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Overpartitions related to the mock theta function $\omega(q)$

by

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1. Introduction. Since Ramanujan introduced mock theta functions in his last letter to Hardy in 1920, they have been the subject of intense study. Along with his third order mock theta function $f(q)$ defined by

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2},$$

there are many studies on the mock theta function

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}$$

in the literature [14], [18], [21], [34]. Throughout the paper, we adopt the following q-series notation:

$$\begin{aligned} (a;q)_0 &:= 1, \\ (a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n \geq 1, \\ (a;q)_\infty &:= \lim_{n\to\infty} (a;q)_n, \qquad |q| < 1. \end{aligned}$$

When the base is q, we sometimes use the short-hand notation $(a)_n := (a;q)_n$ and $(a)_\infty := (a;q)_\infty$.

In a recent paper [11], the first, second and the fourth authors discovered a new partition-theoretic interpretation of $\omega(q)$, namely, the coefficient of q^n in $q\,\omega(q)$ counts $p_\omega(n)$, the number of partitions of n in which all odd

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parts are less than twice the smallest part, that is,

$$\sum_{n=1}^{\infty} p_\omega(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty} = q\omega(q).$$

In the same paper, they also studied some arithmetic properties of the associated smallest parts function $\mathrm{spt}_\omega(n)$ whose generating function is given by

$$\sum_{n=1}^{\infty} \mathrm{spt}_\omega(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}.$$

In this paper, we study the overpartition analogue of $p_\omega(n)$ and its associated smallest parts function. Overpartitions are ordinary partitions extended by allowing a possible overline designation on the first (or equivalently the final) occurrence of a part. For instance, there are eight overpartitions of 3: $3, \overline{3}, 2+1, 2+\overline{1}, \overline{2}+1, \overline{2}+\overline{1}, 1+1+1$, and $\overline{1}+1+1$. Throughout this paper, however, we consider overpartitions in which the smallest part is always overlined, and denote by $\overline{p}(n)$ the number of such overpartitions. For instance, $\overline{p}(3) = 4$ since there are four such overpartitions of 3: $\overline{3}, 2+\overline{1}, \overline{2}+\overline{1}$, and $\overline{1}+1+1$. In an overpartition, a smallest part may or may not be overlined, so the number of overpartitions of n is exactly twice of $\overline{p}(n)$.

Since its introduction in [19], the overpartition construct has been very popular, and has led to a number of studies in q-series, partition theory, modular and mock modular forms.

As remarked earlier, in this paper we study the overpartition analogue of $p_\omega(n)$, namely $\overline{p}_\omega(n)$, which enumerates the number of overpartitions of n such that all odd parts are less than twice the smallest part, and in which the smallest part is always overlined. It is clear that its generating function is given by

$$\sum_{n=1}^{\infty} \overline{p}_\omega(n)q^n = \sum_{n=1}^{\infty} \frac{q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}. \tag{1.1}$$

The series in (1.1) can be written as

$$\begin{aligned}\sum_{n=1}^{\infty} \overline{p}_\omega(n)q^n &= \frac{q(-q^2;q^2)_\infty}{(1-q)(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q^3;q^2)_n(q;q)_n}{(q^3;q^2)_n(-q^2;q)_n} q^n \\ &= \frac{q(-q^2;q^2)_\infty}{(1-q)(q^2;q^2)_\infty}\, {}_4\phi_3\left(\begin{matrix} q, & q, & iq^{3/2}, & -iq^{3/2} \\ & -q^2, & q^{3/2}, & -q^{3/2} \end{matrix}; q, q\right),\end{aligned} \tag{1.2}$$

where the basic hypergeometric series ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, z\right) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_{r+1};q)_n}{(q;q)_n(b_1;q)_n \cdots (b_r;q)_n} z^n.$$

Thus the generating function is essentially a nonterminating ${}_4\phi_3$. The problem of relating this generating function to familiar objects in the theory of basic hypergeometric series, modular forms and mock modular forms is quite difficult. In fact, in order to transform it, we derive a new multi-parameter q-series identity, which generalizes a deep identity due to the first author, and its extension due to R. P. Agarwal (see Theorem 3.1). Basically we need its following variant for our case.

THEOREM 1.1. *Let the Gaussian polynomial be defined by*

$$\begin{bmatrix} n \\ m \end{bmatrix} := \begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_m (q;q)_{n-m}} & \text{if } 0 \le m \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Then, provided $\beta, \delta, f, t \neq q^{-j}$, $j \ge 0$, the following identity holds:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\alpha)_n (\gamma)_n (\epsilon)_n}{(\beta)_n (\delta)_n (f)_n} t^n \\ &= \frac{(\epsilon, \gamma, \beta/\alpha, q, \alpha t, q/(\alpha t), \delta q/\beta, fq/\beta; q)_\infty}{(f, \delta, q/\alpha, \beta, \beta/(\alpha t), \alpha t q/\beta, \gamma q/\beta, \epsilon q/\beta; q)_\infty} \, {}_3\phi_2 \left(\begin{matrix} \frac{\alpha q}{\beta}, \ \frac{\gamma q}{\beta}, \ \frac{\epsilon q}{\beta} \\ \frac{\delta q}{\beta}, \ \frac{fq}{\beta} \end{matrix} ; q, t \right) \\ &\quad + \left(1 - \frac{q}{\beta}\right) \frac{(\epsilon, \gamma, t, \delta q/\beta, fq/\beta; q)_\infty}{(f, \delta, \alpha t/\beta, \gamma q/\beta, \epsilon q/\beta; q)_\infty} \\ &\qquad \times {}_3\phi_2 \left(\begin{matrix} \frac{\alpha q}{\beta}, \ \frac{\gamma q}{\beta}, \ \frac{\epsilon q}{\beta} \\ \frac{\delta q}{\beta}, \ \frac{fq}{\beta} \end{matrix} ; q, t \right) \left({}_2\phi_1 \left(\begin{matrix} q, \ \frac{q}{t} \\ \frac{\beta q}{\alpha t} \end{matrix} ; q, \frac{q}{\alpha} \right) - 1 \right) \\ &\quad + \frac{(\epsilon, \gamma; q)_\infty}{(f, \delta; q)_\infty} \left(1 - \frac{q}{\beta}\right) \sum_{n=0}^{\infty} \frac{(t)_n}{(q)_n (\alpha t/\beta)_{n+1}} \left(\frac{q}{\beta}\right)^n \sum_{p=0}^{n} \frac{(\alpha t/\beta)_p}{(t)_p} \left(\frac{\beta}{q}\right)^p \\ &\qquad \times \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \left(\frac{f}{\epsilon}\right)_m \epsilon^m \left(\frac{\delta}{\gamma}\right)_{n-m} \gamma^{n-m}, \end{aligned} \tag{1.4}$$

where we use the notation

$$(a_1, \ldots, a_m; q)_\infty := (a_1; q)_\infty \cdots (a_m; q)_\infty.$$

This result is then specialized to obtain the following theorem which expresses the generating function in terms of a ${}_3\phi_2$ basic hypergeometric series and an infinite series involving the little q-Jacobi polynomial defined by [9, (3.1)]

$$p_n(x; \alpha, \beta : q) = {}_2\phi_1 \left(\begin{matrix} q^{-n}, \ \alpha\beta q^{n+1} \\ \alpha q \end{matrix} ; qx \right). \tag{1.5}$$

THEOREM 1.2. *The following identity holds for $|q| < 1$:*

$$\sum_{n=0}^{\infty} \frac{q^n(-q^3;q^2)_n(q;q)_n}{(q^3;q^2)_n(-q^2;q)_n} = \frac{1}{2}\left(1-\frac{1}{q}\right)\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\sum_{n=0}^{\infty}\frac{(-1;q)_n(-q;q^2)_n}{(q;q^2)_n(q;q)_n}q^n + \frac{1}{q}\frac{(-q;q^2)_\infty}{(q^3;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}p_{2n}(-1;q^{-2n-1},-1:q). \tag{1.6}$$

Hence the generating function of $\overline{p}_\omega(n)$ is given by

$$\sum_{n=1}^{\infty}\overline{p}_\omega(n)q^n = -\frac{1}{2}\frac{(q;q)_\infty(q;q^2)_\infty}{(-q;q)_\infty(-q;q^2)_\infty}\,{}_3\phi_2\left(\begin{matrix}-1, & iq^{1/2}, & -iq^{1/2} \\ q^{1/2}, & -q^{1/2} & \end{matrix};q,q\right) + \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}p_{2n}(-1;q^{-2n-1},-1:q). \tag{1.7}$$

The series involving the little q-Jacobi polynomials on the right side of (1.7) satisfies a nice congruence modulo 4 given below.

THEOREM 1.3. *The following congruence holds:*

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}p_{2n}(-1;q^{-2n-1},-1:q) \equiv \frac{1}{2}\frac{(q;q^2)_\infty}{(-q;q^2)_\infty} \pmod 4. \tag{1.8}$$

The overpartition function $\overline{p}_\omega(n)$ satisfies some nice congruences. Indeed the two congruences in the following theorem will be proved in Section 4.

THEOREM 1.4. *We have*

$$\overline{p}_\omega(4n+3) \equiv 0 \pmod 4, \tag{1.9}$$

$$\overline{p}_\omega(8n+6) \equiv 0 \pmod 4. \tag{1.10}$$

The smallest parts function $\mathrm{spt}(n)$ counts the total number of appearances of the smallest parts in all partitions of n (see [8]). For the last decade, there have been many papers on $\mathrm{spt}(n)$, and in particular, for generalizations of $\mathrm{spt}(n)$, we refer the reader to [23], [24], [27] and [28].

Bringmann, Lovejoy, and Osburn [16, 17] defined $\overline{\mathrm{spt}}(n)$ as the number of smallest parts in the overpartitions of n and showed that the generating function of $\overline{\mathrm{spt}}(n)$ is a quasimock theta function (see [17, pp. 3–4] for the definition) satisfying simple Ramanujan-type congruences, for instance,

$$\overline{\mathrm{spt}}(3n) \equiv 0 \pmod 3.$$

In this paper, we study $\overline{\mathrm{spt}}_\omega(n)$, the number of smallest parts in the overpartitions of n in which the smallest part is always overlined and all odd parts are less than twice the smallest part. By its definition we see that the generating function of $\overline{\mathrm{spt}}_\omega(n)$ is given by

$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_\omega(n)q^n=\sum_{n=1}^{\infty}\frac{q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty}{(1-q^n)^2(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}. \tag{1.11}$$

The smallest parts function $\overline{\mathrm{spt}}_\omega(n)$ seems to carry arithmetic properties analogous to those of $\overline{\mathrm{spt}}_2(n)$, where $\overline{\mathrm{spt}}_2(n)$ counts the number of smallest parts in the overpartitions of n with smallest parts even. It is known [16] that

$$\overline{\mathrm{spt}}_2(3n)\equiv 0 \pmod 3, \tag{1.12}$$

$$\overline{\mathrm{spt}}_2(3n+1)\equiv 0 \pmod 3, \tag{1.13}$$

$$\overline{\mathrm{spt}}_2(5n+3)\equiv 0 \pmod 5. \tag{1.14}$$

The following are the main congruences satisfied by $\overline{\mathrm{spt}}_\omega(n)$:

THEOREM 1.5. *We have*

$$\overline{\mathrm{spt}}_\omega(3n)\equiv 0 \pmod 3, \tag{1.15}$$

$$\overline{\mathrm{spt}}_\omega(3n+2)\equiv 0 \pmod 3, \tag{1.16}$$

$$\overline{\mathrm{spt}}_\omega(10n+6)\equiv 0 \pmod 5, \tag{1.17}$$

$$\overline{\mathrm{spt}}_\omega(6n+5)\equiv 0 \pmod 6. \tag{1.18}$$

We remark that $\overline{\mathrm{spt}}_\omega(n)$ has recently been studied by Jennings-Shaffer [29], although with a different combinatorial interpretation. The notation in [29] for its generating function is $S_{G2}(q)$. Also, the mod 3 congruences in Theorem 1.5 were established in the same paper.

