ON HURWITZ ZETA FUNCTION AND LOMMEL FUNCTIONS

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Dedicated to Professor Bruce C. Berndt on the occasion of his 80th birthday

Abstract. We obtain a new proof of Hurwitz's formula for the Hurwitz zeta function $\zeta(s,a)$ beginning with Hermite's formula. The aim is to reveal a nice connection between $\zeta(s,a)$ and a special case of the Lommel function $S_{\mu,\nu}(z)$. This connection is used to rephrase a modular-type transformation involving infinite series of Hurwitz zeta function in terms of those involving Lommel functions.

1. Introduction

The Hurwitz zeta function $\zeta(s,a)$ is defined for $\mathrm{Re}(s)>1$ and $a\in\mathbb{C}\backslash\{x\in\mathbb{R}:x\leq 0\}$ by [23, p. 36]

$$\zeta(s,a):=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}.$$

It is well-known that for $\zeta(s,a)$ can be analytically continued to the entire s-complex plane except for a simple pole at $s=1$ with residue 1, and that $\zeta(s,1)=\zeta(s)$.

One of the fundamental results in the theory of $\zeta(s,a)$ is the following formula of Hurwitz [23, p. 37, Equation (2.17.3)].

Theorem 1.1. *For $0<a\leq 1$ and $\mathrm{Re}(s)<0$,*

$$\zeta(s,a)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left\{\sin\left(\frac{1}{2}\pi s\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi na)}{n^{1-s}}+\cos\left(\frac{1}{2}\pi s\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi na)}{n^{1-s}}\right\}. \tag{1.1}$$

The above result also holds[1] *for $\mathrm{Re}(s)<1$ if $0<a<1$.*

We note that when $a=1$, the above formula reduces to the functional equation of $\zeta(s)$ [23, p. 13, Equation (2.1.1)] for $\mathrm{Re}(s)<0$. which can then be seen to be true for all $s\in\mathbb{C}$ by analytic continuation.

Several proofs of (1.1) are available in the literature. For example, Hurwitz himself obtained it by transforming the Mellin transform representation of $\zeta(s,a)$ as a loop integral and then evaluating the latter. This proof can be found, for example, in [23, p. 37]. Berndt [4, Section 5] found a short proof of (1.1) by using the boundedly convergent Fourier series of $\lfloor x\rfloor-x+\frac{1}{2}$. We refer the reader interested in knowing the various proofs of this formula to [9] and the references therein (see also [10]). In [9, Section 4], Kanemitsu, Tanigawa, Tsukada

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[1] See [1, p. 257, Theorem 12.6].

and Yoshimoto obtained a new proof of (1.1). Their proof commences with employing [9, Equation (4.1)] (see also [11, Equation (47)])

$$\zeta(s,a)=\frac{1}{2}a^{-s}+\frac{a^{1-s}}{s-1}+\sum_{n=1}^{\infty}\left\{\frac{e^{-2\pi ina}}{(-2\pi ina)^{1-s}}\Gamma(1-s,-2\pi ina)+\frac{e^{2\pi ina}}{(2\pi ina)^{1-s}}\Gamma(1-s,2\pi ina)\right\},$$

which is a special case of the Ueno-Nishizawa formula [24] and then invoking the Fourier series of the Dirac-delta function $\delta(s)$.

The aim of this note is to give a yet another new proof of (1.1) beginning with Hermite's well-known formula for $\zeta(s,a)$ [16, p. 609, Formula 25.11.29], valid for $\mathrm{Re}(a)>0$ and $s\neq 1$:

$$\zeta(s,a)=\frac{1}{2}a^{-s}+\frac{a^{1-s}}{s-1}+2\int_0^{\infty}\frac{\sin(s\tan^{-1}(x/a))\,dx}{(a^2+x^2)^{s/2}\,(e^{2\pi x}-1)}. \tag{1.2}$$

