NEW PATHWAYS AND CONNECTIONS IN NUMBER THEORY AND ANALYSIS MOTIVATED BY TWO INCORRECT CLAIMS OF RAMANUJAN

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In Memory of W. Keith Moore, Professor of Mathematics at Albion College
Dedicated to Pratibha Kulkarni who first showed me how beautiful Mathematics is
In Memory of Mohanlal S. Roy, Professor at Ramakrishna Mission Vidyamandira
In Memory of Professor Nicolae Popescu

Abstract. The focus of this paper commences with an examination of three (not obviously related) pages in Ramanujan's lost notebook, pages 336, 335, and 332, in decreasing order of attention. On page 336, Ramanujan proposes two identities, but the formulas are wrong – each is vitiated by divergent series. We concentrate on only one of the two incorrect "identities," which may have been devised to attack the extended divisor problem. We prove here a corrected version of Ramanujan's claim, which contains the convergent series appearing in it. The convergent series in Ramanujan's faulty claim is similar to one used by G. F. Voronoï, G. H. Hardy, and others in their study of the classical Dirichlet divisor problem. This now brings us to page 335, which comprises two formulas featuring doubly infinite series of Bessel functions, the first being conjoined with the classical circle problem initiated by Gauss, and the second being associated with the Dirichlet divisor problem. The first and fourth authors, along with Sun Kim, have written several papers providing proofs of these two difficult formulas in different interpretations. In this monograph, we return to these two formulas and examine them in more general settings.

The aforementioned convergent series in Ramanujan's "identity" is also similar to one that appears in a curious identity found in Chapter 15 in Ramanujan's second notebook, written in a more elegant, equivalent formulation on page 332 in the lost notebook. This formula may be regarded as a formula for $\zeta(\frac{1}{2})$, and in 1925, S. Wigert obtained a generalization giving a formula for $\zeta(\frac{1}{k})$ for any even integer $k \geq 2$. We extend the work of Ramanujan and Wigert in this paper.

The Voronoï summation formula appears prominently in our study. In particular, we generalize work of J. R. Wilton and derive an analogue involving the sum of divisors function $\sigma_s(n)$.

The modified Bessel functions $K_s(x)$ arise in several contexts, as do Lommel functions. We establish here new series and integral identities involving modified Bessel functions and modified Lommel functions. Among other results, we establish a modular transformation for an infinite series involving $\sigma_s(n)$ and modified Lommel functions. We also discuss certain obscure related work of N. S. Koshliakov. We define and discuss two new related classes of integral transforms, which we call Koshliakov transforms, because he first found elegant special cases of each.

2010 *Mathematics Subject Classification.* Primary 11M06, 33C10; Secondary 33E30, 11N37.

The first author's research was partially supported by NSA grant H98230-11-1-0200 and a Simons Foundation Collaboration Grant.

The second author is funded in part by the grant NSF-DMS 1112656 of Professor Victor H. Moll of Tulane University and sincerely thanks him for the support.

Contents

1. Introduction

The Dirichlet divisor problem is one of the most notoriously difficult unsolved problems in analytic number theory. Let $d(n)$ denote the number of divisors of n. Define the error term $\Delta(x)$, for $x > 0$, by

$$\sum_{n\leq x}{}' d(n) = x\log x + (2\gamma - 1)x + \frac{1}{4} + \Delta(x), \qquad (1.1)$$

where γ denotes Euler's constant. Here, and in the sequel, a prime $'$ on the summation sign in $\sum_{n\leq x}{}' a(n)$ indicates that only $\frac{1}{2}a(x)$ is counted when x is an integer. The Dirichlet divisor problem asks for the correct order of magnitude of $\Delta(x)$ as $x \to \infty$. At this writing, the best estimate $\Delta(x) = O(x^{131/416+\epsilon})$, for each $\epsilon > 0$, as $x \to \infty$, is due to M. N. Huxley [49] ($\frac{131}{416} = 0.3149\dots$). On the other hand, G. H. Hardy [45] proved that $\Delta(x) \neq O(x^{1/4})$, as $x \to \infty$, with the best result in this direction currently due to K. Soundararajan [76]. It is conjectured that $\Delta(x) = O\left(x^{1/4+\epsilon}\right)$, for each $\epsilon > 0$, as $x \to \infty$.

The conditionally convergent series

$$\frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{3}{4}}}\cos\left(4\pi\sqrt{nx}-\frac{\pi}{4}\right) \tag{1.2}$$

arose in G. F. Voronoï's [82, p. 218] work on the Dirichlet divisor problem, and its importance was further emphasized by Hardy [45, equation (6.32)]. Moreover, J. L. Hafner [44] and Soundararajan [76, equation (1.8)] in their improvements of Hardy's Ω-theorem on the Dirichlet divisor problem also crucially employed (1.2).

Let $\sigma_s(n)=\sum_{d|n}d^s$, and let $\zeta(s)$ denote the Riemann zeta function. For $0<s<1$, define $\Delta_{-s}(x)$ [1] by

$$\sum_{n\leq x}{}^{'}\sigma_{-s}(n)=\zeta(1+s)x+\frac{\zeta(1-s)}{1-s}x^{1-s}-\frac{1}{2}\zeta(s)+\Delta_{-s}(x). \tag{1.3}$$

The problem of determining the correct order of magnitude of the error term $\Delta_{-s}(x)$, as $x\to\infty$, is known as the extended divisor problem [58]. As $x\to\infty$, it is conjectured that for each $\epsilon>0$, $\Delta_{-s}(x)=O(x^{1/4-s/2+\epsilon})$ for $0<s\leq\frac{1}{2}$ and $\Delta_{-s}(x)=O(x^{\epsilon})$ for $\frac{1}{2}\leq s<1$.

In analogy with (1.2), the series

$$\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{\frac{5}{4}+\frac{k}{2}}}\sin\left(4\pi\sqrt{nx}-\frac{\pi}{4}\right), \tag{1.4}$$

for $|k|<\frac{3}{2}$, arises in work [73, p. 282], [52] related to a conjecture of S. Chowla and H. Walum [22], [23, pp. 1058–1063], which is another extension of the Dirichlet divisor problem. It is conjectured that if $a,r\in\mathbb{Z},a\geq 0,r\geq 1$, and if $B_r(x)$ denotes the r-th Bernoulli polynomial, then for every $\epsilon>0$, as $x\to\infty$,

$$\sum_{n\leq\sqrt{x}}n^aB_r\left(\left\{\frac{x}{n}\right\}\right)=O\left(x^{a/2+1/4+\epsilon}\right), \tag{1.5}$$

where $\{x\}$ denotes the fractional part of x. The conjectured correct order of magnitude in the Dirichlet divisor problem is equivalent to (1.5) with $a=0,r=1$.