There are further congruences that both $\overline{\mathrm{spt}}(n)$ and $\overline{\mathrm{spt}}_\omega(n)$ satisfy:

THEOREM 1.6. *For any positive integer n,*

$$\overline{\mathrm{spt}}_\omega(n)\equiv\overline{\mathrm{spt}}(n)\equiv\begin{cases}1 \pmod 2 & \text{if } n=k^2 \text{ or } 2k^2 \text{ for some } k,\\ 0 \pmod 2 & \text{otherwise.}\end{cases} \tag{1.19}$$

THEOREM 1.7. *For any positive integer n,*

$$\overline{\mathrm{spt}}(7n)\equiv\overline{\mathrm{spt}}(n/7) \pmod 4, \tag{1.20}$$

$$\overline{\mathrm{spt}}_\omega(7n)\equiv\overline{\mathrm{spt}}_\omega(n/7) \pmod 4, \tag{1.21}$$

where we follow the convention that $\overline{\mathrm{spt}}(x)=\overline{\mathrm{spt}}_\omega(x)=0$ if x is not a positive integer.

This paper is organized as follows. In Section 2, we recall some basic facts and theorems that are used in the following. Section 3 is devoted to finding an alternative representation for the generating function of $\overline{p}_\omega(n)$ in terms of a ${}_3\phi_2$ basic hypergeometric series and an infinite series involving the little q-Jacobi polynomials. A congruence modulo 4, satisfied by the latter series, is also obtained in that section. In Section 4, we give a proof of the congruences modulo 4 satisfied by $\overline{p}_\omega(n)$ in Theorem 1.4. We recall some facts about $\overline{\mathrm{spt}}(n)$ and $\overline{\mathrm{spt}}_2(n)$ in Section 5 and represent the generating function of $\overline{\mathrm{spt}}_\omega(n)$ in terms of those of these functions. In Section 6, we prove the congruences modulo $3, 5$ and 6 given in Theorem 1.5 based on these representations. We also prove Theorems 1.6 and 1.7 in Section 7. Lastly we conclude our paper by stating two open problems in Section 8.

2. Preliminaries. In this section we collect some important facts and theorems on q-series and partitions. First of all, we assume throughout that $|q| < 1$. The most fundamental theorem in the literature is the q-binomial theorem given for $|z| < 1$ by [5, p. 17, (2.2.1)]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.1}$$

For $|z| < 1$ and $|b| < 1$, Heine's transformation [5, p. 19, Corollary 2.3] is given by

$$ {}_2\phi_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; q, z\right) = \frac{(b, az;q)_\infty}{(c, z;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} c/b,\ z \\ az \end{matrix}; q, b\right), \tag{2.2}$$

and we also note Bailey's ${}_{10}\phi_9$ transformation [7, (2.10)], [12, (6.3)]

(2.3)
$$\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix} a,\ q^2\sqrt{a},\ -q^2\sqrt{a},\ p_1,\ p_1q,\ p_2,\ p_2q,\ f,\ q^{-2N},\ q^{-2N+1} \\ \sqrt{a},\ -\sqrt{a},\ \frac{aq^2}{p_1},\ \frac{aq}{p_1},\ \frac{aq^2}{p_2},\ \frac{aq}{p_2},\ \frac{aq^2}{f},\ aq^{2N+2},\ aq^{2N+1} \end{matrix}; q^2, \frac{a^3q^{4N+3}}{p_1^2p_2^2f}\right)$$
$$= \frac{(aq;q)_\infty\left(\frac{aq}{p_1p_2};q\right)_\infty}{\left(\frac{aq}{p_1};q\right)_\infty\left(\frac{aq}{p_2};q\right)_\infty} \sum_{n=0}^{\infty} \frac{(p_1;q)_n(p_2;q)_n\left(\frac{aq}{f};q^2\right)_n}{(q;q)_n(aq;q^2)_n\left(\frac{aq}{f};q\right)_n}\left(\frac{aq}{p_1p_2}\right)^n.$$

Finally we note a transformation for ${}_2\psi_2$ due to Bailey [25, p. 148, Exercise 5.11]:

(2.4)
$$ {}_2\psi_2\left(\begin{matrix} e, f \\ \frac{aq}{c}, \frac{aq}{d} \end{matrix}; q, \frac{aq}{ef}\right) =$$
$$\frac{\left(\frac{q}{c}, \frac{q}{d}, \frac{aq}{e}, \frac{aq}{f};q\right)_\infty}{\left(aq, \frac{q}{a}, \frac{aq}{cd}, \frac{aq}{ef};q\right)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1-aq^{2n})(c;q)_n(d;q)_n(e;q)_n(f;q)_n}{(1-a)\left(\frac{aq}{c};q\right)_n\left(\frac{aq}{d};q\right)_n\left(\frac{aq}{e};q\right)_n\left(\frac{aq}{f};q\right)_n}\left(\frac{qa^3}{cdef}\right)^n q^{n^2},$$

where ${}_r\psi_r$ is the basic bilateral hypergeometric series defined by [25, p. 137, (5.1.1)]

$$ {}_r\psi_r\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix}; q, z\right) := \sum_{n=-\infty}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(b_1;q)_n \cdots (b_r;q)_n} z^n,$$

where $(a;q)_n := (a;q)_\infty/(aq^n;q)_\infty$ for $n < 0$.

3. The generating function of $\overline{p}_\omega(n)$. First, we recall that $\overline{p}_\omega(n)$ counts the number of overpartitions of n such that all odd parts are less than twice the smallest part, and in which the smallest part is always overlined. To the best of our knowledge, none of the already existing identities from the theory of basic hypergeometric series seems to be capable of handling its generating function. Hence we devise a new q-series identity consisting of seven parameters that transforms (1.2) into a ${}_3\phi_2$ and an infinite series involving little q-Jacobi polynomials defined in (1.5). The motivation and the need for devising such an identity is now given.

In the proof of the representation of the generating function of $p_\omega(n)$ in terms of the third order mock theta function $\omega(q)$ [11, Theorem 3.1], the following four-parameter q-series identity due to the first author [6, Theorem 1] played an instrumental role:

(3.1)
$$\sum_{n=0}^{\infty}\frac{(B;q)_n(-Abq;q)_nq^n}{(-aq;q)_n(-bq;q)_n}=\frac{-a^{-1}(B;q)_\infty(-Abq;q)_\infty}{(-bq;q)_\infty(-aq;q)_\infty}\sum_{m=0}^{\infty}\frac{(A^{-1};q)_m\left(\frac{Abq}{a}\right)^m}{\left(-\frac{B}{a};q\right)_{m+1}}$$
$$+(1+b)\sum_{m=0}^{\infty}\frac{(-a^{-1};q)_{m+1}\left(-\frac{ABq}{a};q\right)_m(-b)^m}{\left(-\frac{B}{a};q\right)_{m+1}\left(\frac{Abq}{a};q\right)_{m+1}}.$$

Agarwal [2, (3.1)] obtained the following 'mild' extension/generalization of (3.1) in the sense that we get (3.1) from the identity below when $t=q$:

(3.2)
$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n}{(\beta)_n(\delta)_n}t^n$$
$$=\frac{(q/(\alpha t),\gamma,\alpha t,\beta/\alpha,q;q)_\infty}{(\beta/(\alpha t),\delta,t,q/\alpha,\beta;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}\delta/\gamma, & t\\ & q\alpha t/\beta\end{matrix};q,\gamma q/\beta\right)$$
$$+\frac{(\gamma)_\infty}{(\delta)_\infty}\left(1-\frac{q}{\beta}\right)\sum_{m=0}^{\infty}\frac{(\delta/\gamma)_m(t)_m}{(q)_m(\alpha t/\beta)_{m+1}}(q\gamma/\beta)^m\left({}_2\phi_1\left(\begin{matrix}q, & q/t\\ & q\beta/(\alpha t)\end{matrix};q,q/\alpha\right)-1\right)$$
$$+\frac{(\gamma)_\infty}{(\delta)_\infty}\left(1-\frac{q}{\beta}\right)\sum_{p=0}^{\infty}\frac{\gamma^p(\delta/\gamma)_p}{(q)_p}\sum_{m=0}^{\infty}\frac{(\delta q^p/\gamma)_m(tq^p)_m}{(q^{1+p})_m(\alpha tq^p/\beta)_{m+1}}(q\gamma/\beta)^m.$$

Since the right side of (1.2) involves three q-shifted factorials (with base q) in the numerator as well as in the denominator of its summand, we need to first generalize (3.2). Indeed, such a generalization will be given in Theorem 3.1. However, we shall first prove its variant, namely Theorem 1.1, that we need for our purpose.

Proof of Theorem 1.1. Let

$$S := S(\alpha,\beta,\gamma,\delta,\epsilon,f;q;t) := \sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n(\epsilon)_n}{(\beta)_n(\delta)_n(f)_n}t^n. \tag{3.3}$$

Then by an application of the q-binomial theorem (2.1),

(3.4)

$$\begin{aligned}
S &= \frac{(\epsilon)_\infty}{(f)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n(fq^n)_\infty}{(\beta)_n(\delta)_n(\epsilon q^n)_\infty}t^n\\
&= \frac{(\epsilon)_\infty}{(f)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n t^n}{(\beta)_n(\delta)_n}\sum_{m=0}^{\infty}\frac{(f/\epsilon)_m}{(q)_m}(\epsilon q^n)^m\\
&= \frac{(\epsilon)_\infty}{(f)_\infty}\sum_{m=0}^{\infty}\frac{(f/\epsilon)_m\epsilon^m}{(q)_m}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n}{(\beta)_n(\delta)_n}(tq^m)^n\\
&= \frac{(\epsilon)_\infty}{(f)_\infty}\sum_{m=0}^{\infty}\frac{(f/\epsilon)_m\epsilon^m}{(q)_m}\\
&\qquad\times\left\{\frac{(q^{1-m}/(\alpha t),\gamma,\alpha tq^m,\beta/\alpha,q;q)_\infty}{(\beta q^{-m}/(\alpha t),\delta,tq^m,q/\alpha,\beta;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}\delta/\gamma, & tq^m\\ & \alpha tq^{m+1}/\beta\end{matrix};q,\gamma q/\beta\right)\right.\\
&\quad\left.+\frac{(\gamma)_\infty}{(\delta)_\infty}\frac{(1-q/\beta)}{(1-\alpha tq^m/\beta)}\sum_{k=0}^{\infty}\frac{(\delta/\gamma)_k(\alpha tq^m/\beta)_k\gamma^k}{(q)_k(\alpha tq^{m+1}/\beta)_k}\sum_{r=0}^{\infty}\frac{(q^{1-k-m}/t)_r}{(\beta q^{1-k-m}/(\alpha t))_r}\left(\frac{q}{\alpha}\right)^r\right\}\\
&=: \frac{(\epsilon)_\infty}{(f)_\infty}\left(\frac{(\gamma,\beta/\alpha,q;q)_\infty}{(\delta,q/\alpha,\beta;q)_\infty}V_1+V_2\right),
\end{aligned}$$

where in the penultimate step, we have used (3.2) in the form given in [2, (3.2)]. Here