The novelty of this proof is that it reveals the connection between Hurwitz zeta function and the Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$ which, to the best of our knowledge, seems to have been unnoticed before. The Lommel functions are defined by [25, p. 346, equation (10)]

$$s_{\mu,\nu}(z)=\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}{}_1F_2\left(1;\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{3}{2},\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{3}{2};-\frac{1}{4}z^2\right). \tag{1.3}$$

and [25, p. 347, equation (2)]

$$S_{\mu,\nu}(z)=s_{\mu,\nu}(z)+\frac{2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\sin(\nu\pi)}\times\left\{\cos\left(\frac{1}{2}(\mu-\nu)\pi\right)J_{-\nu}(z)-\cos\left(\frac{1}{2}(\mu+\nu)\pi\right)J_{\nu}(z)\right\} \tag{1.4}$$

for $\nu\notin\mathbb{Z}$, and

$$S_{\mu,\nu}(z)=s_{\mu,\nu}(z)+2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)\times\left\{\sin\left(\frac{1}{2}(\mu-\nu)\pi\right)J_{\nu}(z)-\cos\left(\frac{1}{2}(\mu-\nu)\pi\right)Y_{\nu}(z)\right\} \tag{1.5}$$

for $\nu\in\mathbb{Z}$, where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions of the first and second kinds respectively. The Lommel functions are the solutions of an inhomogeneous form of the Bessel differential equation [25, p. 345], namely,

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+(z^2-\nu^2)y=z^{\mu+1}.$$

Lommel functions arise in mathematics, for example, in the theory of positive trigonometric sums[12]. Outside of mathematics, Lommel functions have been found to be very useful in physics as well as mathematical physics. See, for example, [2, 7, 20, 22]. Lewis [14] studied a special case of $S_{\mu,\nu}(z)$, that is,

$$\mathcal{C}_s(z)=\sqrt{z}\,\Gamma(2s+1)S_{-2s-\frac{1}{2},\frac{1}{2}}(z), \tag{1.6}$$

and represented it in terms of the incomplete gamma function. Lewis and Zagier [13] represented the period functions for Maass wave forms with spectral parameter s in terms of an infinite series of $\mathcal{C}_s(z)$, and in the course of which they gave different representations for this special case of the Lommel function. See [13, p. 214, Proposition 1].

In the present work, we require a new integral representation for this special case of the Lommel function $S_{\mu,\nu}(z)$ which, to the best of our knowledge, does not seem to have been explicitly stated anywhere including [13]. This is derived in Lemma 2.1.

Another ingredient needed in our proof of (1.1) is a recent result of Maširević [15, Theorem 2.1] (see also [3, p. 176, Theorem 5.23]) which states that for all $m \in \mathbb{N} \cup \{0\},\ \nu \in \mathbb{R},\ x \in (0, 2\pi)$ and $\mu > \max\left\{-\nu - 1, \nu - 2, -\frac{1}{2}\right\}$,

$$\sum_{k=1}^{\infty} \frac{s_{\mu,\nu}(kx)}{k^{2m+\mu+1}} = \frac{x^{\mu+1}}{4}\Gamma\left(\frac{1+\mu-\nu}{2}\right)\Gamma\left(\frac{1+\mu+\nu}{2}\right)$$
$$\times\left(\frac{(-1)^m\pi}{2\Gamma(m+1+(\mu-\nu)/2)\Gamma(m+1+(\mu+\nu)/2)}\left(\frac{x}{2}\right)^{2m-1}\right.$$
$$\left.+\sum_{n=0}^{m}\frac{(-1)^n\zeta(2m-2n)}{\Gamma(n+1+(1+\mu-\nu)/2)\Gamma(n+1+(1+\mu+\nu)/2)}\left(\frac{x}{2}\right)^{2n}\right). \quad (1.7)$$

2. A new proof of Hurwitz's formula using Hermite's formula (1.2)

Here we prove Theorem 1.1. To do that, however, we first need a lemma which evaluates an integral in terms of the Lommel function $S_{\mu,\nu}(z)$. This lemma seems to be new.

Lemma 2.1. *Let the Lommel function $S_{\mu,\nu}(z)$ be defined in* (1.4) *and* (1.5)*. For $k \in \mathbb{N}$ and $a > 0$, we have*

$$\int_0^{\infty} \frac{e^{-2\pi kx}\sin(s\tan^{-1}(x/a))}{(a^2+x^2)^{s/2}}\ dx = s\sqrt{a}(2\pi k)^{s-\frac{1}{2}}S_{-s-\frac{1}{2},\frac{1}{2}}(2\pi ak). \quad (2.1)$$

Proof. We first prove the above result for $\mathrm{Re}(s) < 0$ and then extend it to all $s \in \mathbb{C}$ by analytic continuation.

