)\alpha^{\frac{z-1}{2}}}{(2\pi)^z} - \frac{\Gamma(z+1)\zeta(z+1)}{2\alpha^{\frac{z+1}{2}}(2\pi)^z}$$

$$+ 2\alpha^{\frac{z+1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2\pi\alpha)^{2m+z+2}} \Gamma(2m+2+z)\zeta(2m+2)\zeta(2m+z+2).$$

Oloa's asymptotic expansion[3] [24, Equation 1.5] of $\mathfrak{I}_1(0,\alpha)$, namely, as $\alpha \to \infty$,

$$\frac{1}{\pi^{3/2}}\mathfrak{I}_1(0,\alpha) \sim \frac{1}{2}\frac{\log\alpha}{\sqrt{\alpha}} + \frac{1}{2\sqrt{\alpha}}(\log 2\pi - \gamma) + \frac{\pi^2}{72\alpha^{3/2}} - \frac{\pi^4}{10800\alpha^{7/2}} + \cdots,$$

can be readily obtained by letting $z \to 0$ in (3.7).

4. Concluding remarks and further questions

We hope to have demonstrated the usefulness of modular-type transformations along with the associated integrals involving $\Xi(t)$. It would be remarkable if one is able to associate at least some of them to modular forms.

While it may seem from the variety of examples considered here that one can always associate an integral involving $\Xi(t)$ to a modular-type transformation, there are some conjectured modular-type transformations for which there are no such integral representations. For example, consider the following remarkable conjecture of Hardy and Littlewood [22, p. 158, Equation (2.516)] suggested to them by work of Ramanujan.

Conjecture 4.1. *Let $\mu(n)$ denote the Möbius function. Let α and β be two positive numbers such that $\alpha\beta = 1$. Assume that the series $\sum_\rho \left(\Gamma((1-\rho)/2)/\zeta^{'}(\rho)\right) a^\rho$ converges, where ρ runs through the non-trivial zeros of $\zeta(s)$ and a denotes a positive real number, and that the non-trivial zeros of $\zeta(s)$ are simple. Then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}(\mu(n)/n)e^{-\pi\alpha^2/n^2} - \frac{1}{4\sqrt{\pi}\sqrt{\alpha}}\sum_\rho \frac{\Gamma((1-\rho)/2)}{\zeta^{'}(\rho)}\pi^{\frac{\rho}{2}}\alpha^\rho$$
$$= \sqrt{\beta}\sum_{n=1}^{\infty}(\mu(n)/n)e^{-\pi\beta^2/n^2} - \frac{1}{4\sqrt{\pi}\sqrt{\beta}}\sum_\rho \frac{\Gamma((1-\rho)/2)}{\zeta^{'}(\rho)}\pi^{\frac{\rho}{2}}\beta^\rho.$$

A generalization of this conjecture was obtained in [9, Theorem 1.6] which led to a Riesz-type criterion for the Riemann Hypothesis in [16, Theorem 1.1].

Let $\mathrm{erf}(w)$ and $\mathrm{erfi}(w)$ denote the error function and the complementary error function respectively. In view of the remark made before the conjecture (4.1), we do like to point out that there is a modular-type transformation obtained in [17, Equation (1.18)], namely

$$\sqrt{\alpha}e^{\frac{w^2}{8}}\left(\mathrm{erf}\left(\frac{w}{2}\right) + 4\int_{-\infty}^{0}\frac{e^{-\pi\alpha^2x^2}\sin(\sqrt{\pi}\alpha xw)}{e^{2\pi x}-1}\,dx\right)$$
$$= \sqrt{\beta}e^{\frac{-w^2}{8}}\left(\mathrm{erfi}\left(\frac{w}{2}\right) + 4\int_{-\infty}^{0}\frac{e^{-\pi\beta^2x^2}\sinh(\sqrt{\pi}\beta xw)}{e^{2\pi x}-1}\,dx\right), \tag{4.1}$$

[3]There is a slight misprint in this asymptotic expansion given in Oloa's paper. The minus sign in front of the second expression on the right-hand side there should be a plus. This has been corrected here.

whose expressions, we believe, are equal to an integral involving $\Xi(t)$. However, we are unable to find this integral. If it exists, it would be significant, as it would enable us to find an integral involving $\Xi(t)$ for the modular-type transformation corresponding to an integral analogue of the Jacobi theta function. See [17, p. 32] for a discussion on this topic.

In [17, Section 7], two questions were posed regarding the exact evaluation of

$$\int_0^\infty \frac{xe^{-\pi x^2}}{e^{2\pi x}-1} {}_1F_1(-2k; \tfrac{3}{2}; 2\pi x^2)\, dx$$

for $k \in \mathbb{Z}^+ \cup \{0\}$, and an exact evaluation of, or at least an approximation to

$$\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1} {}_1F_1\left(-2k-1; \frac{3}{2}; 2\alpha x^2\right) dx$$

when $\alpha \neq \pi$ is a positive real number and $k \in \mathbb{Z}^+ \cup \{0\}$. These integrals resulted from differentiating some modular type transformations of the form $F(w, \alpha) = F(iw, \beta), \alpha\beta = 1$, involving the error functions. These questions were recently solved partially by Paris [25] who obtained approximations of the integrals to within exponentially small accuracy when k is large and $\alpha = O(1)$.

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Atul Dixit
Discipline of Mathematics, Indian Institute of Technology Gandhinagar
Palaj, Gandhinagar-382355, Gujarat, India
E-mail: *adixit@iitgn.ac.in*