(3.5)

$$\begin{aligned}
V_1 &:= \sum_{m=0}^{\infty}\frac{(f/\epsilon)_m(q^{1-m}/(\alpha t),\alpha tq^m;q)_\infty\epsilon^m}{(q)_m(\beta q^{-m}/(\alpha t),tq^m;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}\delta/\gamma, & tq^m\\ & \alpha tq^{m+1}/\beta\end{matrix};q,\gamma q/\beta\right),\\
V_2 &:= \frac{(\gamma)_\infty}{(\delta)_\infty}\left(1-\frac{q}{\beta}\right)\sum_{m=0}^{\infty}\frac{\left(\frac{f}{\epsilon}\right)_m\epsilon^m}{(q)_m\left(1-\frac{\alpha tq^m}{\beta}\right)}\\
&\qquad\times\sum_{k=0}^{\infty}\frac{\left(\frac{\delta}{\gamma}\right)_k\left(\frac{\alpha tq^m}{\beta}\right)_k\gamma^k}{(q)_k\left(\frac{\alpha tq^{m+1}}{\beta}\right)_k}\sum_{r=0}^{\infty}\frac{\left(\frac{q^{1-k-m}}{t}\right)_r}{\left(\frac{\beta q^{1-k-m}}{\alpha t}\right)_r}\left(\frac{q}{\alpha}\right)^r.
\end{aligned}$$

Rewriting V_1 using

$$(aq^{-m})_\infty = (a)_\infty(q/a)_m q^m q^{-m(m+1)/2},$$

and then applying Heine's transformation (2.2) in the second step below, we see that

(3.6)

$$V_1 = \frac{(\alpha t)_\infty (q/(at))_\infty}{(t)_\infty (\beta/(at))_\infty} \sum_{m=0}^{\infty} \frac{(f/\epsilon)_m (t)_m}{(\alpha tq/\beta)_m (q)_m} \left(\frac{q\epsilon}{\beta}\right)^m {}_2\phi_1\left(\begin{matrix}\delta/\gamma, & tq^m \\ & \alpha tq^{m+1}/\beta\end{matrix}; q, \gamma q/\beta\right)$$

$$= \frac{(\alpha t)_\infty (q/(at))_\infty (\delta q/\beta)_\infty}{(\alpha tq/\beta)_\infty (\gamma q/\beta)_\infty (\beta/(at))_\infty} \sum_{m=0}^{\infty} \frac{(f/\epsilon)_m}{(q)_m} \left(\frac{q\epsilon}{\beta}\right)^m {}_2\phi_1\left(\begin{matrix}\alpha q/\beta, & \gamma q/\beta \\ & \delta q/\beta\end{matrix}; q, tq^m\right)$$

$$= \frac{(\alpha t)_\infty (q/(at))_\infty (\delta q/\beta)_\infty}{(\alpha tq/\beta)_\infty (\gamma q/\beta)_\infty (\beta/(at))_\infty} \sum_{k=0}^{\infty} \frac{(\alpha q/\beta)_k (\gamma q/\beta)_k (fq^{k+1}/\beta)_\infty t^k}{(\delta q/\beta)_k (q)_k (\epsilon q^{k+1}/\beta)_\infty},$$

where in the last step we used (2.1) after interchanging the order of summation. Hence

$$V_1 = \frac{(\alpha t, q/(\alpha t), \delta q/\beta, fq/\beta; q)_\infty}{(\beta/(\alpha t), \alpha tq/\beta, \gamma q/\beta, \epsilon q/\beta; q)_\infty} {}_3\phi_2\left(\begin{matrix}\alpha q/\beta, & \gamma q/\beta, & \epsilon q/\beta \\ & \delta q/\beta, fq/\beta & \end{matrix}; q, t\right). \tag{3.7}$$

Let us now consider V_2. Since

(3.8)

$$\sum_{r=0}^{\infty} \frac{(q^{1-k-m}/t)_r}{(\beta q^{1-k-m}/(\alpha t))_r} \left(\frac{q}{\alpha}\right)^r$$

$$= \frac{(t)_{m+k}}{(\alpha t/\beta)_{m+k}} \left(\frac{q}{\beta}\right)^{m+k} \sum_{r=0}^{\infty} \frac{(\alpha t/\beta)_{m+k-r}}{(t)_{m+k-r}} \left(\frac{\beta}{q}\right)^{m+k-r}$$

$$= \frac{(t)_{m+k}}{\left(\frac{\alpha t}{\beta}\right)_{m+k}} \left(\frac{q}{\beta}\right)^{m+k} \left(\sum_{p=1}^{\infty} \frac{\left(\frac{q}{t}\right)_p}{\left(\frac{\beta q}{\alpha t}\right)_p} \left(\frac{q}{\alpha}\right)^p + \sum_{p=0}^{m+k} \frac{(\alpha t/\beta)_p}{(t)_p} \left(\frac{\beta}{q}\right)^p\right),$$

we find that

$$V_2 = \frac{(\gamma)_\infty}{(\delta)_\infty} \left(1 - \frac{q}{\beta}\right) (V_3 + V_3^*), \tag{3.9}$$

where

$$V_3 := \sum_{m=0}^{\infty} \frac{(f/\epsilon)_m (t)_m (\epsilon q/\beta)^m}{(q)_m (\alpha t/\beta)_m (1 - \alpha tq^m/\beta)} \sum_{k=0}^{\infty} \frac{(\delta/\gamma)_k (tq^m)_k (\gamma q/\beta)^k}{(q)_k (\alpha tq^{m+1}/\beta)_k} \times \sum_{p=1}^{\infty} \frac{(q/t)_p}{(\beta q/(\alpha t))_p} \left(\frac{q}{\alpha}\right)^p,$$

(3.10)

$$V_3^* := \sum_{m=0}^{\infty} \frac{(f/\epsilon)_m (\epsilon q/\beta)^m}{(q)_m (1 - \alpha tq^m/\beta)} \sum_{k=0}^{\infty} \frac{(\delta/\gamma)_k (t)_{m+k} (\alpha tq^m/\beta)_k (\gamma q/\beta)^k}{(q)_k (\alpha tq^{m+1}/\beta)_k (\alpha t/\beta)_{m+k}} \times \sum_{p=0}^{m+k} \frac{(\alpha t/\beta)_p}{(t)_p} \left(\frac{\beta}{q}\right)^p.$$

Consider V_3. Again using Heine's transformation (2.2) for the middle series followed by (2.1), we have

$$V_3 = \frac{(t, \delta q/\beta, fq/\beta; q)_\infty}{(\alpha t/\beta, \gamma q/\beta, \epsilon q/\beta; q)_\infty} {}_3\phi_2\left(\begin{matrix}\alpha q/\beta,\ \gamma q/\beta,\ \epsilon q/\beta \\ \delta q/\beta,\ fq/\beta\end{matrix}; q, t\right) \times \left({}_2\phi_1\left(\begin{matrix}q,\ \ q/t \\ \beta q/(\alpha t)\end{matrix}; q, q/\alpha\right) - 1\right). \tag{3.11}$$

Also,

$$V_3^* = \sum_{n=0}^{\infty} \frac{(t)_n}{(q)_n(\alpha t/\beta)_{n+1}}\left(\frac{q}{\beta}\right)^n \sum_{p=0}^{n} \frac{(\alpha t/\beta)_p}{(t)_p}\left(\frac{\beta}{q}\right)^p \times \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \left(\frac{f}{\epsilon}\right)_m \epsilon^m \left(\frac{\delta}{\gamma}\right)_{n-m} \gamma^{n-m}. \tag{3.12}$$

Finally, from (3.4), (3.7), (3.9), (3.11) and (3.12), we arrive at (1.4). ∎

We now give the aforementioned generalization of (3.2) which can also be viewed as a corollary of Theorem 1.1.

THEOREM 3.1. *Provided $\beta, \delta, f, t \neq q^{-j}$, $j \geq 0$, the following identity holds:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\alpha)_n(\gamma)_n(\epsilon)_n}{(\beta)_n(\delta)_n(f)_n} t^n \\
&= \frac{(\epsilon, \gamma, \beta/\alpha, q, \alpha t, q/(\alpha t), \delta q/\beta, fq/\beta; q)_\infty}{(f, \delta, q/\alpha, \beta, \beta/(\alpha t), \alpha t q/\beta, \gamma q/\beta, \epsilon q/\beta; q)_\infty} {}_3\phi_2\left(\begin{matrix}\alpha q/\beta,\ \gamma q/\beta,\ \epsilon q/\beta \\ \delta q/\beta,\ fq/\beta\end{matrix}; q, t\right) \\
&\quad + \left(1 - \frac{q}{\beta}\right) \frac{(\epsilon, \gamma, t, \delta q/\beta, fq/\beta; q)_\infty}{(f, \delta, \alpha t/\beta, \gamma q/\beta, \epsilon q/\beta; q)_\infty} {}_3\phi_2\left(\begin{matrix}\alpha q/\beta,\ \gamma q/\beta,\ \epsilon q/\beta \\ \delta q/\beta,\ fq/\beta\end{matrix}; q, t\right) \\
&\qquad \times \left({}_2\phi_1\left(\begin{matrix}q,\ \ q/t \\ \beta q/(\alpha t)\end{matrix}; q/\alpha\right) - 1\right) \\
&\quad + \left(1 - \frac{q}{\beta}\right) \frac{(\epsilon, \gamma, t, fq/\beta; q)_\infty}{(f, \delta, \alpha t/\beta, \epsilon q/\beta; q)_\infty} \sum_{p=0}^{\infty} \frac{(\delta/\gamma)_p(\alpha t/\beta)_p \gamma^p}{(t)_p(q)_p} \sum_{k=0}^{\infty} \frac{(\delta q^p/\gamma)_k (q\gamma/\beta)^k}{(q^{1+p})_k} \\
&\qquad \times {}_2\phi_1\left(\begin{matrix}\alpha q/\beta,\ \epsilon q/\beta \\ fq/\beta\end{matrix}; q, tq^{k+p}\right) \\
&\quad + \left(1 - \frac{q}{\beta}\right) \frac{(\epsilon, \gamma; q)_\infty}{(f, \delta; q)_\infty} \sum_{p=1}^{\infty} \frac{(f/\epsilon)_p \epsilon^p}{(q)_p} \sum_{k=0}^{\infty} \frac{(\delta/\gamma)_k \gamma^k}{(q)_k} \\
&\qquad \times \sum_{m=0}^{\infty} \frac{(fq^p/\epsilon)_m (tq^{p+k})_m}{(q^{1+p})_m (\alpha t q^{p+k}/\beta)_{m+1}} (\epsilon q/\beta)^m.
\end{aligned} \tag{3.13}$$

Proof. Write V_3^* in (3.10) as

$$V_3^* = V_4 + V_5, \tag{3.14}$$

where

$$V_4 = \sum_{m=0}^{\infty} \frac{(f/\epsilon)_m(\epsilon q/\beta)^m}{(q)_m(1-\alpha t q^m/\beta)} \sum_{k=0}^{\infty} \frac{(\delta/\gamma)_k(t)_{m+k}(\alpha t q^m/\beta)_k(\gamma q/\beta)^k}{(q)_k(\alpha t q^{m+1}/\beta)_k(\alpha t/\beta)_{m+k}} \times \sum_{p=0}^{k} \frac{(\alpha t/\beta)_p}{(t)_p}\left(\frac{\beta}{q}\right)^p,$$