Using the inverse Mellin transform representation of the exponential function, for $c_1 := \mathrm{Re}(\xi) > 0$ and $k > 0$,

$$\frac{1}{2\pi i}\int_{(c_1)} \frac{\Gamma(\xi)}{(2\pi k)^{\xi}} y^{-\xi}\ d\xi = e^{-2\pi ky}, \quad (2.2)$$

where here, and throughout the paper, $\int_{(d)}$ will always mean the integral $\int_{d-i\infty}^{d+i\infty}$.

Also, from [17, p. 193, Formula 5.19], for $-1 < c_2 := \mathrm{Re}(\xi) < \mathrm{Re}(s)$,

$$\frac{1}{2\pi i}\int_{(c_2)} \frac{\Gamma(s-\xi)\Gamma(\xi)}{\Gamma(s)}\sin\left(\frac{\pi\xi}{2}\right)a^{\xi-s}y^{-\xi}\ d\xi = \frac{\sin\left(s\tan^{-1}\left(\frac{y}{a}\right)\right)}{(y^2+a^2)^{\frac{s}{2}}}. \quad (2.3)$$

From (2.2), (2.3) and Parseval's identity [18, p. 82, Equation (3.1.11)]

$$\int_0^{\infty} g(x)h(x)\,dx = \frac{1}{2\pi i}\int_{(c)} \mathfrak{G}(1-s)\mathfrak{H}(s)\,ds,$$

where $\mathfrak{G}$ and $\mathfrak{H}$ are Mellin transforms of g and h respectively, for $c := \mathrm{Re}(\xi) < \mathrm{Re}(s)$ and $-1 < \mathrm{Re}(\xi) < 1$,

$$2\int_0^{\infty} \frac{e^{-2\pi kx}\sin(s\tan^{-1}(x/a))}{(a^2+x^2)^{s/2}}\ dx = \frac{2}{2\pi i}\int_{(c)} \frac{\Gamma(\xi)\Gamma(1-\xi)\Gamma(s-\xi)}{(2\pi k)^{1-\xi}\Gamma(s)}\sin\left(\frac{\pi\xi}{2}\right)a^{\xi-s}\ d\xi$$
$$= \frac{a^{-s}}{\Gamma(s)}\frac{1}{2\pi i}\int_{(c)} \frac{\pi\Gamma(s-\xi)}{\cos\left(\frac{\pi\xi}{2}\right)}(2\pi k)^{\xi-1}a^{\xi}\ d\xi$$

$$= \frac{a^{-s}}{\Gamma(s)} \frac{1}{2\pi i} \int_{(c)} \Gamma\left(\frac{1+\xi}{2}\right) \Gamma\left(\frac{1-\xi}{2}\right) \Gamma(s-\xi) (2\pi k)^{\xi-1} a^{\xi}\ d\xi,$$

where we used the reflection formula for the gamma function. Replacing ξ by $-\xi$, we see that for $c := \mathrm{Re}(\xi) > -\mathrm{Re}(s)$ and $-1 < \mathrm{Re}(\xi) < 1$,

$$2\int_0^\infty \frac{e^{-2\pi k x}\sin(s\tan^{-1}(x/a))}{(a^2+x^2)^{s/2}}\ dx = \frac{a^{-s}}{2\pi k\Gamma(s)} \frac{1}{2\pi i} \int_{(c)} \Gamma\left(\frac{1+\xi}{2}\right) \Gamma\left(\frac{1-\xi}{2}\right) \Gamma(s+\xi)(2\pi a k)^{-\xi}\ d\xi. \tag{2.4}$$

To evaluate the integral on the right-hand side of the above equation, first consider

$$I_s(z) := \frac{1}{2\pi i} \int_{(c)} \Gamma\left(\frac{1+\xi}{2}\right) \Gamma\left(\frac{1-\xi}{2}\right) \Gamma(s+\xi) z^{-\xi}\ d\xi, \tag{2.5}$$

where $c = \mathrm{Re}(\xi)$.