Our last example is as famous as the Dirichlet divisor problem with which we opened this paper. Let $r_2(n)$ denote the number of representations of n as a sum of two squares. The equally celebrated circle problem asks for the precise order of magnitude of the error term $P(x)$, as $x\to\infty$, where

$$\sum_{n\leq x}{}^{'}r_2(n)=\pi x+P(x).$$

During the five years that Ramanujan visited Hardy at Cambridge, there is considerable evidence, from Hardy in his papers and from Ramanujan in his lost notebook [71], that the two frequently discussed both the circle and divisor problems. For details of Ramanujan's contributions to these problems, see either the first author's book with G. E. Andrews [4, Chapter 2] or the survey paper by the first author, S. Kim, and the last author [17].

It is possible that Ramanujan also thought of the extended divisor problem, for on page 336 in his lost notebook [71], we find the following claim.

[1] We use $\Delta_{-s}(x)$ instead of $\Delta_s(x)$, as is customarily used, so as to be consistent with the results in this paper, most of which require Re $s>0$.

Let $\sigma_s(n)=\sum_{d|n} d^s$, *and let* $\zeta(s)$ *denote the Riemann zeta function. Then*

$$\Gamma\left(s+\frac{1}{2}\right)\left\{\frac{\zeta(1-s)}{(s-\frac{1}{2})x^{s-\frac{1}{2}}}+\frac{\zeta(-s)\tan\frac{1}{2}\pi s}{2x^{s+\frac{1}{2}}}\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{2i}\left((x-in)^{-s-\frac{1}{2}}-(x+in)^{-s-\frac{1}{2}}\right)\right\}$$
$$=(2\pi)^s\left\{\frac{\zeta(1-s)}{2\sqrt{\pi x}}-2\pi\sqrt{\pi x}\zeta(-s)\tan\tfrac{1}{2}\pi s\right.$$
$$\left.+\sqrt{\pi}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right)\right\}. \quad (1.6)$$

In view of the identities for (1.2) and (1.4), it is possible that Ramanujan developed the series on the right-hand side of (1.6) to study a generalized divisor problem. Unfortunately, (1.6) is incorrect, since the series on the left-hand side, which can be written as

$$\sum_{n=1}^{\infty}\frac{\sigma_s(n)\sin\left(\left(s+\frac{1}{2}\right)\tan^{-1}\left(\frac{n}{x}\right)\right)}{(x^2+n^2)^{\frac{s}{2}+\frac{1}{4}}},$$

diverges for all real values of s since $\sigma_s(n)\geq n^s$. See [13] for further discussion. In this paper, we obtain a corrected version of Ramanujan's claim, where we start with the series on the right-hand side, since we know that it converges.

Before stating our version, we need to define a general hypergeometric function. Define the rising or shifted factorial $(a)_n$ by

$$(a)_n=a(a+1)(a+2)\cdots(a+n-1),\qquad n\geq 1,\qquad (a)_0=1. \quad (1.7)$$

Let p and q be non-negative integers, with $q\leq p+1$. Then, the generalized hypergeometric function ${}_qF_p$ is defined by

$${}_qF_p(a_1,a_2,\ldots,a_q;b_1,b_2,\ldots,b_p;z):=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_q)_n}{(b_1)_n(b_2)_n\cdots(b_p)_n}\frac{z^n}{n!}, \quad (1.8)$$

where $|z|<1$, if $q=p+1$, and $|z|<\infty$, if $q<p+1$.

We emphasize further notation. Throughout the paper, $s=\sigma+it$, with σ and t both real. We also set $R_a(f)=R_a$ to denote the residue of a meromorphic function $f(z)$ at a pole $z=a$.

Theorem 1.1. *Let* ${}_3F_2$ *be defined by* (1.8). *Fix* s *such that* $\sigma>0$. *Let* $x\in\mathbb{R}^+$. *Let* a *be the number defined by*

$$a=\begin{cases}0, & \text{if } s \text{ is an odd integer,}\\ 1, & \text{otherwise.}\end{cases} \quad (1.9)$$

Then,

$$\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right)$$
$$=4\pi\left(\frac{\zeta(1-s)}{8\pi^2\sqrt{x}}+\frac{1}{4\sqrt{2}\pi}\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{1}{2}-s\right)-\frac{2^{-s-3}}{\pi^{s+\frac{3}{2}}}\frac{\Gamma(s+1/2)\cot\left(\frac{\pi s}{2}\right)\zeta(-s)}{x^{s+\frac{1}{2}}}\right)$$

$$
\begin{aligned}
&+\frac{\sqrt{x}}{\pi^s}\Bigg\{\sum_{n<x}\frac{\sigma_s(n)}{n^{s+1}}\Bigg[-\frac{\sqrt{n}\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)}{\sqrt{2x}\Gamma\left(\frac{1}{4}-\frac{s}{2}\right)}\\
&-\frac{a\Gamma\left(s+\frac{1}{2}\right)\cot\left(\frac{\pi s}{2}\right)}{2^{s+1}\sqrt{\pi}}\left(\frac{n}{x}\right)^{s+1}\left\{\left(1+\frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)}+\left(1-\frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)}\right\}\\
&+\frac{n2^{-s}}{x\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)}{}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1-s}{2},1-\frac{s}{2}\end{matrix};-\frac{n^2}{x^2}\right)\Bigg]\\
&+\sum_{n\geq x}\frac{\sigma_s(n)}{n^{s+1}}\Bigg[-\frac{n\Gamma(s)\cos\left(\frac{\pi s}{2}\right)}{2^{s-1}\pi x}\left\{{}_3F_2\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2},1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{x^2}{n^2}\right)-1\right\}\\
&-\frac{i\sqrt{n}\Gamma\left(s+\frac{1}{2}\right)}{2^{s+1}\sqrt{\pi x}}\Bigg\{\sin\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left(\left(1+\frac{ix}{n}\right)^{-(s+\frac{1}{2})}-\left(1-\frac{ix}{n}\right)^{-(s+\frac{1}{2})}\right)\\
&+i\cos\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left(\left(1+\frac{ix}{n}\right)^{-(s+\frac{1}{2})}+\left(1-\frac{ix}{n}\right)^{-(s+\frac{1}{2})}-2\right)\Bigg\}\Bigg]\Bigg\},
\end{aligned}
\tag{1.10}
$$

where, if x is an integer, we additionally require that $\sigma<\frac{1}{2}$.

The following lemma, which is interesting in its own right, is the main ingredient of our proof. We use the notation $\int_{(c)}$ to designate $\int_{c-i\infty}^{c+i\infty}$.