(3.15)

$$V_5 = \sum_{m=0}^{\infty} \frac{(f/\epsilon)_m(\epsilon q/\beta)^m}{(q)_m(1-\alpha t q^m/\beta)} \sum_{k=0}^{\infty} \frac{(\delta/\gamma)_k(t)_{m+k}(\alpha t q^m/\beta)_k(\gamma q/\beta)^k}{(q)_k(\alpha t q^{m+1}/\beta)_k(\alpha t/\beta)_{m+k}} \times \sum_{p=1}^{m} \frac{(\alpha t/\beta)_{k+p}}{(t)_{k+p}}\left(\frac{\beta}{q}\right)^{k+p}.$$

Note that V_4 can be written as

(3.16)

$$\begin{aligned}
V_4 &= \sum_{m=0}^{\infty} \frac{(f/\epsilon)_m(t)_m(\epsilon q/\beta)^m}{(q)_m(\alpha t/\beta)_{m+1}} \sum_{p=0}^{\infty} \frac{(\alpha t/\beta)_p}{(t)_p}\left(\frac{\beta}{q}\right)^p \sum_{k=p}^{\infty} \frac{(\delta/\gamma)_k(tq^m)_k}{(q)_k(\alpha t q^{m+1}/\beta)_k}\left(\frac{\gamma q}{\beta}\right)^k \\
&= \sum_{m=0}^{\infty} \frac{(f/\epsilon)_m(t)_m(\epsilon q/\beta)^m}{(q)_m(\alpha t/\beta)_{m+1}} \sum_{p=0}^{\infty} \frac{(\alpha t/\beta)_p(\delta/\gamma)_p(tq^m)_p\gamma^p}{(t)_p(\alpha t q^{m+1}/\beta)_p(q)_p} \\
&\qquad \times \sum_{k=0}^{\infty} \frac{(\delta q^p/\gamma)_k(tq^{m+p})_k(\gamma q/\beta)^k}{(q^{p+1})_k(\alpha t q^{m+p+1}/\beta)_k} \\
&= \sum_{p=0}^{\infty} \frac{(\delta/\gamma)_p\gamma^p}{(q)_p} \sum_{m=0}^{\infty} \frac{(f/\epsilon)_m(tq^p)_m}{(q)_m(\alpha t q^p/\beta)_{m+1}}\left(\frac{\epsilon q}{\beta}\right)^m \sum_{k=0}^{\infty} \frac{(\delta q^p/\gamma)_k(tq^{m+p})_k(\gamma q/\beta)^k}{(q^{p+1})_k(\alpha t q^{m+p+1}/\beta)_k} \\
&= \sum_{p=0}^{\infty} \frac{(\delta/\gamma)_p\gamma^p}{(q)_p} \sum_{k=0}^{\infty} \frac{(\delta q^p/\gamma)_k(tq^p)_k}{(q^{p+1})_k(\alpha t q^p/\beta)_{k+1}}\left(\frac{\gamma q}{\beta}\right)^k \sum_{m=0}^{\infty} \frac{(f/\epsilon)_m(tq^{p+k})_m}{(q)_m(\alpha t q^{p+k+1}/\beta)_m}\left(\frac{\epsilon q}{\beta}\right)^m \\
&= \frac{(fq/\beta)_\infty(t)_\infty}{(\alpha t/\beta)_\infty(\epsilon q/\beta)_\infty} \sum_{p=0}^{\infty} \frac{(\delta/\gamma)_p(\alpha t/\beta)_p\gamma^p}{(t)_p(q)_p} \\
&\qquad \times \sum_{k=0}^{\infty} \frac{(\delta q^p/\gamma)_k(q\gamma/\beta)^k}{(q^{1+p})_k}\, {}_2\phi_1\left(\begin{matrix}\alpha q/b,\ \epsilon q/\beta \\ fq/\beta\end{matrix}; q, tq^{k+p}\right),
\end{aligned}$$

where in the last step, we have used (2.2) to transform the innermost series.

Lastly, V_5 can be simplified to

$$(3.17)\qquad V_5 = \sum_{k=0}^{\infty}\frac{(\delta/\gamma)_k\gamma^k}{(q)_k}\sum_{m=1}^{\infty}\frac{(f/\epsilon)_m(tq^k)_m(\epsilon q/\beta)^m}{(q)_m(\alpha tq^k/\beta)_{m+1}}\sum_{p=1}^{m}\frac{(\alpha tq^k/\beta)_p}{(tq^k)_p}\left(\frac{\beta}{q}\right)^p$$
$$= \sum_{k=0}^{\infty}\frac{(\delta/\gamma)_k\gamma^k}{(q)_k}\sum_{p=1}^{\infty}\frac{(\alpha tq^k/\beta)_p}{(tq^k)_p}\left(\frac{\beta}{q}\right)^p\sum_{m=0}^{\infty}\frac{(f/\epsilon)_{m+p}(tq^k)_{m+p}}{(q)_{m+p}(\alpha tq^k/\beta)_{m+p+1}}\left(\frac{\epsilon q}{\beta}\right)^{m+p}$$
$$= \sum_{p=1}^{\infty}\frac{(f/\epsilon)_p\epsilon^p}{(q)_p}\sum_{k=0}^{\infty}\frac{(\delta/\gamma)_k\gamma^k}{(q)_k}\sum_{m=0}^{\infty}\frac{(fq^p/\epsilon)_m(tq^{p+k})_m}{(q^{1+p})_m(\alpha tq^{p+k}/\beta)_{m+1}}\left(\frac{\epsilon q}{\beta}\right)^m.$$

Now (3.4), (3.7), (3.9), (3.11), (3.14), (3.16) and (3.17) give (3.13). ∎

REMARKS. 1. Agarwal's identity (3.2) can be obtained from (3.13) by letting $\epsilon = f = 0$ in (3.13), and then applying (2.2) to each of the resulting ${}_3\phi_2$'s and to the ${}_2\phi_1$ in the third expression on the right side.

2. The fact that the identity [1, (4.5)]

$$(3.18)\qquad \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}t^n = \frac{(\beta/\alpha, q, \alpha t, q/(\alpha t); q)_\infty}{(q/\alpha, \beta, t, \beta/(\alpha t); q)_\infty} + \frac{(1-(q/\beta))}{(1-(\alpha t/\beta))}\,{}_2\phi_1(q, q/t; q\beta/(\alpha t); q, q/\alpha)$$

was used in the proof of (3.2) (see [2, p. 294]), and (3.2) was used in the proof of (3.13) given above, suggests that a generalization of (3.2) for the series

$$(3.19)\qquad \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(b_1,\ldots,b_r;q)_n}t^n,$$

for $r \in \mathbb{N}$, is not inconceivable. A generalization of (3.13), with $r = 4$ in (3.19), has recently been obtained in [13, Theorem 1.3], which, together with (3.18) and (3.2), shows a nice pattern among the expressions as r increases. Actually, Gupta [26] has already obtained a generalization of Agarwal's result (3.2) for the series in (3.19) for any $r \in \mathbb{N}$, however, his general result, and hence also its specialization when $r = 3$ (and $r = 4$), is not explicit, and is in terms of q-Lauricella functions. Our Theorem 3.1, its variant Theorem 1.1 as well as Theorem 1.3 from [13] are, on the other hand, quite explicit.

We now prove Theorem 1.2 using Theorem 1.1.

LEMMA 3.2. *If m is a positive integer, we have*

$$(3.20)\qquad \sum_{j=0}^{m}\begin{bmatrix} m\\ j\end{bmatrix}(-a)_j(-a)_{m-j}(-1)^j = \begin{cases}(q;q^2)_n(a^2;q^2)_n & \text{if } m = 2n,\\ 0 & \text{if } m \text{ is odd.}\end{cases}$$

Proof. We note that

$$\sum_{j=0}^{m}\begin{bmatrix}m\\ j\end{bmatrix}(-a)_j(-a)_{m-j}(-1)^j=(-a)_m\sum_{j=0}^{m}\frac{(q^{-m})_j(-a)_j(q/a)^j}{(q)_j(-a^{-1}q^{1-m})_j}$$
$$=\begin{cases}(q;q^2)_n(a^2;q^2)_n & \text{if } m=2n,\\ 0 & \text{if } m \text{ is odd,}\end{cases}$$

by [4, p. 526, (1.7)]. ■

Remark. Ismail and Zhang [32, Lemma 4.1] have obtained several interesting results of a similar type.

Proof of Theorem 1.2. Let

$$\beta=-q^2,\quad \gamma=iq^{3/2},\quad \delta=q^{3/2},\quad \epsilon=-iq^{3/2},\quad f=-q^{3/2},\quad t=q$$

in Theorem 1.1, and then let $\alpha\to q$. Note that the second expression on the right side of (1.4) vanishes. Hence

$$\begin{aligned}(3.21)\quad &\sum_{n=0}^{\infty}\frac{q^n(-q^3;q^2)_n(q;q)_n}{(q^3;q^2)_n(-q^2;q)_n}\\ &=\frac{1}{2}\left(1-\frac{1}{q}\right)\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\sum_{n=0}^{\infty}\frac{(-1;q)_n(-q;q^2)_n}{(q;q^2)_n(q;q)_n}q^n\\ &\quad+\frac{(-q^3;q^2)_\infty}{(q^3;q^2)_\infty}\left(1+\frac{1}{q}\right)\sum_{n=0}^{\infty}\frac{(-i\sqrt{q})^n}{(-1)_{n+1}}\sum_{p=0}^{n}\frac{(-1)_p(-q)^p}{(q)_p}\\ &\qquad\times\sum_{m=0}^{n}\begin{bmatrix}n\\ m\end{bmatrix}(-i)_m(-i)_{n-m}(-1)^m\\ &=\frac{1}{2}\left(1-\frac{1}{q}\right)\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\sum_{n=0}^{\infty}\frac{(-1;q)_n(-q;q^2)_n}{(q;q^2)_n(q;q)_n}q^n\\ &\quad+\frac{1}{2q}\frac{(-q;q^2)_\infty}{(q^3;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1;q^2)_n(-q)^n}{(-q)_{2n}}\sum_{p=0}^{2n}\frac{(-1)_p(-q)^p}{(q)_p},\end{aligned}$$

where in the last step, we have applied Lemma 3.2 with $a=i$. This proves (1.6) upon observing that

$$(3.22)\qquad p_{2n}(-1;q^{-2n-1},-1:q)=\sum_{p=0}^{2n}\frac{(-1)_p(-q)^p}{(q)_p},$$

which follows from (1.5). Now (1.2) and (1.6) imply (1.7). ■

Proof of Theorem 1.3. Let

$$\begin{aligned} S_1(q) &:= \sum_{n=0}^{\infty} \frac{q^n(-q^3;q^2)_n(q)_n}{(q^3;q^2)_n(-q^2)_n}, \qquad S_2(q) := -\frac{1}{2}\left(1-\frac{1}{q}\right)\frac{(q)_\infty^2}{(-q)_\infty^2}, \\ S_3(q) &:= -\frac{1}{2}\left(1-\frac{1}{q}\right)\frac{(q)_\infty^2}{(-q)_\infty^2}\sum_{n=1}^{\infty}\frac{(-1)_n(-q;q^2)_n}{(q;q^2)_n(q)_n}q^n. \end{aligned} \tag{3.23}$$