We evaluate the above integral first for $|z| < 1, z \notin (-1, 0]$ and then extend it later to all $z \in \mathbb{C}\backslash(-\infty, 0]$ by analytic continuation. Consider the contour formed by the line segments $[c-iT, c+iT]$, $[c+iT, -\lambda+iT]$, $[-\lambda+iT, -\lambda-iT]$ and $[-\lambda-iT, c-iT]$, where $\lambda \notin \mathbb{Z}, \lambda > 1$. Observe that the integrand on the right-hand side of (2.5) has simple poles at $\xi = -s-m, 0 \leq m \leq \lfloor \lambda - \mathrm{Re}(s) \rfloor$, and $\xi = -2n-1$, where $0 \leq n \leq \lfloor \frac{\lambda-1}{2} \rfloor$ due to $\Gamma(s-\xi)$ and $\Gamma\left(\frac{1+\xi}{2}\right)$ respectively. (Note that since $\mathrm{Re}(s) < 0$, there will not be any pole of order 2.) The residues of the integrand at these poles can be easily calculated to be $\frac{(-1)^m z^{m+s}\pi}{m!\cos\left(\frac{\pi}{2}(m+s)\right)}$ and $2(-1)^n z^{2n+1}\Gamma(-1-2n+s)$ respectively. By employing Stirling's formula in a vertical strip $\alpha \leq c \leq \beta$ [16, p. 141, Formula **5.11.9**], namely,

$$|\Gamma(c+iT)| = (2\pi)^{\frac{1}{2}} |T|^{c-\frac{1}{2}} e^{-\frac{1}{2}\pi|T|} \left(1 + O\left(\frac{1}{|T|}\right)\right), \tag{2.6}$$

as $|T| \to \infty$, we see that the integrals along horizontal segments go to 0 as $T \to \infty$. Hence by Cauchy's residue theorem, we obtain

$$\begin{aligned} I_s(z) = \pi z^s \sum_{m=0}^{\lfloor \lambda - \mathrm{Re}(s) \rfloor} \frac{(-z)^m}{m!\cos\left(\frac{\pi}{2}(m+s)\right)} + 2z \sum_{n=0}^{\lfloor \frac{\lambda-1}{2} \rfloor} (-1)^n z^{2n} \Gamma(-1-2n+s) \\ + \frac{1}{2\pi i} \int_{(-\lambda)} \Gamma\left(\frac{1+\xi}{2}\right) \Gamma\left(\frac{1-\xi}{2}\right) \Gamma(s+\xi) z^{-\xi}\ d\xi. \end{aligned} \tag{2.7}$$

Next, we show that as $\lambda \to \infty$,

$$\frac{1}{2\pi i} \int_{(-\lambda)} \Gamma\left(\frac{1+\xi}{2}\right) \Gamma\left(\frac{1-\xi}{2}\right) \Gamma(s+\xi) z^{-\xi}\ d\xi \to 0. \tag{2.8}$$

By (2.6), we find that as $|t| \to \infty$,

$$\left|\Gamma\left(\frac{1 \pm \lambda - it}{2}\right)\right| = O_\lambda\left(|t|^{\pm\lambda/2} e^{-\frac{\pi}{4}|t|}\right), \tag{2.9}$$

and

$$|\Gamma(s-\lambda+it)| = O\left(|(t+\mathrm{Im}(s))|^{-\lambda+\mathrm{Re}(s)-1/2} e^{-\frac{\pi}{2}|(t+\mathrm{Im}(s))|}\right). \tag{2.10}$$

Upon making change of variable $\xi = -\lambda + it$ and then using (2.9) and (2.10), we see that

$$\begin{aligned}&\left|\int_{(-\lambda)} \Gamma\left(\frac{1+\xi}{2}\right)\Gamma\left(\frac{1-\xi}{2}\right)\Gamma(s+\xi)z^{-\xi}\ d\xi\right|\\ &= \left|\frac{1}{i}\int_{-\infty}^{\infty} \Gamma\left(\frac{1-\lambda+it}{2}\right)\Gamma\left(\frac{1+\lambda-it}{2}\right)\Gamma(s-\lambda+it)z^{\lambda-it}\ dt\right|\\ &= |z|^{\lambda}\int_{-M}^{M} O(1)\ dt + |z|^{\lambda}\int_{|t|\geq M} O\left(|(t+\mathrm{Im}(s))|^{-\lambda+\mathrm{Re}(s)-1/2}\, e^{-\frac{\pi}{2}|t|-\frac{\pi}{2}|(t+\mathrm{Im}(s))|}\right)\ dt\\ &= O(|z|^{\lambda}),\end{aligned}$$