Lemma 1.2. *Fix s such that $\sigma>0$. Fix $x\in\mathbb{R}^+$. Let $-1<\lambda<0$ and let a be defined by (1.9). Define $I(s,x)$ by*

$$
\begin{aligned}
I(s,x):=\frac{1}{2\pi i}\int_{(\lambda)}&\Gamma(z-1)\Gamma\left(1-\frac{z}{2}\right)\Gamma\left(1-\frac{z}{2}+s\right)\\
&\times\sin^2\left(\frac{\pi z}{4}\right)\sin\left(\frac{\pi z}{4}-\frac{\pi s}{2}\right)(4x)^{-\frac{1}{2}z}\,dz.
\end{aligned}
\tag{1.11}
$$

Then,
(i) *for $x>1$,*

$$
\begin{aligned}
I(s,x)=-\frac{\pi}{2^{2-s}}&\Bigg[\frac{\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)}{\sqrt{2x}\Gamma\left(\frac{1}{4}-\frac{s}{2}\right)}+\frac{ax^{-s-1}\cot\left(\frac{\pi s}{2}\right)}{2^{s+1}\sqrt{\pi}}\Gamma\left(s+\frac{1}{2}\right)\left\{\left(1+\frac{i}{x}\right)^{-\left(s+\frac{1}{2}\right)}\right.\\
&\left.+\left(1-\frac{i}{x}\right)^{-\left(s+\frac{1}{2}\right)}\right\}-\frac{1}{x2^s\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)}{}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1-s}{2};1-\frac{s}{2}\end{matrix};-\frac{1}{x^2}\right)\Bigg];
\end{aligned}
\tag{1.12}
$$

(ii) *for $x\leq1$,*

$$
\begin{aligned}
I(s,x)=-\frac{\pi}{2^{2-s}}\Bigg[&\frac{\Gamma(s)\cos\left(\frac{\pi s}{2}\right)}{2^{s-1}\pi x}\left\{{}_3F_2\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2},1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-x^2\right)-1\right\}\\
&+\frac{i\Gamma\left(s+\frac{1}{2}\right)}{2^{s+1}\sqrt{\pi x}}\left\{\sin\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left((1+ix)^{-(s+\frac{1}{2})}-(1-ix)^{-(s+\frac{1}{2})}\right)\right.\\
&\left.+i\cos\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left((1+ix)^{-(s+\frac{1}{2})}+(1-ix)^{-(s+\frac{1}{2})}-2\right)\right\}\Bigg],
\end{aligned}
\tag{1.13}
$$

where, if $x=1$, we additionally require that $\sigma<\frac{1}{2}$.

We note in passing that each ${}_3F_2$ in Theorem 1.1, as well as in Lemma 1.2, can be written, using the duplication formula for the Gamma function (see (2.4) below), as a sum of two ${}_2F_1$'s.

If we replace the '+' sign in the argument of the sine function in the series on the left-hand side of (1.10) by a '−' sign, then we obtain the following theorem.

Theorem 1.3. *Fix s such that $\sigma > 0$. Let $x \in \mathbb{R}^+$. Then,*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\sigma_s(n)}{\sqrt{n}} e^{-2\pi\sqrt{2nx}} \sin\left(\frac{\pi}{4} - 2\pi\sqrt{2nx}\right) \\
&= 4\pi \left(\frac{\sqrt{x}}{2}\zeta(-s) + \frac{\zeta\left(\frac{1}{2}\right)}{4\pi\sqrt{2}}\zeta\left(\frac{1}{2} - s\right) + \frac{\Gamma\left(s+\frac{1}{2}\right)\zeta(-s)}{2^{s+3}\pi^{s+\frac{3}{2}}x^{s+\frac{1}{2}}} \right) \\
&\quad + \frac{\Gamma\left(s+\frac{1}{2}\right)}{2^s\pi^{s+\frac{1}{2}}} \Bigg\{ \sum_{n<x} \frac{\sigma_s(n)}{n^{s+\frac{1}{2}}} \Bigg[-\sin\left(\frac{\pi}{4} - \frac{\pi s}{2}\right) + \frac{n^{s+\frac{1}{2}}}{2x^{s+\frac{1}{2}}} \\
&\qquad\qquad \times \left(\left(1 + \frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)} + \left(1 - \frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)} \right) \Bigg] \\
&\quad + \frac{1}{2}\sum_{n \geq x} \frac{\sigma_s(n)}{n^{s+\frac{1}{2}}} \Bigg[\cos\left(\frac{\pi}{4} + \frac{\pi s}{2}\right) \left(\left(1 + \frac{ix}{n}\right)^{-\left(s+\frac{1}{2}\right)} + \left(1 - \frac{ix}{n}\right)^{-\left(s+\frac{1}{2}\right)} - 2 \right) \\
&\quad + i\sin\left(\frac{\pi}{4} + \frac{\pi s}{2}\right) \left(\left(1 + \frac{ix}{n}\right)^{-\left(s+\frac{1}{2}\right)} - \left(1 - \frac{ix}{n}\right)^{-\left(s+\frac{1}{2}\right)} \right) \Bigg] \Bigg\}. \qquad (1.14)
\end{aligned}$$

On page 332 in his lost notebook [71], Ramanujan gives an elegant reformulation of a formula for $\zeta\left(\frac{1}{2}\right)$ that appears in Chapter 15 of his second notebook [70], [12, p. 314, Entry 8].

Let α and β be two positive numbers such that $\alpha\beta = 4\pi^3$. If $\phi(n)$, $n \geq 1$, and $\psi(n), n \geq 1$, are defined by

$$\sum_{j=1}^{\infty} \frac{x^{j^2}}{1 - x^{j^2}} = \sum_{n=1}^{\infty} \phi(n)x^n$$

and

$$\sum_{j=1}^{\infty} \frac{jx^{j^2}}{1 - x^{j^2}} = \sum_{n=1}^{\infty} \psi(n)x^n, \qquad (1.15)$$

respectively, then[2]

$$\sum_{n=1}^{\infty} \phi(n)e^{-n\alpha} = \frac{\pi^2}{6\alpha} + \frac{1}{4} + \frac{\sqrt{\beta}}{\pi\sqrt{2}} \left(\frac{1}{2\sqrt{2}}\zeta\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{\psi(n)}{\sqrt{n}} e^{-\sqrt{n\beta}} \sin\left(\frac{\pi}{4} - \sqrt{n\beta}\right) \right). \qquad (1.16)$$

Recall that [47, p. 340, Theorem 3.10]

$$\sum_{n=1}^{\infty} d(n)x^n = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}.$$

[2]Ramanujan inadvertently omitted the term $\frac{1}{4}$ on the right-hand side of (1.16).

A similar elementary argument shows that

$$\sum_{n=1}^{\infty} \sigma(n)x^n = \sum_{n=1}^{\infty} \frac{nx^n}{1-x^n}.$$

Hence, we see that $\phi(n)$ and $\psi(n)$ are analogues of $d(n)$ and $\sigma(n)$, respectively. The identity (1.16) was rediscovered by S. Wigert [84, p. 9], who actually gave a general formula for $\zeta\left(\frac{1}{k}\right)$ for each positive even integer k. See [4, pp. 191–193] for more details about (1.16).