By (1.6), proving (1.8) is equivalent to showing

$$S_1(q)+S_2(q)+S_3(q) \equiv \frac{1}{2q}-\frac{1}{2} \pmod 4. \tag{3.24}$$

Note that $(q)_\infty^2 \equiv (-q)_\infty^2 \pmod 4$ since $(1-x)^2 \equiv (1+x)^2 \pmod 4$. Hence

$$\begin{aligned} S_1(q)+S_3(q) &\equiv S_1(q)+\sum_{n=1}^{\infty}\frac{(-q)_{n-1}(-q;q^2)_n q^{n-1}}{(q^3;q^2)_{n-1}(q)_n} \pmod 4 \\ &= S_1(q)+\sum_{n=0}^{\infty}\frac{(-q)_n(-q;q^2)_{n+1}q^n}{(q^3;q^2)_n(q)_{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{q^n(-q;q^2)_{n+1}(q)_n}{(q^3;q^2)_n(-q)_{n+1}}+\sum_{n=0}^{\infty}\frac{(-q)_n(-q;q^2)_{n+1}q^n}{(q^3;q^2)_n(q)_{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+1}q^n}{(q^3;q^2)_n}\left(\frac{(q)_n}{(-q)_{n+1}}+\frac{(-q)_n}{(q)_{n+1}}\right) \\ &= \sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+1}q^n}{(q^2;q)_{2n+1}}\left((q)_n^2(1-q^{n+1})+(-q)_n^2(1+q^{n+1})\right) \\ &\equiv 2\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+1}(q)_n^2q^n}{(q^2;q)_{2n+1}} \pmod 4, \end{aligned} \tag{3.25}$$

since $(q)_n^2 \equiv (-q)_n^2 \pmod 4$. Now

$$\begin{aligned} S_2(q) &= -\frac{1}{2}\left(1-\frac{1}{q}\right)\left(1+2\sum_{n=1}^{\infty}(-1)^nq^{n^2}\right)^2 \\ &= \frac{1}{2q}-\frac{1}{2}+\frac{2}{q}(1-q)\left(\sum_{n=1}^{\infty}(-1)^nq^{n^2}+\left(\sum_{n=1}^{\infty}(-1)^nq^{n^2}\right)^2\right). \end{aligned} \tag{3.26}$$

From (3.25) and (3.26), it suffices to show that

$$\begin{aligned} &2\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+1}(q)_n^2q^n}{(q^2;q)_{2n+1}} \\ &\qquad \equiv -\frac{2}{q}(1-q)\left(\sum_{n=1}^{\infty}(-1)^nq^{n^2}+\left(\sum_{n=1}^{\infty}(-1)^nq^{n^2}\right)^2\right) \pmod 4, \end{aligned} \tag{3.27}$$

or equivalently

$$\text{(3.28)}\qquad \sum_{n=0}^{\infty}\frac{(q^3;q^2)_n(q^2;q^2)_nq^{n+1}}{(q^2;q^2)_{n+1}(q^3;q^2)_n} \equiv -\Big(\sum_{n=1}^{\infty}(-1)^nq^{n^2}+\Big(\sum_{n=1}^{\infty}(-1)^nq^{n^2}\Big)^2\Big) \pmod 4.$$

Now

$$\text{(3.29)}\qquad \sum_{n=0}^{\infty}\frac{(q^3;q^2)_n(q^2;q^2)_nq^{n+1}}{(q^2;q^2)_{n+1}(q^3;q^2)_n}=\sum_{n=0}^{\infty}\frac{q^{n+1}}{1-q^{2n+2}}=\sum_{N=1}^{\infty}d_o(N)q^N,$$

where $d_o(N)$ is the number of odd divisors of N. Also,

$$\text{(3.30)}\qquad -\Big(\sum_{n=1}^{\infty}(-1)^nq^{n^2}+\Big(\sum_{n=1}^{\infty}(-1)^nq^{n^2}\Big)^2\Big) \equiv \sum_{n=1}^{\infty}q^{n^2}+\Big(\sum_{n=1}^{\infty}q^{n^2}\Big)^2 \pmod 4.$$

Let

$$\text{(3.31)}\qquad \sum_{N=1}^{\infty}a(N)q^N:=\sum_{n=1}^{\infty}q^{n^2}+\Big(\sum_{n=1}^{\infty}q^{n^2}\Big)^2.$$

Let $r_2(m)$ denote the number of representations of m as a sum of two squares, where representations with different orders or different signs of the summands are regarded as distinct. Now if N is not a square, then the number of representations of N as a sum of two positive squares is equal to

$$\text{(3.32)}\qquad \tfrac{1}{4}r_2(N)=d_1(N)-d_3(N)\equiv d_o(N) \pmod 2,$$

where $d_j(N)$ denotes the number of divisors of N congruent to j modulo 4 for $j=1,3$. Note that Jacobi's formula on $r_2(N)$ was employed in the penultimate step. Thus, $a(N)\equiv d_o(N) \pmod 2$. If, however, N is a square, then the number of representations of N as a sum of two positive squares is equal to $\frac{1}{4}r_2(n)-1$. Hence,

$$\text{(3.33)}\qquad a(N)=\tfrac{1}{4}r_2(N)=d_1(N)-d_3(N)\equiv d_o(N) \pmod 2.$$

This implies that (3.28) always holds, and this proves the theorem. ∎

4. Congruences for $\overline{p}_\omega(n)$. This section is devoted to proving Theorem 1.4. We start with the following series $S(q)$:

$$\text{(4.1)}\qquad S(q):=\sum_{n=1}^{\infty}\frac{q^n(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}{(1+q^n)(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty}.$$

Lemma 4.1. *The following identity holds:*

$$S(q) = -\sum_{n=1}^{\infty}(-1)^n q^{2n^2} + \sum_{k=1}^{\infty} \frac{q^k(q;q^2)_k}{(-q;q^2)_k(1+q^{2k})}. \tag{4.2}$$

Proof. We have

$$\begin{aligned}
S(q) &= \sum_{n=1}^{\infty} \frac{q^n(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}{(1+q^n)(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty} \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{n-1}(-q^{2n+1};q^2)_\infty}{(q;q)_n(q^{2n+1};q^2)_\infty} \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{n-1}}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(-1;q^2)_k}{(q^2;q^2)_k} q^{(2n+1)k} \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^k(-1;q^2)_k}{(q^2;q^2)_k} \sum_{n=1}^{\infty} \frac{(-q;q)_{n-1}}{(q;q)_n} q^{(2k+1)n} \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^k(-1;q^2)_k}{(q^2;q^2)_k} \left(-\frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1;q)_n}{(q;q)_n} q^{(2k+1)n} \right) \\
&= -\frac{(q;q)_\infty}{2(-q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^k(-1;q^2)_k}{(q^2;q^2)_k} \\
&\quad + \frac{(q;q)_\infty}{2(-q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^k(-1;q^2)_k}{(q^2;q^2)_k} \sum_{n=0}^{\infty} \frac{(-1;q)_n}{(q;q)_n} q^{(2k+1)n} \\
&= -\frac{(q;q)_\infty}{2(-q;q)_\infty} \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} + \frac{(q;q)_\infty}{2(-q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^k(-1;q^2)_k}{(q^2;q^2)_k} \frac{(-q^{2k+1};q)_\infty}{(q^{2k+1};q)_\infty} \\
&= -\frac{(q^2;q^2)_\infty}{2(-q^2;q^2)_\infty} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{q^k(-1;q^2)_k}{(q^2;q^2)_k} \frac{(q;q)_{2k}}{(-q;q)_{2k}} \\
&= -\frac{1}{2} - \sum_{n=1}^{\infty}(-1)^n q^{2n^2} + \frac{1}{2} + \sum_{k=1}^{\infty} \frac{q^k(q;q^2)_k}{(-q;q^2)_k(1+q^{2k})},
\end{aligned}$$

where (2.1) has been used for the third and seventh equalities. ■

Let

$$A(q) := \sum_{k=1}^{\infty} \frac{q^k(q;q^2)_k}{(-q;q^2)_k(1+q^{2k})}. \tag{4.3}$$

Lemma 4.2. *We have*

$$A(q) + A(-q) = -\frac{1}{2} + \frac{1}{2} \frac{(q^2,q^2;q^2)_\infty}{(-q^2,-q^2;q^2)_\infty}.$$

Proof. We now extend the sum in (4.3) to negative infinity:

$$\sum_{k=-\infty}^{\infty} \frac{q^k (q;q^2)_k}{(-q;q^2)_k(1+q^{2k})} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{q^k (q;q^2)_k}{(-q;q^2)_k(1+q^{2k})} + \sum_{k=1}^{\infty} \frac{(-1)^k q^k (-q;q^2)_k}{(q;q^2)_k(1+q^{2k})}, \tag{4.4}$$

where $(a;q)_n := (a;q)_\infty/(aq^n;q)_\infty$. Thus

$$\sum_{k=-\infty}^{\infty} \frac{q^k (q;q^2)_k}{(-q;q^2)_k(1+q^{2k})} = \frac{1}{2} + A(q) + A(-q). \tag{4.5}$$

Set $q \to q^2$, $a = -1, c = q, d = 1, e = q$, and $f = -1$ in (2.4). Then we obtain

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \frac{q^k (q;q^2)_k}{(-q;q^2)_k(1+q^{2k})} &= \frac{1}{2} \sum_{k=-\infty}^{\infty} \frac{q^k (q;q^2)_k(-1;q^2)_k}{(-q;q^2)_k(-q^2;q^2)_k} \\ &= \frac{1}{2} \frac{(q,q^2,-q,q^2;q^2)_\infty}{(-q^2,-q^2,-q,q;q^2)_\infty} \\ &= \frac{1}{2} \frac{(q^2,q^2;q^2)_\infty}{(-q^2,-q^2;q^2)_\infty},\end{aligned}$$

which with (4.5) completes the proof. ∎

Lemma 4.3. *We have*

$$\frac{1}{2}\big(A(q) - A(-q)\big) \equiv q \frac{(q^8;q^8)_\infty^4}{(q^4;q^4)_\infty^2} \pmod{4}. \tag{4.6}$$

Proof. First note that using Alladi's identity [3, p. 215] we obtain

$$\frac{(q;q^2)_k}{(-q;q^2)_k} = 1 - 2\sum_{j=1}^{k} \frac{q^{2j-1}(q;q^2)_{j-1}}{(-q;q^2)_j}.$$

Thus

$$\begin{aligned}A(q) &= \sum_{k=1}^{\infty} \frac{q^k (q;q^2)_k}{(-q;q^2)_k(1+q^{2k})} \\ &= \sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}} - 2\sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}} \sum_{j=1}^{k} \frac{q^{2j-1}(q;q^2)_{j-1}}{(-q;q^2)_j} \\ &\equiv \sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}} + 2\sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}} \sum_{j=1}^{k} \frac{q^{2j-1}}{1+q^{2j-1}} \pmod{4}.\end{aligned} \tag{4.7}$$

Now

$$\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}\sum_{j=1}^{k}\frac{q^{2j-1}}{1+q^{2j-1}}=\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}\sum_{j=1}^{k}\sum_{m=1}^{\infty}(-1)^{m-1}q^{(2j-1)m} \tag{4.8}$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}\sum_{m=1}^{\infty}\sum_{j=1}^{k}(-1)^{m-1}q^{(2j-1)m}$$

$$=-\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}\sum_{m=1}^{\infty}(-q)^m\sum_{j=0}^{k-1}q^{2mj}$$

$$=-\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}\sum_{m=1}^{\infty}(-q)^m\frac{(1-q^{2km})}{1-q^{2m}}$$

$$\equiv\sum_{k,m=1}^{\infty}\frac{q^{k+m}}{(1+q^{2k})(1+q^{2m})}+\sum_{k,m=1}^{\infty}\frac{q^{2km+k+m}}{(1+q^{2k})(1+q^{2m})} \pmod 2$$

$$\equiv\sum_{k=1}^{\infty}\frac{q^{2k}}{(1-q^{2k})^2}+\sum_{k=1}^{\infty}\frac{q^{2k^2+2k}}{(1-q^{2k})^2} \pmod 2,$$

where the last congruence follows since each of the double summations is symmetric in k and m. Thus, by (4.7) and (4.8),

$$A(q)\equiv\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}+2\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{4k}}+2\sum_{k=1}^{\infty}\frac{q^{2k(k+1)}}{1-q^{4k}} \pmod 4$$

$$\equiv\sum_{k=1}^{\infty}\frac{q^{2k-1}}{1+q^{4k-2}}+\sum_{k=1}^{\infty}\frac{q^{2k}}{1+q^{4k}}+2\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{4k}}+2\sum_{k=1}^{\infty}\frac{q^{2k(k+1)}}{1-q^{4k}} \pmod 4.$$