where M is a large enough positive real number. Since $|z| < 1$, as $\lambda \to \infty$, we arrive at (2.8). Therefore by (2.7) and (2.8), for $|z| < 1, z \notin (-1, 0]$,

$$I_s(z) = \pi z^s \sum_{m=0}^{\infty} \frac{(-z)^m}{m! \cos\left(\frac{\pi}{2}(m+s)\right)} + 2z\sum_{n=0}^{\infty}(-1)^n z^{2n}\Gamma(-1-2n+s). \tag{2.11}$$

Now observe that both sides of above equation are analytic, as functions of z, in $\mathbb{C}\backslash(-\infty, 0]$. Therefore (2.11) holds for all $z \in \mathbb{C}\backslash(-\infty, 0]$ by analytic continuation. Hence letting $z = 2\pi ak, a > 0, k \in \mathbb{N}$, in (2.11) and simplifying the resulting first sum by splitting it into two sums, one over even m and other over odd m and by rephrasing the resulting second sum into a ${}_1F_2$ using the reflection and duplication formulas of the gamma function, we obtain

$$I_s(2\pi ak) = \frac{2\pi(2\pi ak)^s}{\sin(\pi s)}\sin\left(\frac{\pi}{2}(4ak+s)\right) + 4\pi ak\Gamma(s-1){}_1F_2\left(1; 1-\frac{s}{2}, \frac{3-s}{2}; -a^2\pi^2k^2\right). \tag{2.12}$$

Equation (2.1) now follows from (1.3), (1.4), (2.4), (2.5) and (2.12). This completes the proof for $\mathrm{Re}(s) < 0$. Now using an argument similar to that in [26, p. 269-270], one can show that the left-hand side of (2.1) is an entire function of s. The right-hand side of (2.1) is also analytic in the whole s-complex plane except for possible poles at $s = 1, 2, 3\cdots$. However, as shown in [25, p. 347-349], the Lommel function $S_{\mu,\nu}(z)$ has a limit when $\mu+\nu$ or $\mu-\nu$ are odd negative integers. Since the positive integer values of s render the $\mu+\nu$ and $\mu-\nu$ of our special case of the Lommel function, namely, $S_{-s-\frac{1}{2},\frac{1}{2}}(2\pi ak)$, to fall precisely in this category, we see that $s = 1, 2, 3, \cdots$ are indeed removable singularities of the right-hand side. Therefore both sides of (2.1) are entire functions of s, and hence the equality follows for all $s \in \mathbb{C}$ by analytic continuation. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The proof is divided into two cases.

Case 1: $0 < a < 1$.

We first prove the result for $s < 0$ and then extend it by analytic continuation. For $a > 0$ and $s \neq 1$, from (1.2),

$$\zeta(s, a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + 2\sum_{k=1}^{\infty}\int_0^{\infty} \frac{e^{-2\pi kx}\sin(s\tan^{-1}(x/a))}{(a^2+x^2)^{s/2}}\ dx, \tag{2.13}$$

where the interchange of the order of summation and integration is justified by absolute and uniform convergence (see, for example, [21, p. 30, Theorem 2.1]). Invoking Lemma 2.1 in

(2.13), we have, for $a>0$ and $s\neq 1$,

$$\begin{aligned}\zeta(s,a)&=\frac{1}{2}a^{-s}+\frac{a^{1-s}}{s-1}+2s\sqrt{a}(2\pi)^{s-\frac{1}{2}}\sum_{k=1}^{\infty}\frac{S_{-s-\frac{1}{2},\frac{1}{2}}(2\pi ak)}{k^{\frac{1}{2}-s}}\\&=\frac{1}{2}a^{-s}+\frac{a^{1-s}}{s-1}+\frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}\left\{2\sqrt{a}(2\pi)^{s-\frac{1}{2}}\Gamma(s+1)\frac{s_{-s-\frac{1}{2},\frac{1}{2}}(2\pi ak)}{k^{\frac{1}{2}-s}}\right.\\&\qquad\left.+\frac{(2\pi)^s}{\sin(\pi s)}\frac{\sin\left(\frac{\pi}{2}(4ak+s)\right)}{k^{1-s}}\right\},\end{aligned}\tag{2.14}$$

where in the second step, we used (1.4). This is the first instance where the infinite series on the right-hand side of (1.1) makes its conspicuous presence.