We have included (1.16) here to demonstrate the similarity in the structure of the series on its right-hand side with the series on the left-hand sides of (1.10) and (1.14). One might therefore ask if other arithmetic functions, analogous to $\sigma_s(n)$ in (1.10) and $\psi(n)$ in (1.16), produce interesting series identities like those in (1.10) and (1.16).

The special case $s = \frac{1}{2}$ of Theorem 1.1 (see (5.15)) is very interesting, since the two sums, one over $n < x$ and the other over $n \geq x$, coalesce into a single infinite sum. If $K_s(x)$ denotes the modified Bessel function or the Macdonald function [83, p. 78] of order s, and if we use the identities [83, p. 80, equation (13)]

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \tag{1.17}$$

and [83, p. 79, equation (8)]

$$K_{-s}(z) = K_s(z), \tag{1.18}$$

we see that this special case of the series on the left-hand side of (1.10) can be realized as a special case of the series

$$2\sum_{n=1}^{\infty} \sigma_{-s}(n) n^{\frac{1}{2}s} \left(e^{\pi i s/4} K_s\left(4\pi e^{\pi i/4}\sqrt{nx}\right) - e^{-\pi i s/4} K_s\left(4\pi e^{-\pi i/4}\sqrt{nx}\right)\right) \tag{1.19}$$

when $s = -\frac{1}{2}$. If we replace the minus sign by a plus sign between the Bessel functions in the summands of (1.19), then the resulting series is a generalization of the series

$$\varphi(x) := 2\sum_{n=1}^{\infty} d(n) \left(K_0\left(4\pi e^{i\pi/4}\sqrt{nx}\right) + K_0\left(4\pi e^{-i\pi/4}\sqrt{nx}\right)\right), \tag{1.20}$$

extensively studied by N. S. Koshliakov (also spelled N. S. Koshlyakov) [53, 54, 55, 56]. See also [30] for properties of this series and some integral transformations involving it. The authors of this paper feel that Koshliakov's work has not earned the respect that it deserves in the mathematical community. Some of his best work was achieved under extreme hardship, as these excerpts from a paper written for the centenary of his birth clearly demonstrate [20].

> The repressions of the thirties which affected scholars in Leningrad continued even after the outbreak of the Second World War. In the winter of 1942 at the height of the blockade of Leningrad, Koshlyakov along with a group ... was arrested on fabricated ... dossiers and condemned to 10 years correctional hard labour. After the verdict he was exiled to one of the camps in the Urals. ... On the grounds of complete exhaustion and complicated pellagra, Koshlyakov was classified in the camp as an invalid and was not sent to do any of the usual jobs. ... very serious shortage of paper. He was forced to carry out calculations on a piece of plywood, periodically scraping off what he had written with a piece of glass. Nevertheless, between 1943 and 1944 Koshlyakov wrote two long memoirs ...

A natural question arises – what may have motivated Ramanujan to consider the series

$$\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right)? \tag{1.21}$$

We provide a plausible answer to this question in Section 6, demonstrating that (1.21) is related to a generalization of the famous Voronoï summation formula and also to the generalization of Koshliakov's series (1.20) discussed above and its analogue.

This paper is organized as follows. The preliminary results are given in Section 2. The proof of Theorem 1.1 appears in Section 3. We do not give a proof of Theorem 1.3 since it is similar to that of Theorem 1.1. Lemma 1.2, which is crucial in the proof of Theorem 1.1, is derived in Section 4. Special cases of Theorems 1.1 and 1.3 are examined in Section 5, and connections with modified Bessel functions are made. In Section 6, we relate (1.21) and Theorem 1.1 to Voronoï's formula for ${\sum_{n\leq x}}' d(n)$ and work of Hardy, Koshliakov, and A. Oppenheim. In the following section, we examine an analogue of the Voronoï summation formula with $d(n)$ replaced by $\sigma_s(n)$. The work of Ramanujan [71] and Wigert [84], evinced in (1.16), is extended in Section 10. On page 335 in his lost notebook [71], Ramanujan stated two beautiful identities connected, respectively, with the circle and divisor problems. We extend these identities in Sections 11–13. The linear combination of Bessel functions appearing in our representation for $\sum_{n\leq x}\sigma_{-s}(n)$ was remarkably shown by Koshliakov [57] to be the kernel of an integral transform for which the modified Bessel function $K_\nu(x)$ is self-reciprocal. We study these transforms in Section 15.

2. Preliminary Results

We recall below the functional equation, the reflection formula (along with a variant), and Legendre's duplication formula for the Gamma function $\Gamma(s)$. To that end,

$$\Gamma(s+1)=s\Gamma(s), \tag{2.1}$$

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin(\pi s)}, \tag{2.2}$$

$$\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right)=\frac{\pi}{\cos(\pi s)}, \tag{2.3}$$

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2s-1}}\Gamma(2s). \tag{2.4}$$

Throughout the paper, we shall need Stirling's formula for the Gamma function in a vertical strip [26, p. 224]. Thus, for $\sigma_1\leq\sigma\leq\sigma_2$, as $|t|\to\infty$,

$$|\Gamma(s)|=\sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}\left(1+O\left(\frac{1}{|t|}\right)\right). \tag{2.5}$$

The functional equation of the Riemann zeta function $\zeta(s)$ in its asymmetric form is given by [79, p. 24]

$$\zeta(1-s)=2^{1-s}\pi^{-s}\cos\left(\tfrac{1}{2}\pi s\right)\Gamma(s)\zeta(s), \tag{2.6}$$

whereas its symmetric form takes the shape

$$\pi^{-s/2}\Gamma\left(\tfrac{1}{2}s\right)\zeta(s)=\pi^{-(1-s)/2}\Gamma\left(\tfrac{1}{2}(1-s)\right)\zeta(1-s). \tag{2.7}$$

Since $\zeta(s)$ has a simple pole at $s=1$ with residue 1, i.e.,

$$\lim_{s\to 1}(s-1)\zeta(s)=1, \tag{2.8}$$

from (2.6) and (2.8), we find the value [79, p. 19]

$$\zeta(0) = -\tfrac{1}{2}.$$

The Riemann ξ-function $\xi(s)$ is defined by

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\tfrac{1}{2}s\right)\zeta(s), \tag{2.9}$$

where $\Gamma(s)$ and $\zeta(s)$ are the Gamma and the Riemann zeta functions respectively. The Riemann Ξ-function is defined by

$$\Xi(t) := \xi\left(\tfrac{1}{2}+it\right). \tag{2.10}$$

For $0 < c = \text{Re } w < \sigma$ [41, p. 908, formula **8.380.3**; p. 909, formula **8.384.1**],

$$\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(w)\Gamma(s-w)}{\Gamma(s)}x^{-w}\,dw = \frac{1}{(1+x)^s}. \tag{2.11}$$