Since the odd powers of q appear only in the first sum on the right hand side above, we see that

$$\frac{1}{2}\big(A(q)-A(-q)\big)\equiv\sum_{k=1}^{\infty}\frac{q^{2k-1}}{1+q^{4k-2}} \pmod 4$$

$$=q\frac{(q^8;q^8)_\infty^4}{(q^4;q^4)_\infty^2},$$

where the last equality follows from [20, (32.26)]. It can also be derived by letting $q\to q^4$ and then substituting $a=-q^{-2}, b=-q^2, z=q^2$ in Ramanujan's ${}_1\psi_1$ summation formula [25, p. 239, (II 29)]

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n=\frac{(az;q)_\infty(q/(az);q)_\infty(q;q)_\infty(b/a;q)_\infty}{(z;q)_\infty(b/(az);q)_\infty(b;q)_\infty(q/a;q)_\infty}, \tag{4.9}$$

valid for $|b/a|<|z|<1$ and $|q|<1$. ∎

By Lemmas 4.2 and 4.3, we have

$$A(q) \equiv -\frac{1}{4} + \frac{1}{4}\frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2} + q\frac{(q^8;q^8)_\infty^4}{(q^4;q^4)_\infty^2} \pmod 4.$$

Also, we recall that

$$\phi(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = (q;q)_\infty (q;q^2)_\infty = \frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{4.10}$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{4.11}$$

Thus,

$$A(q) \equiv -\frac{1}{4} + \frac{1}{4}\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{2(m^2+n^2)} + q\sum_{m,n=0}^{\infty} q^{2m(m+1)+2n(n+1)} \pmod 4. \tag{4.12}$$

We are now ready to prove Theorem 1.4. First, note that

$$\frac{1+x}{1-x} \equiv \frac{1-x}{1+x} \pmod 4. \tag{4.13}$$

Thus from (1.1),

$$\sum_{n=1}^{\infty} \overline{p}_\omega(n) q^n \equiv \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}{(1-q^n)(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty} \pmod 4. \tag{4.14}$$

Now, modulo 4,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}{(1-q^n)(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty} \\
&\quad = \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}{(1+q^n)(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty} \\
&\qquad + 2\sum_{n=1}^{\infty} \frac{q^{2n} (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}{(1-q^{2n})(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty} \\
&\quad \equiv \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}{(1+q^n)(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty} + 2\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \\
&\quad \equiv \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}{(1+q^n)(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty} + 2\sum_{n=1}^{\infty} q^{2n^2},
\end{aligned} \tag{4.15}$$

where the second-to-last congruence follows from the fact that

$$1+x \equiv 1-x \pmod 2. \tag{4.16}$$

For the last congruence above, we have used Clausen's identity [20, p. 16, (14.51)]

$$\sum_{n=1}^{\infty} d(n)q^n = \sum_{n=1}^{\infty} \frac{1+q^n}{1-q^n} q^{n^2},$$

which implies that

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \equiv \sum_{n=1}^{\infty} q^{n^2} \pmod 2. \tag{4.17}$$

Thus, from (4.1), (4.14) and (4.15), we have

$$\sum_{n=1}^{\infty} \overline{p}_\omega(n)q^n \equiv S(q) + 2\sum_{n=1}^{\infty} q^{2n^2} \pmod 4. \tag{4.18}$$

Theorem 4.4. *We have*

$$\sum_{n=1}^{\infty} \overline{p}_\omega(n)q^n \equiv -1 + \sum_{m,n=0}^{\infty} (-1)^{m+n} q^{2(m^2+n^2)} + q \sum_{m,n=0}^{\infty} q^{2m(m+1)+2n(n+1)} \pmod 4. \tag{4.19}$$

Proof. The congruence follows from (4.2), (4.3), (4.12), and (4.18). ∎

Thus it immediately follows from Theorem 4.4 that $\overline{p}_\omega(4n+3) \equiv 0 \pmod 4$. Since $8n+6 = 2(4n+3)$ and $4n+3$ cannot be written as a sum of two squares, this also proves $\overline{p}_\omega(8n+6) \equiv 0 \pmod 4$.

5. Different representations of the generating function of $\overline{\mathrm{spt}}_\omega(n)$. In this section, we will show some relationships between $\overline{\mathrm{spt}}_\omega(n)$ and $\overline{\mathrm{spt}}(n)$ and $\overline{\mathrm{spt}}_2(n)$.

First, note that $(a;q)_{-n} = 1/(aq^{-n};q)_n$ for any $n \geq 0$. Now, by taking $d=1$, $e=0$, and $v=1$ in [17, (1.1)], we obtain

$$\begin{aligned}
&\mathcal{N}_2(1,0;q) \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n^2+n}(-1;q)_n}{(1-q^n)^2(-q;q)_n} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n^2+n}(-q^{1-n};q)_n}{(1-q^n)^2(-q^{-n};q)_n}\right) \\
&= 2\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n^2+n}}{(1-q^n)^2}.
\end{aligned}$$

Also, by [17, (1.2)],

$$\mathrm{Spt}(1,0;q) = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - \mathcal{N}_2(1,0;q),$$

where $\mathrm{Spt}(1,0;q)$ is the generating function $\overline{\mathrm{spt}}(n)$ by [17, Theorem 7.1]. Thus,

$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}(n)q^n=\sum_{n=1}^{\infty}\frac{q^n(-q^{n+1};q)_\infty}{(1-q^n)^2(q^{n+1};q)_\infty}$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+2\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}}{(1-q^n)^2}. \tag{5.1}$$

We also define

$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_2(n)q^n:=\sum_{n=1}^{\infty}\frac{q^{2n}(-q^{2n+1};q)_\infty}{(1-q^{2n})^2(q^{2n+1};q)_\infty}.$$

We replace q by q^2, and then set $d=1$ and $e=q^{-1}$ in [17, (1.1) and (1.2)]. Then, by [17, Theorem 7.1], we have

$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_2(n)q^n=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}+2\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+2n}}{(1-q^{2n})^2}$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}$$
$$+\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+n}(1+q^{2n})}{(1-q^{2n})^2}+\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}}{(1-q^n)^2}\right)$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}$$
$$+\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}+\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}}{(1-q^n)^2},$$

where the last equality follows from [22, (1.4) and Theorem 2.2], i.e.,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}(1+q^n)}{(1-q^n)^2}=\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}.$$

Also,

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}=\sum_{n,k=1}^{\infty}kq^{kn}=\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}.$$

Therefore,

$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_2(n)q^n=\sum_{n=1}^{\infty}\frac{q^{2n}(-q^{2n+1};q)_\infty}{(1-q^{2n})^2(q^{2n+1};q)_\infty}$$
$$=2\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}+\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}}{(1-q^n)^2}. \tag{5.2}$$

Theorem 5.1. *We have*

(5.3)
$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_{\omega}(n)q^n=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+2\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^nq^{2n(n+1)}}{(1-q^{2n})^2}.$$

Proof. In (2.3), we set $a=1$, $p_1=z=p_2^{-1}$, and $f=-1$. Then we obtain

(5.4)
$$\sum_{n=0}^{\infty}\frac{(z;q)_n(z^{-1};q)_n(-q;q^2)_nq^n}{(q;q)_n(q;q^2)_n(-q;q)_n}=\frac{(zq;q)_\infty(z^{-1}q;q)_\infty}{(q;q)_\infty^2}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{2n(n+1)}}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right).$$

Recalling the facts [8, (2.1), (2.4)]

$$-\frac{1}{2}\left[\frac{d^2}{dz^2}(1-z)(1-z^{-1})f(z)\right]_{z=1}=f(1),$$
$$-\frac{1}{2}\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty\right]_{z=1}=(q;q)_\infty^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},$$

we now take the second derivative on both sides of (5.4) with respect to z, then set $z=1$ to obtain

$$\sum_{n=1}^{\infty}\frac{q^n(q;q)_n(-q;q^2)_n}{(1-q^n)^2(-q;q)_n(q;q^2)_n}=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+2\sum_{n=1}^{\infty}\frac{(-1)^nq^{2n(n+1)}}{(1-q^{2n})^2}.$$

Multiply both sides of the above identity by $(-q^2;q^2)_\infty/(q^2;q^2)_\infty$ to get

(5.5)
$$\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(q;q)_n(-q;q^2)_n}{(1-q^n)^2(-q;q)_n(q;q^2)_n}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+2\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^nq^{2n(n+1)}}{(1-q^{2n})^2}.$$

Note that the left hand side of (5.5) is

$$\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(q;q)_n(-q;q^2)_n}{(1-q^n)^2(-q;q)_n(q;q^2)_n}$$
$$=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(q;q)_n(-q;q)_{2n}(q^2;q^2)_n}{(1-q^n)^2(-q;q)_n(q;q)_{2n}(-q^2;q^2)_n}$$
$$=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q^{n+1};q)_n(q^2;q^2)_n}{(1-q^n)^2(q^{n+1};q)_n(-q^2;q^2)_n}$$

$$= \sum_{n=1}^{\infty} \frac{q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty}{(1-q^n)^2(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}$$
$$= \sum_{n=1}^{\infty} \overline{\mathrm{spt}}_\omega(n)q^n,$$

where the last equality follows from the definition of $\overline{\mathrm{spt}}_\omega(n)$ in (1.11). ∎

We now relate our $\overline{\mathrm{spt}}_\omega(n)$ to $\overline{\mathrm{spt}}(n)$ and $\overline{\mathrm{spt}}_2(n)$.

Corollary 5.2. *We have*

$$\sum_{n=1}^{\infty} \overline{\mathrm{spt}}_\omega(n)q^n$$
$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + \sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right) + \sum_{n=1}^{\infty}\overline{\mathrm{spt}}(n)q^{2n} \tag{5.6}$$
$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + 2\sum_{n=1}^{\infty}\frac{(2n-1)q^{4n-2}}{1-q^{4n-2}}\right) \tag{5.7}$$
$$+ 2\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_2(n)q^{2n}.$$

Proof. From (5.3),

$$\sum_{n=1}^{\infty} \overline{\mathrm{spt}}_\omega(n)q^n$$
$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}$$
$$+ 2\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n(n+1)}}{(1-q^{2n})^2}$$
$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} + \sum_{n=1}^{\infty}\overline{\mathrm{spt}}(n)q^{2n}$$
$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + \sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right) + \sum_{n=1}^{\infty}\overline{\mathrm{spt}}(n)q^{2n},$$

where the second last equality follows from (5.1). Also, by (5.2),

$$\sum_{n=1}^{\infty} \overline{\mathrm{spt}}_\omega(n)q^n = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - \sum_{n=1}^{\infty}\frac{4nq^{4n}}{1-q^{4n}}\right) + 2\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_2(n)q^{2n},$$

which yields (5.7). ∎

6. Congruences for $\overline{\mathrm{spt}}_\omega(n)$. The congruences satisfied by $\overline{\mathrm{spt}}_\omega(n)$, which are given in Theorem 1.5, are proved in this section.