Now let $m=0$, $\mu=-s-\frac{1}{2}$, $\nu=\frac{1}{2}$ and $x=2\pi a$ in (1.7) and then use $\zeta(0)=-1/2$ to deduce that for $s<0$ and $0<a<1$,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{s_{-s-\frac{1}{2},\frac{1}{2}}(2\pi ak)}{k^{\frac{1}{2}-s}}&=\frac{(2\pi a)^{\frac{1}{2}-s}}{4}\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{\pi}{2\pi a\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}+\frac{-1/2}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{3-s}{2}\right)}\right)\\&=\frac{(2\pi a)^{\frac{1}{2}-s}}{4}\frac{\sqrt{\pi}\Gamma(-s)}{2^{-s-1}}\left(\frac{2^{-s}}{2a\sqrt{\pi}\Gamma(1-s)}-\frac{2^{1-s}}{2\sqrt{\pi}\Gamma(2-s)}\right),\end{aligned}$$

where in the last step, we used the duplication formula for the gamma function twice. Thus,

$$\sum_{k=1}^{\infty}\frac{s_{-s-\frac{1}{2},\frac{1}{2}}(2\pi ak)}{k^{\frac{1}{2}-s}}=\frac{-1}{2s\sqrt{a}(2\pi)^{s-\frac{1}{2}}}\left(\frac{1}{2}a^{-s}+\frac{a^{1-s}}{s-1}\right).\tag{2.15}$$

Hence from (2.14) and (2.15),

$$\zeta(s,a)=\frac{(2\pi)^s}{\Gamma(s)\sin(\pi s)}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{\pi}{2}(4ak+s)\right)}{k^{1-s}},$$

which readily gives (1.1). This completes the proof of Theorem 1.1 for $s<0$. Since both sides of (1.1) are analytic for $\mathrm{Re}(s)<0$, we conclude that (1.1) is valid for $\mathrm{Re}(s)<0$.

If $0<a<1$, note that in addition to being absolutely convergent for $\mathrm{Re}(s)<0$, the series on the right-hand side of (1.1) are conditionally convergent for $0<\mathrm{Re}(s)<1$ whence we see that for $0<a<1$, the result (1.1) actually holds for $\mathrm{Re}(s)<1$.

Case 2: $a=1$.
We first prove the result for $s<-1$ and then extend it to all complex s by analytic continuation. From (2.14) and the fact that $\zeta(s,1)=\zeta(s)$ for all $s\in\mathbb{C}$, we have

$$\zeta(s)=\frac{1}{2}+\frac{1}{s-1}+\frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}\left\{2(2\pi)^{s-\frac{1}{2}}\Gamma(s+1)\frac{s_{-s-\frac{1}{2},\frac{1}{2}}(2\pi k)}{k^{\frac{1}{2}-s}}+\frac{(2\pi)^s}{\sin(\pi s)}\frac{\sin\left(\frac{\pi}{2}(4k+s)\right)}{k^{1-s}}\right\}.\tag{2.16}$$

We now wish to evaluate in closed form the series $\displaystyle\sum_{k=1}^{\infty}\frac{s_{-s-\frac{1}{2},\frac{1}{2}}(2\pi k)}{k^{\frac{1}{2}-s}}$, which is indeed convergent for $s<-1$ since $\sum_{k=1}^{\infty}k^{s-1}\sin\left(\frac{\pi}{2}(4k+s)\right)$ converges for $s<-1$.