We note Parseval's identity [68, pp. 82–83]

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathfrak{F}(1-w)\mathfrak{G}(w)\,dw, \tag{2.12}$$

where $\mathfrak{F}$ and $\mathfrak{G}$ are Mellin transforms of f and g, and which is valid for Re $w = c$ lying in the common strip of analyticity of $\mathfrak{F}(1-w)$ and $\mathfrak{G}(w)$. A variant of the above identity [68, p. 83, equation (3.1.13)] is

$$\frac{1}{2\pi i}\int_{(k)} \mathfrak{F}(w)\mathfrak{G}(w)t^{-w}\,dw = \int_0^\infty f(x)g\left(\frac{t}{x}\right)\frac{dx}{x}. \tag{2.13}$$

We close this section by recalling facts about Bessel functions. The ordinary Bessel function $J_\nu(z)$ of order ν is defined by [83, p. 40]

$$J_\nu(z) = \sum_{m=0}^\infty \frac{(-1)^m(z/2)^{2m+\nu}}{m!\Gamma(m+1+\nu)}, \quad |z| < \infty. \tag{2.14}$$

As customary, $Y_\nu(z)$ denotes the Bessel function of order ν of the second kind. Its relation to $J_\nu(z)$ is given in the identity [83, p. 64]

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin\pi\nu}. \tag{2.15}$$

and, as above, $K_\nu(z)$ denotes the modified Bessel function of order ν. The asymptotic formulas of the Bessel functions $J_\nu(z), Y_\nu(z)$, and $K_\nu(z)$, as $|z| \to \infty$, are given by [83, p. 199 and p. 202]

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(\cos w\sum_{n=0}^\infty \frac{(-1)^n(\nu,2n)}{(2z)^{2n}} - \sin w\sum_{n=0}^\infty \frac{(-1)^n(\nu,2n+1)}{(2z)^{2n+1}}\right), \tag{2.16}$$

$$Y_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(\sin w\sum_{n=0}^\infty \frac{(-1)^n(\nu,2n)}{(2z)^{2n}} + \cos w\sum_{n=0}^\infty \frac{(-1)^n(\nu,2n+1)}{(2z)^{2n+1}}\right), \tag{2.17}$$

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\sum_{n=0}^\infty \frac{(\nu,n)}{(2z)^n}, \tag{2.18}$$

for $|\arg z| < \pi$. Here $w = z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi$ and

$$(\nu,n) = \frac{\Gamma(\nu+n+1/2)}{\Gamma(n+1)\Gamma(\nu-n+1/2)}.$$

3. Proof of Theorem 1.1

Let

$$S(s,x) := \sum_{n=1}^{\infty} \frac{\sigma_s(n)}{\sqrt{n}} e^{-2\pi\sqrt{2nx}} \sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right). \tag{3.1}$$

From [64, p. 45, equations (5.19), (5.20)], we have

$$\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(z)}{(a^2+b^2)^{z/2}} \sin\left(z\tan^{-1}\left(\frac{a}{b}\right)\right) x^{-z}\, dz = e^{-bx}\sin(ax), \tag{3.2}$$

$$\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(z)}{(a^2+b^2)^{z/2}} \cos\left(z\tan^{-1}\left(\frac{a}{b}\right)\right) x^{-z}\, dz = e^{-bx}\cos(ax), \tag{3.3}$$

where $a, b > 0$, and Re $z > 0$ for (3.2) and Re $z > -1$ for (3.3). Let $a = b = 2\pi\sqrt{2n}$, replace x by $\sqrt{x}$, add (3.2) and (3.3), and then simplify, so that for $c =$ Re $z > 0$,

$$\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(z)}{(16\pi^2 n)^{z/2}} \sin\left(\frac{\pi(z+1)}{4}\right) x^{-z/2}\, dz = e^{-2\pi\sqrt{2nx}} \sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right). \tag{3.4}$$

Now replace z by $z-1$ in (3.4), so that for $c =$ Re $z > 1$,

$$\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(z-1)}{(4\pi)^{z-1} n^{z/2}} \sin\left(\frac{\pi z}{4}\right) x^{(1-z)/2}\, dz = \frac{e^{-2\pi\sqrt{2nx}}}{\sqrt{n}} \sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right). \tag{3.5}$$

Now substitute (3.5) in (3.1) and interchange the order of summation and integration to obtain

$$S(s,x) = \frac{2}{i}\int_{(c)} \left(\sum_{n=1}^{\infty} \frac{\sigma_s(n)}{n^{z/2}}\right) \frac{\Gamma(z-1)}{(4\pi)^z} \sin\left(\frac{\pi z}{4}\right) x^{(1-z)/2}\, dz. \tag{3.6}$$

It is well-known [79, p. 8, equation (1.3.1)] that for Re $\nu > 1$ and Re $\nu > 1+$ Re μ,

$$\zeta(\nu)\zeta(\nu-\mu) = \sum_{n=1}^{\infty} \frac{\sigma_\mu(n)}{n^\nu}. \tag{3.7}$$

Invoking (3.7) in (3.6), we see that

$$S(s,x) = \frac{2}{i}\int_{(c)} \Omega(z,s,x)\, dz, \tag{3.8}$$

where $c > 2\sigma + 2$ (since $\sigma > 0$) and

$$\Omega(z,s,x) := \zeta\left(\frac{z}{2}\right)\zeta\left(\frac{z}{2} - s\right) \frac{\Gamma(z-1)}{(4\pi)^z} \sin\left(\frac{\pi z}{4}\right) x^{(1-z)/2}. \tag{3.9}$$

We want to shift the line of integration from Re $z = c$ to Re $z = \lambda$, where $-1 < \lambda < 0$. Note that the integrand in (3.8) has poles at $z = 1$, 2, and $2s+2$. Consider the positively oriented rectangular contour formed by $[c-iT, c+iT]$, $[c+iT, \lambda+iT]$, $[\lambda+iT, \lambda-iT]$, and $[\lambda-iT, c-iT]$, where T is any positive real number. By Cauchy's residue theorem,

$$\begin{aligned}\frac{1}{2\pi i}&\left\{\int_{c-iT}^{c+iT} + \int_{c+iT}^{\lambda+iT} + \int_{\lambda+iT}^{\lambda-iT} + \int_{\lambda-iT}^{c-iT}\right\}\Omega(z,s,x)\, dz \\ &= R_1(\Omega) + R_2(\Omega) + R_{2s+2}(\Omega),\end{aligned} \tag{3.10}$$

where we recall that $R_a(f)$ denotes the residue of a function f at the pole $z=a$. The residues are now calculated. First,