6.1. Congruences modulo 3. We prove (1.15) and (1.16) here. Let

$$S(q) := \sum_{n=1}^{\infty} c_n q^n = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + 2\sum_{n=1}^{\infty}\frac{(2n-1)q^{4n-2}}{1-q^{4n-2}}\right). \tag{6.1}$$

Then, by (5.7),

$$\overline{\mathrm{spt}}_\omega(n) = c_n + 2\overline{\mathrm{spt}}_2(n/2), \tag{6.2}$$

where we follow the convention that $\overline{\mathrm{spt}}_2(x) = 0$ if x is not a positive integer.

By (1.12) and (1.13), it suffices to show that $c_{3n} \equiv c_{3n+2} \equiv 0 \pmod 3$. Now

$$\begin{aligned} S(q) &\equiv \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} - \sum_{n=1}^{\infty}\frac{(2n-1)q^{4n-2}}{1-q^{4n-2}}\right) \pmod 3 \\ &= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} \\ &= q(-q^2;q^2)_\infty^9(q^2;q^2)_\infty^3 \equiv q(-q^6;q^6)_\infty^3(q^6;q^6)_\infty \pmod 3, \end{aligned}$$

where the third equality follows from [20, (32.31)]. Hence $c_{3n} \equiv c_{3n+2} \equiv 0 \pmod 3$.

6.2. Another proof of (1.15). Let

$$M_1(q) := \sum_{n=1}^{\infty} d_n q^n = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + \sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right).$$

Since [23, Thm. 1.2] implies $\overline{\mathrm{spt}}(3n) \equiv 0 \pmod 3$, it suffices to show $d_{3n} \equiv 0 \pmod 3$ by (5.6). Now

$$\begin{aligned} M_1(q) &= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + \sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right) \\ &\equiv \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} - \sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}\right) \pmod 3 \\ &= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{1-q^n} \\ &\equiv \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{\chi(n)q^n}{1-q^n} \pmod 3, \end{aligned}$$

where $\chi(n)=1$ if $n\equiv 1$ or $2 \pmod 6$, is -1 if $n\equiv 4$ or $5 \pmod 6$, and is 0 if $n\equiv 0 \pmod 3$. Thus,

$$M_1(q)\equiv \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}E_{1,2}(n;6)q^n \pmod 3,$$

where

$$E_{1,2}(n;6)=\sum_{\substack{d\mid n\\ d\equiv 1,2 \,(\mathrm{mod}\, 6)}}1-\sum_{\substack{d\mid n\\ d\equiv -1,-2 \,(\mathrm{mod}\, 6)}}1.$$

By (4.10), we see that

$$\phi(-q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2}-2q\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2+6n}=\phi(-q^9)-2qW(q^3).$$

Hence, by [20, p. 80, (32.39)] we have, modulo 3,

$$\begin{aligned}
M_1(q)&\equiv \frac{1}{\phi(-q^2)}\sum_{n=1}^{\infty}E_{1,2}(n;6)q^n\\
&\equiv \frac{1}{\phi(-q^2)}\left(1-\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty^6}{(q;q)_\infty^2(q^6;q^6)_\infty^3}\right)\\
&=\frac{1}{\phi(-q^2)}\left(1-\frac{\phi(-q^3)^3}{\phi(-q)}\right)\\
&\equiv \frac{(\phi(-q)-\phi(-q^9))}{\phi(-q^2)\phi(-q)}\\
&=\frac{-2qW(q^3)(-q;q)_\infty(-q^2;q^2)_\infty}{(q^2;q^2)_\infty(q;q)_\infty}=\frac{-2qW(q^3)(q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty^3}\\
&\equiv \frac{qW(q^3)}{(q^3;q^3)_\infty}(q^4;q^4)_\infty(q;q^2)_\infty=\frac{qW(q^3)}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2-n}.
\end{aligned}$$

Now $2n^2-n$ is only congruent to 0 or 1 modulo 3. Hence this last expression has no nonzero coefficients for terms where q is a power of 3. Hence $3\mid d_{3n}$.

6.3. Congruence modulo 6. The congruence in (1.18) can be reduced to (1.16). Using (4.16), we see that, modulo 2,

$$\begin{aligned}
\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_\omega(n)q^n&=\sum_{n=1}^{\infty}\frac{q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty}{(1-q^n)^2(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}\\
&\equiv\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}=\sum_{n=1}^{\infty}\sigma(n)q^n,
\end{aligned}\tag{6.3}$$

where $\sigma(n)$ denotes the sum of all positive divisors of n.

Now any number of the form $6n+5$ has all its prime divisors odd. The sum of the divisors of an odd prime raised to an odd power is even. For any number congruent to 5 modulo 6 there must be at least one odd prime congruent to 5 modulo 6 raised to an odd power in its prime factorization (otherwise the number would be congruent to 1 modulo 6). Since $\sigma(n)$ is multiplicative, it follows that $\sigma(6n+5)$ is even. Hence the coefficients of q^{6n+5} in both series are all even.

6.4. Congruence modulo 5. The congruence (1.17) is proved here. Since $\overline{\mathrm{spt}}_2(5n+3) \equiv 0 \pmod 5$, using (6.2), it suffices to show that $c_{10n+6} \equiv 0 \pmod 5$, where c_n is defined in (6.1). By (6.1),

$$S(q) = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + 2\sum_{n=1}^{\infty}\frac{(2n-1)q^{4n-2}}{1-q^{4n-2}}\right) \tag{6.4}$$
$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}$$
$$\times\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}} + \sum_{n=1}^{\infty}\frac{(2n-1)q^{4n-2}}{1-q^{4n-2}} + 2\sum_{n=1}^{\infty}\frac{(2n-1)q^{4n-2}}{1-q^{4n-2}}\right)$$
$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}} + 3\sum_{n=1}^{\infty}\frac{(2n-1)q^{4n-2}}{1-q^{4n-2}}\right)$$
$$=: qE_1(q^2) + 3E_2(q^2).$$

Thus, it suffices to show that the coefficient of q^{5n+3} in $E_2(q)$ is congruent to 0 modulo 5, which follows from the following lemma.

LEMMA 6.1. *Let $r(q)$ be the Rogers–Ramanujan continued fraction given by*

$$r(q) = q^{1/5}\frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty},$$

and let $E_2(q)$ be defined as above. Then, modulo 5,

$$E_2(q^{1/5}) \equiv \frac{q(q;q)_\infty^2(q^{10};q^{10})_\infty}{(q^2,q^3;q^5)_\infty^5(q^5;q^5)_\infty^2}\left(\frac{r(q^2)}{r(q)^2} + \frac{1}{r(q)^2r(q^2)} + \frac{3}{r(q)^3} + \frac{r(q^2)}{r(q)^3}\right).$$

Proof. As in [30], set

$$A(q) = q^{1/5}\frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty^{3/5}}, \qquad B(q) = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty^{3/5}}.$$

Although $A(q)^{\pm 1}$ and $B(q)^{\pm 1}$ do not have integer coefficients, all of the series $A(q)^{\pm 5}$, $B(q)^{\pm 5}$, and $r(q)^{\pm 1}$ have integer coefficients. We will use the following properties:

$$r(q) = \frac{A(q)}{B(q)}, \tag{6.5}$$

$$A(q)B(q) = q^{1/5}\frac{(q^5;q^5)_\infty}{(q;q)_\infty^{1/5}}, \tag{6.6}$$

$$2A(q)^5 + B(q)^5 \equiv 1 \pmod 5, \tag{6.7}$$

$$A(q^{1/5})^5 \equiv \frac{r(q)}{1+2r(q)} \pmod 5. \tag{6.8}$$

Identities (6.5) and (6.6) follow directly from the definitions of $A(q)$ and $B(q)$. By multiplying through by $(q;q)_\infty^3$, we see that (6.7) has the equivalent formulation

$$(q^2,q^3,q^5;q^5)_\infty^5 + 2q(q,q^4,q^5;q^5)_\infty^5 \equiv (q;q)_\infty^3 \pmod 5.$$

After applying Jacobi's triple product identity [5, p. 21, Theorem 2.8] to each of the products on the left hand side, applying the fact that the characteristic is 5, and using Jacobi's identity [5, p. 176] on the right side, we must show that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n}{2}(5n-1)} + 2q\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5n}{2}(5n-3)} \equiv \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n}{2}(n+1)} \pmod 5,$$

which is easily seen to be true by breaking n into residue classes modulo 10 on the right hand side. Next, from [30, Theorem 3.3], we have

$$A(q^{1/5})^5 = A(q)^5 - 3A(q)^4B(q) + 4A(q)^3B(q)^2 - 2A(q)^2B(q)^3 + A(q)B(q)^4,$$

so, by (6.5),

$$\begin{aligned} A(q^{1/5})^5 &= A(q)^5 - 3A(q)^4B(q) + 4A(q)^3B(q)^2 - 2A(q)^2B(q)^3 + A(q)B(q)^4 \\ &= B(q)^5 r(q)\big(1 - 2r(q) + 4r(q)^2 - 3r(q)^3 + r(q)^4\big) \\ &\equiv B(q)^5 r(q)(1+2r(q))^4 \equiv \frac{r(q)(1+2r(q))^4}{1+2r(q)^5} \equiv \frac{r(q)}{1+2r(q)} \pmod 5, \end{aligned}$$

where (6.5) and (6.7) have been used to obtain the penultimate equality. Thus, (6.8) is clear. We will also require the two identities

$$(q^{1/5};q^{1/5})_\infty = q^{1/5}(q^5;q^5)_\infty\left(\frac{1}{r(q)} - r(q) - 1\right), \tag{6.9}$$

$$A(q)^5 = \sum_{\substack{n=1\\5\nmid n}}^{\infty} \frac{q^n}{1-q^n}\cdot\begin{cases} 1, & n\equiv 1 \pmod 5,\\ -3, & n\equiv 2 \pmod 5,\\ 3, & n\equiv 3 \pmod 5,\\ -1, & n\equiv 4 \pmod 5.\end{cases} \tag{6.10}$$

The first identity can be found in [15, p. 270] and the second is the first equality in [30, Lemma 2.4]. Let us save space by writing $r = r(q)$ and

$R = r(q^2)$. Now, by (6.10),

$$\begin{aligned}
E_2(q^{1/5}) &= \frac{(-q^{1/5};q^{1/5})_\infty}{(q^{1/5};q^{1/5})_\infty} \sum_{n=1}^{\infty} \frac{(2n-1)q^{(2n-1)/5}}{1-q^{(2n-1)/5}} \\
&= \frac{(q^{2/5};q^{2/5})_\infty}{(q^{1/5};q^{1/5})_\infty^2} \sum_{n=1}^{\infty} \left(\frac{nq^{n/5}}{1-q^{n/5}} - \frac{2nq^{2n/5}}{1-q^{2n/5}} \right) \\
&\equiv \frac{(q^{2/5};q^{2/5})_\infty (q^{1/5};q^{1/5})_\infty^3}{(q;q)_\infty} \sum_{n=1}^{\infty} \left(\frac{nq^{n/5}}{1-q^{n/5}} - \frac{2nq^{2n/5}}{1-q^{2n/5}} \right) \pmod 5 \\
&\equiv \frac{(q^{2/5};q^{2/5})_\infty (q^{1/5};q^{1/5})_\infty^3}{(q;q)_\infty} (A(q^{1/5})^5 - 2A(q^{2/5})^5) \pmod 5.
\end{aligned}$$

By the dissection formulas (6.9) and (6.8) we have, modulo 5,

$$\begin{aligned}
&E_2(q^{1/5}) \\
&\equiv \frac{q(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty}{(q;q)_\infty} \left(\frac{1}{R} - R - 1 \right) \left(\frac{1}{r} - r - 1 \right)^3 \left(\frac{r}{1+2r} - \frac{2R}{1+2R} \right) \\
&\equiv \frac{q(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty}{(q;q)_\infty} \frac{(1+2R)^2}{R} \frac{(1+2r)^6}{r^3} \left(\frac{r}{1+2r} - \frac{2R}{1+2R} \right) \\
&\equiv \frac{q(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty (1+2r^5)}{(q;q)_\infty} \frac{(1+2R)^2}{R} \frac{1+2r}{r^3} \left(\frac{r}{1+2r} - \frac{2R}{1+2R} \right) \\
&= \frac{q(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty (1+2r^5)}{(q;q)_\infty} \left(-\frac{4R}{r^2} + \frac{1}{r^2 R} - \frac{2}{r^3} - \frac{4R}{r^3} \right) \\
&\equiv \frac{q(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty}{(q;q)_\infty B(q)^5} \left(\frac{R}{r^2} + \frac{1}{r^2 R} + \frac{3}{r^3} + \frac{R}{r^3} \right). \qquad \blacksquare
\end{aligned}$$

REMARKS. 1. $E_2(q)$ defined in (6.4) is the function $Ov(q)$ in [16, (1.9)], and the desired congruence modulo 5 then follows from [16, (1.10)]. However, the above lemma is much more general.