However, one should be careful as (1.7) cannot be applied here. This is because, it requires $x\in(0,2\pi)$, whereas here we need x in (1.7) to be 2π. Thankfully, Maširević has also obtained the following result [15, Theorem 2.2] where $0 \leq x \leq 2\pi, m \in \mathbb{N}$ and $\mu > 0$:

$$\sum_{k=1}^{\infty} \frac{s_{\mu-\frac{3}{2},\frac{1}{2}}(kx)}{k^{2m+\mu-\frac{1}{2}}} = \frac{1}{2}(-1)^{m-1}x^{2m+\mu-\frac{1}{2}}\Gamma(\mu-1)\left(\frac{-\pi}{x\Gamma(2m+\mu)} + 2\sum_{n=0}^{m} \frac{(-1)^{n-1}\zeta(2n)}{\Gamma(2m+\mu+1-2n)x^{2n}}\right). \tag{2.17}$$

Even though x can equal 2π in the above result, note that m has to be a natural number, whereas, to evaluate the series $\sum_{k=1}^{\infty} \frac{s_{-s-\frac{1}{2},\frac{1}{2}}(2\pi k)}{k^{\frac{1}{2}-s}}$ using (2.17), we would require $m = 0$. To circumvent this problem, we first employ the well-known result [8, p. 946, Formula **8.575.1**],

$$s_{\mu+2,\nu}(z) = z^{\mu+1} - [(\mu+1)^2 - \nu^2]s_{\mu,\nu}(z). \tag{2.18}$$

Use (2.18) with $\mu = -s - 5/2$, $\nu = 1/2$ and $z = 2\pi k$ so that

$$\sum_{k=1}^{\infty} \frac{s_{-s-\frac{1}{2},\frac{1}{2}}(2\pi k)}{k^{\frac{1}{2}-s}} = \sum_{k=1}^{\infty} \frac{(2\pi k)^{-s-\frac{3}{2}}}{k^{\frac{1}{2}-s}} - \left\{\left(s+\frac{3}{2}\right)^2 - \frac{1}{4}\right\}\sum_{k=1}^{\infty} \frac{s_{-s-\frac{5}{2},\frac{1}{2}}(2\pi k)}{k^{\frac{1}{2}-s}}. \tag{2.19}$$

We now transform the second series on the right-hand side of the above equation using (2.17). Before we do that, however, we need the well-known result $\zeta(0) = -1/2$, which can be proved *without* using the functional equation of $\zeta(s)$ so that circular reasoning is avoided. For example, one can put $s = 0$ in the following formula [23, p. 14, Equation (2.1.4)]

$$\zeta(s) = s\int_1^{\infty} \frac{[x] - x + 1/2}{x^{s+1}}\, dx + \frac{1}{s-1} + \frac{1}{2} \qquad (\mathrm{Re}(s) > -1),$$

to conclude that $\zeta(0) = -1/2$.

Let $m = 1$, $x = 2\pi$ and $\mu = -s - 1$ in (2.17) so that for $s < -1$,

$$\sum_{k=1}^{\infty} \frac{s_{-s-\frac{5}{2},\frac{1}{2}}(2\pi k)}{k^{\frac{1}{2}-s}} = \frac{1}{2}(2\pi)^{\frac{1}{2}-s}\Gamma(-s-2)\left(-\frac{1}{2\Gamma(1-s)} + \frac{1}{\Gamma(2-s)} + \frac{1}{12\Gamma(-s)}\right), \tag{2.20}$$

where we used $\zeta(0) = -1/2$ and $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6$.

Substitute (2.20) in (2.19) and simplify using the functional equation of the gamma function to arrive at

$$\sum_{k=1}^{\infty} \frac{s_{-s-\frac{1}{2},\frac{1}{2}}(2\pi k)}{k^{\frac{1}{2}-s}} = -\frac{1}{2s(2\pi)^{s-\frac{1}{2}}}\left(\frac{1}{2} + \frac{1}{s-1}\right). \tag{2.21}$$

Comparing the above equation with (2.15), we see that (2.15) holds for $a = 1$ too.

Using (2.21) in (2.16), we arrive at

$$\zeta(s) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)$$

for $s < -1$. The result then follows for all complex s by analytic continuation. □

3. A modular-type transformation involving the Lommel function $S_{-s-\frac{1}{2},\frac{1}{2}}(z)$

Modular-type transformations are the ones governed by the map $\alpha \to \beta$, where $\alpha\beta = 1$. An equivalent way to say this using the language of modular forms is that they consist of functions which transform nicely under $z \to -1/z$, $\mathrm{Im}(z) > 0$. But they may not transform nicely under $z \to z+1$, hence the nomenclature *modular-type* transformations. For a detailed survey on modular-type transformations, the reader is referred to [6].