$$\begin{aligned}R_{2s+2}(\Omega) &= \lim_{z\to 2s+2}(z-2s-2)\zeta\left(\frac{z}{2}-s\right)\zeta\left(\frac{z}{2}\right)\frac{\Gamma(z-1)}{(4\pi)^z}\sin\left(\frac{\pi z}{4}\right)x^{(1-z)/2}\\ &= 2\zeta(s+1)\frac{\Gamma(2s+1)}{(4\pi)^{2s+2}}\sin\left(\frac{\pi(2s+2)}{4}\right)x^{-s-\frac12}\\ &= -\frac{2^{-s-3}}{\pi^{s+\frac32}}\frac{\Gamma(s+\frac12)\cot\left(\frac12\pi s\right)\zeta(-s)}{x^{s+\frac12}}, \end{aligned}\tag{3.11}$$

where in the first step we used (2.8), and in the last step we employed (2.4) and (2.6) with s replaced by $s+1$. Second and third,

$$\begin{aligned}R_1(\Omega) &= \lim_{z\to 1}(z-1)\frac{\Gamma(z-1)}{(4\pi)^z}\zeta\left(\frac{z}{2}\right)\zeta\left(\frac{z}{2}-s\right)\sin\left(\frac{\pi z}{4}\right)x^{(1-z)/2}\\ &= \frac{1}{4\sqrt{2}\pi}\zeta\left(\frac12\right)\zeta\left(\frac12-s\right), \end{aligned}\tag{3.12}$$

$$\begin{aligned}R_2(\Omega) &= \lim_{z\to 2}(z-2)\zeta\left(\frac{z}{2}\right)\zeta\left(\frac{z}{2}-s\right)\frac{\Gamma(z-1)}{(4\pi)^z}\sin\left(\frac{\pi z}{4}\right)x^{(1-z)/2}\\ &= \frac{\zeta(1-s)}{8\pi^2\sqrt{x}}, \end{aligned}\tag{3.13}$$

where, in (3.12) we utilized (2.1), and in (3.13) we used (2.8). Next, we show that as $T\to\infty$, the integrals along the horizontal segments $[c+iT,\lambda+iT]$ and $[\lambda-iT,c-iT]$ tend to zero. To that end, note that if $s=\sigma+it$, for $\sigma\geq-\delta$ [79, p. 95, equation (5.1.1)],

$$\zeta(s)=O(t^{\frac32+\delta}).\tag{3.14}$$

Also, as $|t|\to\infty$,

$$\left|\sin\left(\frac{\pi s}{4}\right)\right| = \left|\frac{e^{\frac14 i\pi s}-e^{-\frac14 i\pi s}}{2i}\right| = O\left(e^{\frac14\pi|t|}\right).\tag{3.15}$$

Thus from (3.14), (2.5), and (3.15), we see that the integrals along the horizontal segments tend to zero as $T\to\infty$. Along with (3.10), this implies that

$$\begin{aligned}\int_{(c)}\Omega(z,s,x)\,dz &= \int_{(\lambda)}\Omega(z,s,x)\,dz\\ &+2\pi i\left(\frac{\zeta(1-s)}{8\pi^2\sqrt{x}}+\frac{1}{4\sqrt{2}\pi}\zeta\left(\frac12\right)\zeta\left(\frac12-s\right)-\frac{2^{-s-3}}{\pi^{s+\frac32}}\frac{\Gamma(s+1/2)\cot\left(\frac{\pi s}{2}\right)\zeta(-s)}{x^{s+\frac12}}\right).\end{aligned}\tag{3.16}$$

We now evaluate the integral along the vertical line $\mathrm{Re}\, z=\lambda$. Using (2.6) twice, we have

$$\begin{aligned}\int_{(\lambda)}\Omega(z,s,x)\,dz &= \int_{(\lambda)}2^{z-s}\pi^{z-s-2}\zeta\left(1-\frac{z}{2}\right)\zeta\left(1-\frac{z}{2}+s\right)\Gamma\left(1-\frac{z}{2}\right)\\ &\quad\times\Gamma\left(1-\frac{z}{2}+s\right)\frac{\Gamma(z-1)}{(4\pi)^z}\sin^2\left(\frac{\pi z}{4}\right)\sin\left(\frac{\pi z}{4}-\frac{\pi s}{2}\right)x^{(1-z)/2}\,dz\\ &= \frac{\sqrt{x}}{2^s\pi^{s+2}}\sum_{n=1}^\infty\frac{\sigma_s(n)}{n^{s+1}}\int_{(\lambda)}\Gamma(z-1)\Gamma\left(1-\frac{z}{2}\right)\Gamma\left(1-\frac{z}{2}+s\right)\end{aligned}$$

$$\times \sin^2\left(\frac{\pi z}{4}\right)\sin\left(\frac{\pi z}{4}-\frac{\pi s}{2}\right)\left(\frac{4x}{n}\right)^{-z/2} dz$$

$$= \frac{i\sqrt{x}}{2^{s-1}\pi^{s+1}}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s+1}} I\left(s, \frac{x}{n}\right), \tag{3.17}$$

where in the penultimate step we used (3.7), since $\lambda < 0$, and used the notation for $I(s,x)$ in the lemma. From (3.8), (3.16), and (3.17), we deduce that

$$S(s,x) = \frac{\sqrt{x}}{2^{s-2}\pi^{s+1}}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s+1}} I\left(s, \frac{x}{n}\right)$$
$$+ 4\pi\left(\frac{\zeta(1-s)}{8\pi^2\sqrt{x}} + \frac{1}{4\sqrt{2}\pi}\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{1}{2}-s\right) - \frac{2^{-s-3}}{\pi^{s+\frac{3}{2}}}\frac{\Gamma(s+1/2)\cot\left(\frac{1}{2}\pi s\right)\zeta(-s)}{x^{s+\frac{1}{2}}}\right).$$

The final result follows by substituting the expressions for $I\left(s, \frac{x}{n}\right)$ from the lemma, accordingly as $n < x$ or $n \geq x$. This completes the proof.

4. Proof of Lemma 1.2

Multiplying and dividing the integrand in (1.11) by $\Gamma\left(\frac{1}{2}(3-z)\right)$ and then applying (2.4) and (2.2), we see that

$$I(s,x) = -\frac{\pi^{\frac{3}{2}}}{4\pi i}\int_{(\lambda)}\frac{\sin^2\left(\frac{1}{4}\pi z\right)\sin\left(\frac{1}{4}\pi z-\frac{1}{2}\pi s\right)}{\sin \pi z}\frac{\Gamma\left(1-\frac{1}{2}z+s\right)}{\Gamma\left(1-\frac{1}{2}z+\frac{1}{2}\right)}x^{-\frac{1}{2}z}\, dz. \tag{4.1}$$

We now apply (2.2), (2.3), and (2.4) repeatedly to simplify the integrand in (4.1). This gives

$$I(s,x) = \frac{1}{2\pi i}\frac{-\pi}{2^{2-s}}\int_{(\lambda)} F(z,s,x)\, dz, \tag{4.2}$$

where

$$F(z,s,x) := \frac{\tan\left(\frac{1}{4}\pi z\right)}{2^{z/2}(1-z)}\frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)}x^{-z/2}. \tag{4.3}$$

The poles of $F(z,s,x)$ are at $z=1$, at $z = 2(2k+1+s)$, $k \in \mathbb{N}\cup\{0\}$, at $z = 2(2m+1)$, $m \in \mathbb{Z}$, and at $z = -(2j+1)$, $j \in \mathbb{N}\cup\{0\}$.