2. In [11, proof of Theorem 6.4], it is written that

$$\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{2n+1}} = q\frac{(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4},$$

which is not correct. What the authors meant is

$$\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{4n+2}} = q\frac{(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}.$$

The proof can be easily fixed as the congruence concerns the odd power terms only. However, using the functions $A(q)$, $B(q)$ and $r(q)$ from the proof of the lemma above, we can also correct the proof of [11, Theorem 6.4]. With

$k = r(q)r(q^2)^2$, we have the parameterizations [10, Entry 24]

$$r(q)^5 = k\left(\frac{1-k}{1+k}\right)^2, \qquad r(q^2)^5 = k^2\left(\frac{1-k}{1+k}\right),$$

and the dissection of the relevant q-series is found to be

$$\begin{aligned}\frac{1}{(q^{1/5};q^{1/5})_\infty}&\sum_{n=1}^{\infty}\frac{nq^{n/5}}{1-q^{n/5}}\\ &\equiv \frac{r(q^2)B(q)^5B(q^2)^5}{q^{2/5}(q^{10};q^{10})_\infty}\frac{(2+k)^3}{(1+k)^2}\big(4k+2r(q)(1+k)+r(q^2)\big) \pmod 5.\end{aligned}$$

7. Congruences involving $\overline{\mathrm{spt}}(n)$ and $\overline{\mathrm{spt}}_\omega(n)$. This section is devoted to proving Theorems 1.6 and 1.7. We first need the following lemmas.

LEMMA 7.1. *We have*

$$\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{1-q^n} = \sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}}. \tag{7.1}$$

Proof. This is proved in [33, p. 28]. ∎

LEMMA 7.2. *We have*

$$\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}} \equiv \sum_{n=1}^{\infty}(q^{n^2}+q^{2n^2}) \pmod 2. \tag{7.2}$$

Proof. We have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}} &= \sum_{n=1}^{\infty}\frac{q^n}{1-q^n} - \sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\\ &\equiv \sum_{n=1}^{\infty}q^{n^2} + \sum_{n=1}^{\infty}q^{2n^2} \pmod 2,\end{aligned}$$

by (4.17). ∎

7.1. Proof of Theorem 1.6. By (5.1) and (5.3), we know that

$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}(n)q^n \equiv \sum_{n=1}^{\infty}\overline{\mathrm{spt}}_\omega(n)q^n \equiv \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} \equiv \sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}} \pmod 2.$$

Therefore, the congruences in (1.19) follow from (7.2).

7.2. Proof of Theorem 1.7. Let us introduce the series

$$T(q) = \sum_{n=1}^{\infty}q^{n^2}.$$

Several identities satisfied by this series are

$$1+2T(-q)=\frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{7.3}$$

$$T(q)+T(q)^2=\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n+1}}{1-q^{2n+1}}, \tag{7.4}$$

where the first identity is a restatement of (4.10) and the second is [20, p. 59, (26.63)]. By applying (5.1), (7.1), (7.3), and (7.4) in the same sequence, we find that, modulo 4,

$$\begin{aligned}
\sum_{n=1}^{\infty}\overline{\mathrm{spt}}(n)q^n &= \frac{(-q;q)_\infty}{(q;q)_\infty}\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+2\sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2+n}}{(1-q^n)^2}\right)\\
&\equiv \frac{(-q;q)_\infty}{(q;q)_\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n+1}}{1-q^{2n+1}}+2\sum_{n=0}^{\infty}\frac{q^{4n+2}}{1-q^{4n+2}}+2\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{1-q^{2n}}\right)\\
&\equiv \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n+1}}{1-q^{2n+1}} \equiv \frac{1}{1+2T(-q)}(T(q)+T(q)^2)\\
&\equiv (1+2T(q))(T(q)+T(q)^2)\equiv 2T(q)^3+3T(q)^2+T(q).
\end{aligned}$$

Next, set

$$t=\sum_{n=1}^{\infty}q^{(7n+0)^2/7}=T(q^7), \qquad b=\sum_{n=-\infty}^{\infty}q^{(7n+2)^2/7},$$

$$a=\sum_{n=-\infty}^{\infty}q^{(7n+1)^2/7}, \qquad c=\sum_{n=-\infty}^{\infty}q^{(7n+3)^2/7},$$

and note that $T(q^{1/7})=t+a+b+c$. Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty}\overline{\mathrm{spt}}(n)q^{n/7} &\equiv 2T(q^{1/7})^3+3T(q^{1/7})^2+T(q^{1/7}) \pmod 4\\
&= 12abc+2t^3+3t^2+t\\
&\quad +6at^2+6at+a+6b^2t+3b^2+6bc^2\\
&\quad +6a^2t+3a^2+6ab^2+6ct^2+6ct+c\\
&\quad +2a^3+12act+6ac+6b^2c\\
&\quad +6a^2c+6bt^2+6bt+b+6c^2t+3c^2\\
&\quad +12abt+6ab+6ac^2+2b^3\\
&\quad +6a^2b+12bct+6bc+2c^3,
\end{aligned}$$

where the terms in the expansion have been collected according to the residue classes modulo 1 of the exponents on q. Taking the terms involving only integral powers of q gives

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{\mathrm{spt}}(7n)q^n &\equiv 12abc+2t^3+3t^2+t\\ &\equiv 2t^3+3t^2+t\equiv\sum_{n=1}^{\infty}\overline{\mathrm{spt}}(n)q^{7n} \pmod 4.\end{aligned}$$

This proves the congruence (1.20). Similarly, by (5.3), modulo 4,

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_\omega(n)q^n &= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+2\sum_{n=1}^{\infty}\frac{(-1)^nq^{2n(n+1)}}{(1-q^{2n})^2}\bigg)\\ &\equiv \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{n=0}^{\infty}\frac{(-1)^nq^{2n+1}}{1-q^{2n+1}}+2\sum_{n=0}^{\infty}\frac{(-1)^nq^{4n+2}}{1-q^{4n+2}}+2\bigg(\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{1-q^n}\bigg)^4\bigg)\\ &= \frac{1}{1+2T(-q^2)}\big(T(q)+T(q)^2+2T(q^2)+2T(q^2)^2+2(T(q)+T(q)^2)^4\big)\\ &\equiv (1+2T(q)^2)(T(q)+3T(q)^2+2T(q)^8)\\ &\equiv 2T(q)^8+2T(q)^4+2T(q)^3+3T(q)^2+T(q).\end{aligned}$$

This polynomial in $T(q)$ may be dissected in the same fashion, and we find that

$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_\omega(7n)q^n\equiv 2t^8+2t^4+2t^3+3t^2+t\equiv\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_\omega(n)q^{7n} \pmod 4,$$

which proves the congruence (1.21).

8. Concluding remarks and some open problems. In conclusion, we remark that the study involving the partition function $\overline{p}_\omega(n)$ and its associated smallest parts function $\overline{\mathrm{spt}}_\omega(n)$ is more difficult than the one involving $p_\omega(n)$ and $\mathrm{spt}_\omega(n)$. Nonetheless, these functions are as interesting as their aforementioned counterparts. This certainly merits further study of these functions. In particular, we give two open problems below. First, it is not difficult to relate (1.1) to indefinite theta functions. Using the following Bailey pair relative to (q,q):

$$\alpha_n=\frac{(1-q^{2n+1})q^{n^2}}{1-q}\sum_{j=-n}^{n}(-1)^jz^jq^{-j(j+1)/2},\qquad \beta_n=\frac{(z;q)_n(q/z;q)_n}{(q;q)_{2n}},$$

we apply the Bailey Lemma

$$\begin{aligned}&\sum_{n=0}^{\infty}(\rho_1,\rho_2;q)_n\bigg(\frac{q^2}{\rho_1\rho_2}\bigg)^n\beta_n\\ &\qquad=\frac{(q^2/\rho_1,q^2/\rho_2;q)_\infty}{(q^2,q^2/\rho_1\rho_2;q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n}{(q^2/\rho_1,q^2/\rho_2;q)_n}\bigg(\frac{q^2}{\rho_1\rho_2}\bigg)^n\alpha_n,\end{aligned}$$

and then with $\rho_1 = -\rho_2 = i\sqrt{q}$, we take $\frac{d}{dz}\big|_{z=1}$ on both sides of the resulting identity to obtain

$$\sum_{n=1}^{\infty} \frac{q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty} = \frac{-1}{(q;q)_\infty^3} \sum_{n=1}^{\infty} \sum_{|j|\leq n} j(-1)^j q^{n^2+n-j(j+1)/2} \frac{1-q^{2n+1}}{1+q^{2n+1}},$$

where the left hand side is the generating function of $\overline{p}_\omega(n)$ in (1.1).

Problem 1. Give the precise modular behavior of the generating function of $\overline{p}_\omega(n)$.

We have been able to obtain another proof of the mod 4 congruences in Theorem 1.4 assuming the conjecture that the function $Y(q)$ defined by

$$Y(q) := \sum_{n,m\geq 1} \frac{(-1)^m q^{2nm+m}}{(1+q^n)(1-q^{2m-1})} \tag{8.1}$$

is an odd function of q. Some coefficients in the expansion of $Y(q)$ are

$$\begin{aligned} Y(q) = & -q^3 - 2q^5 - 3q^7 - 5q^9 - 4q^{11} - 7q^{13} - 9q^{15} - \cdots \\ & - 53q^{91} - 62q^{93} - 38q^{95} - 55q^{97} - \cdots . \end{aligned}$$

Unfortunately we are unable to prove that indeed it is an odd function, hence we state it below as another open problem.

Problem 2. Prove that the function $Y(q)$ defined in (8.1) is an odd function of q.

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Abstract (will appear on the journal's web site only)

It was recently shown that $q\omega(q)$, where $\omega(q)$ is one of the third order mock theta functions, is the generating function of $p_\omega(n)$, the number of partitions of a positive integer n such that all odd parts are less than twice the smallest part. In this paper, we study the overpartition analogue of $p_\omega(n)$, and express its generating function in terms of a ${}_3\phi_2$ basic hypergeometric series and an infinite series involving little q-Jacobi polynomials. This is accomplished by obtaining a new seven-parameter q-series identity which generalizes a deep identity due to the first author as well as its generalization by R. P. Agarwal. We also derive two interesting congruences satisfied by the overpartition analogue, and some congruences satisfied by the associated smallest parts function.