The following modular-type transformation involving infinite series of Hurwitz zeta function was obtained by the first author in [5, Theorem 1.4] as a generalization of a transformation of Ramanujan [5, Theorem 1.1] on page 220 of the Lost Notebook [19].

Theorem 3.1. *Let* $0 < \mathrm{Re}(s) < 2$. *Define* $\varphi(s,x)$ *by*

$$\varphi(s,x) := \zeta(s,x) - \frac{1}{2}x^{-s} + \frac{x^{1-s}}{1-s}.$$

If α *and* β *are any positive numbers such that* $\alpha\beta = 1$,

$$\alpha^{\frac{s}{2}}\left(\sum_{n=1}^{\infty}\varphi(s,n\alpha) - \frac{\zeta(s)}{2\alpha^s} - \frac{\zeta(s-1)}{(s-1)\alpha}\right) = \beta^{\frac{s}{2}}\left(\sum_{n=1}^{\infty}\varphi(s,n\beta) - \frac{\zeta(s)}{2\beta^s} - \frac{\zeta(s-1)}{(s-1)\beta}\right)$$
$$= \frac{8(4\pi)^{\frac{s-4}{2}}}{\Gamma(s)}\int_0^{\infty}\Gamma\left(\frac{s-2+it}{4}\right)\Gamma\left(\frac{s-2-it}{4}\right)\Xi\left(\frac{t+i(s-1)}{2}\right)$$
$$\times\,\Xi\left(\frac{t-i(s-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{s^2+t^2}\,dt,$$

where $\Xi(t) := \xi\left(\frac{1}{2}+it\right)$ *with* $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$.

In view of the first equality in (2.14), the modular-type transformation in the above result can be rephrased in the following form.

Corollary 3.2. *Let* $0 < \mathrm{Re}(s) < 2$ *and let* $\sigma_s(n) = \sum_{d|n} d^s$. *Let* $S_{\mu,\nu}(z)$ *be defined in* (1.4). *If* α *and* β *are any positive numbers such that* $\alpha\beta = 1$,

$$\alpha^{\frac{s}{2}}\left(2s(2\pi)^{s-\frac{1}{2}}\sqrt{\alpha}\sum_{m=1}^{\infty}\sigma_{1-s}(m)m^{s-\frac{1}{2}}S_{-s-\frac{1}{2},\frac{1}{2}}(2\pi m\alpha) - \frac{\zeta(s)}{2\alpha^s} - \frac{\zeta(s-1)}{(s-1)\alpha}\right)$$
$$= \beta^{\frac{s}{2}}\left(2s(2\pi)^{s-\frac{1}{2}}\sqrt{\beta}\sum_{m=1}^{\infty}\sigma_{1-s}(m)m^{s-\frac{1}{2}}S_{-s-\frac{1}{2},\frac{1}{2}}(2\pi m\beta) - \frac{\zeta(s)}{2\beta^s} - \frac{\zeta(s-1)}{(s-1)\beta}\right).$$

Proof. The result follows at once if we observe that from (2.14),

$$\sum_{n=1}^{\infty}\varphi(s,n\alpha) = 2s(2\pi)^{s-\frac{1}{2}}\sqrt{\alpha}\sum_{n=1}^{\infty}\sqrt{n}\sum_{k=1}^{\infty}\frac{S_{-s-\frac{1}{2},\frac{1}{2}}(2\pi nk\alpha)}{k^{\frac{1}{2}-s}}$$
$$= 2s(2\pi)^{s-\frac{1}{2}}\sqrt{\alpha}\sum_{n,k=1}^{\infty}n^{1-s}(nk)^{s-\frac{1}{2}}S_{-s-\frac{1}{2},\frac{1}{2}}(2\pi nk\alpha)$$
$$= 2s(2\pi)^{s-\frac{1}{2}}\sqrt{\alpha}\sum_{m=1}^{\infty}\sigma_{1-s}(m)m^{s-\frac{1}{2}}S_{-s-\frac{1}{2},\frac{1}{2}}(2\pi m\alpha).$$

□

Remark 1. *The series in Corollary 3.2 should be compared with the series considered by Lewis and Zagier in* [13, Equation (2.11)], *namely,* $\sum_{n=1}^{\infty} n^{s-1/2}A_n\mathcal{C}_s(2\pi na)$, *where* $\mathcal{C}_s(z)$ *is defined by* (1.6).

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