Case (i): When $x > 1$, we would like to move the vertical line of integration to $+\infty$. To that end, let $X > \lambda$ be such that the line $(X - i\infty, X + i\infty)$ does not pass through any poles of $F(z)$. Consider the positively oriented rectangular contour formed by $[\lambda - iT, X - iT]$, $[X - iT, X + iT]$, $[X + iT, \lambda + iT]$, and $[\lambda + iT, \lambda - iT]$, where T is any positive real number. Then by Cauchy's residue theorem,

$$\frac{1}{2\pi i}\left\{\int_{\lambda-iT}^{X-iT} + \int_{X-iT}^{X+iT} + \int_{X+iT}^{\lambda+iT} + \int_{\lambda+iT}^{\lambda-iT}\right\} F(z,s,x)\, dz$$
$$= R_1(F) + \sum_{0\leq k<\frac{1}{2}\left(\frac{1}{2}X-1-\mathrm{Re}\, s\right)} R_{2(2k+1+s)}(F) + \sum_{0\leq m<\frac{1}{2}\left(\frac{1}{2}X-1\right)} R_{2(2m+1)}(F).$$

We now calculate the residues. First,

$$R_1(F) = \lim_{z\to 1}(z-1)\frac{\tan\left(\frac{1}{4}\pi z\right)}{2^{\frac{z}{2}}(1-z)}\frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)}x^{-\frac{1}{2}z}$$

$$= -\frac{1}{\sqrt{2x}}\frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{4}-\frac{1}{2}s\right)}. \tag{4.4}$$

Second,

$$\begin{aligned}
&R_{2(2k+1+s)}(F)\\
&= \lim_{z\to 2(2k+1+s)}\{z-2(2k+1+s)\}\frac{\tan\left(\frac{1}{4}\pi z\right)}{2^{\frac{z}{2}}(1-z)}\frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)}x^{-\frac{1}{2}z}\\
&= \frac{4(-1)^{k+1}\cot\left(\frac{1}{2}\pi s\right)}{k!2^{2k+2+s}}\frac{\Gamma\left(\frac{1}{2}+2k+s\right)}{\Gamma\left(\frac{1}{2}(2k+1)\right)}x^{-(2k+1+s)}\\
&= \frac{(-1)^{k+1}\cot\left(\frac{1}{2}\pi s\right)}{(2k)!2^s\sqrt{\pi}}\Gamma\left(s+\frac{1}{2}\right)\left(s+\frac{1}{2}\right)_{2k}x^{-(2k+1+s)},
\end{aligned} \tag{4.5}$$

where in the second calculation, we used the fact $\lim_{z\to -n}(z+n)\Gamma(z)=(-1)^n/n!$, followed by (2.1) and (2.4). Here $(y)_n$ denotes the rising factorial defined in (1.7). Note that we do not have a pole at $2(2k+1+s)$ when s is an odd integer. Also,

$$\begin{aligned}
&R_{2(2m+1)}(F)\\
&= \lim_{z\to 2(2m+1)}\{z-2(2m+1)\}\frac{\tan\left(\frac{1}{4}\pi z\right)}{2^{z/2}(1-z)}\frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)}x^{-z/2}\\
&= \frac{1}{\pi 2^{2m}}\frac{\Gamma\left(\frac{1}{2}s-m\right)\Gamma\left(2m+\frac{1}{2}\right)}{\Gamma\left(m-\frac{1}{2}s+\frac{1}{2}\right)}x^{-(2m+1)}\\
&= \frac{(-1)^m}{2^s\sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s)}\frac{\left(\frac{1}{2}\right)_{2m}}{(1-s)_{2m}}x^{-(2m+1)},
\end{aligned} \tag{4.6}$$

where we used (2.2) and (2.4). As in the proof of Theorem 1.1, using Stirling's formula (2.5), we see that the integrals along the horizontal segments tend to zero as $T\to\infty$. Thus,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{(X)}F(z,s,x)\,dz &= \frac{1}{2\pi i}\int_{(\lambda)}F(z,s,x)\,dz \\
&+ R_1(F) + a\sum_{0\le k\le\frac{1}{2}\left(\frac{1}{2}X-1-\operatorname{Re}s\right)}R_{2(2k+1+s)}(F) + \sum_{0\le m<\frac{1}{2}\left(\frac{1}{2}X-1\right)}R_{2(2m+1)}(F),
\end{aligned} \tag{4.7}$$

where a is defined in (1.9). From (4.3), we see that

$$F(z+4,s,x) = -\frac{F(z,s,x)(z-1)\left(\frac{1}{2}(z+1)\right)\left(\frac{1}{2}(z+3)\right)}{4x^2(z+3)\left(\frac{1}{4}z-\frac{1}{2}(s-1)\right)\left(\frac{1}{4}z-\frac{1}{2}s\right)}, \tag{4.8}$$

so that

$$|F(z+4,s,x)| = \frac{|F(z,s,x)|}{x^2}\left(1+O_s\left(\frac{1}{|z|}\right)\right). \tag{4.9}$$

Applying (4.8) and (4.9) repeatedly, we find that

$$|F(z+4\ell,s,x)| = \frac{|F(z,s,x)|}{x^{2\ell}}\left(1+O_s\left(\frac{1}{|z|}\right)\right)^{\ell},$$

for any positive integer ℓ and $\operatorname{Re}z>0$. Therefore,

$$\left|\int_{(X+4\ell)}F(z,s,x)\,dz\right| = \left|\int_{(X)}\frac{F(z,s,x)}{x^{2\ell}}\left(1+O_s\left(\frac{1}{|z|}\right)\right)^{\ell}dz\right|$$

$$= \frac{1}{|x|^{2\ell}}\left(1+O_s\left(\frac{1}{|X|}\right)\right)^{\ell}\left|\int_{(X)}F(z,s,x)\,dz\right|. \tag{4.10}$$

Since $x>1$, we can choose X large enough so that

$$|x| > \sqrt{1+O_s\left(\frac{1}{|X|}\right)}.$$

With this choice of X and the fact that $\left|\int_{(X)}F(z,s,x)\,dz\right|$ is finite, if we let $\ell\to\infty$, then, from (4.10), we find that

$$\lim_{\ell\to\infty}\int_{X+4\ell-i\infty}^{X+4\ell+i\infty}F(z,s,x)\,dz = 0. \tag{4.11}$$

Hence, if we shift the vertical line (X) through the sequence of vertical lines $\{(X+4\ell)\}_{\ell=1}^{\infty}$, then, from (4.7) and (4.11), we arrive at

$$\frac{1}{2\pi i}\int_{(\lambda)}F(z,s,x)\,dz = -R_1(F) - a\sum_{k=0}^{\infty}R_{2(2k+1+s)}(F) - \sum_{m=0}^{\infty}R_{2(2m+1)}(F). \tag{4.12}$$

Since $x>1$, from (4.5) and the binomial theorem, we deduce that

$$\begin{aligned}a\sum_{k=0}^{\infty}R_{2(2k+1+s)}(F) &= -a\frac{x^{-s-1}\cot\left(\frac12\pi s\right)}{2^s\sqrt{\pi}}\Gamma\left(s+\frac12\right)\sum_{k=0}^{\infty}\frac{\left(s+\frac12\right)_{2k}}{(2k)!}\left(\frac{i}{x}\right)^{2k} \\ &= -a\frac{x^{-s-1}\cot\left(\frac12\pi s\right)}{2^{s+1}\sqrt{\pi}}\Gamma\left(s+\frac12\right)\left\{\left(1+\frac{i}{x}\right)^{-\left(s+\frac12\right)}+\left(1-\frac{i}{x}\right)^{-\left(s+\frac12\right)}\right\}.\end{aligned} \tag{4.13}$$

From (4.6),

$$\begin{aligned}\sum_{m=0}^{\infty}R_{2(2m+1)}(F) &= \frac{1}{x2^s\sin\left(\frac12\pi s\right)\Gamma(1-s)}\sum_{m=0}^{\infty}\frac{\left(\frac12\right)_{2m}}{(1-s)_{2m}}\left(\frac{i}{x}\right)^{2m} \\ &= \frac{1}{x2^s\sin\left(\frac12\pi s\right)\Gamma(1-s)}\,{}_3F_2\left(\begin{matrix}\frac14,\frac34,1\\ \frac12(1-s),1-\frac12 s\end{matrix};-\frac{1}{x^2}\right).\end{aligned} \tag{4.14}$$

Therefore from (4.4), (4.12), (4.13), and (4.14) we deduce that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(\lambda)}F(z,s,x)\,dz \\ &= a\frac{x^{-s-1}\cot\left(\frac12\pi s\right)}{2^{s+1}\sqrt{\pi}x}\Gamma\left(s+\frac12\right)\left\{\left(1+\frac{i}{x}\right)^{-\left(s+\frac12\right)}+\left(1-\frac{i}{x}\right)^{-\left(s+\frac12\right)}\right\} \\ &\quad-\frac{1}{x2^s\sin\left(\frac12\pi s\right)\Gamma(1-s)}\,{}_3F_2\left(\begin{matrix}\frac14,\frac34,1\\ \frac12(1-s),1-\frac12 s\end{matrix};-\frac{1}{x^2}\right)+\frac{1}{\sqrt{2x}}\frac{\Gamma\left(\frac14+\frac12 s\right)}{\Gamma\left(\frac14-\frac12 s\right)}.\end{aligned}$$

Using (4.2), we complete the proof of (1.12).

Case (ii): Now consider $x\le 1$. We would like to shift the line of integration all the way to $-\infty$. Let $X<\lambda$ be such that the line $[X-i\infty, X+i\infty]$ again does not pass through any pole of $F(z)$. Consider a positively oriented rectangular contour formed by

$[\lambda - iT, \lambda + iT], [\lambda + iT, X + iT]$, $[X + iT, X - iT]$, and $[X - iT, \lambda - iT]$, where T is any positive real number. Again, by Cauchy's residue theorem,

$$\begin{aligned}&\frac{1}{2\pi i}\left[\int_{\lambda-iT}^{\lambda+iT} + \int_{\lambda+iT}^{X+iT} + \int_{X+iT}^{X-iT} + \int_{X-iT}^{\lambda-iT}\right] F(z,s,x)\,dz \\ &= \sum_{0\leq k<\frac{1}{2}\left(-\frac{1}{2}X-1\right)} R_{-2(2k+1)}(F) + \sum_{0\leq j<\frac{1}{2}(-X-1)} R_{-(2j+1)}(F).\end{aligned}$$

The residues in this case are calculated below. First,

$$\begin{aligned}&R_{-2(2k+1)}(F) \\ &= \lim_{z\to-2(2k+1)} \left\{(z+2(2k+1))\tan\left(\frac{\pi z}{4}\right)\right\}\frac{1}{2^{z/2}}(1-z) \\ &\quad\times \frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)}x^{-\frac{1}{2}z} \\ &= \frac{(-1)^{k+1}}{\sqrt{\pi}2^s\sin\left(\frac{1}{2}\pi s\right)}\frac{\Gamma\left(\frac{1}{2}-2(k+1)\right)}{\Gamma\left(1-2(k+1)-s\right)}x^{(2k+1)} \\ &= \frac{(-1)^{k+1}}{\sqrt{\pi}2^s\sin\left(\frac{1}{2}\pi s\right)}\frac{\Gamma\left(\frac{1}{2}+2(k+1)\right)\Gamma\left(\frac{1}{2}-2(k+1)\right)}{\Gamma\left(2(k+1)+s\right)\Gamma\left(1-2(k+1)-s\right)}x^{(2k+1)}\frac{\Gamma\left(2(k+1)+s\right)}{\Gamma\left(\frac{1}{2}+2(k+1)\right)} \\ &= \frac{(-1)^{k+1}\cos\left(\frac{1}{2}\pi s\right)\Gamma(s)}{2^{s-1}\pi x}\frac{(s)_{2(k+1)}}{\left(\frac{1}{2}\right)_{2(k+1)}}x^{2(k+1)},\end{aligned} \tag{4.15}$$

where in the last step we used (2.2) and (2.3). Second,

$$\begin{aligned}&R_{-(2j+1)}(F) \\ &= \lim_{z\to-(2j+1)}(z+(2j+1))\frac{\tan\left(\frac{1}{4}\pi z\right)}{2^{z/2}(1-z)}\frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)}x^{-z/2} \\ &= -\frac{2^{j+\frac{1}{2}}}{(j+1)!}\frac{\Gamma\left(\frac{5}{4}+\frac{1}{2}j+\frac{1}{2}s\right)}{\Gamma\left(-\frac{1}{4}-\frac{1}{2}j-\frac{1}{2}s\right)}x^{j+\frac{1}{2}} \\ &= \frac{1}{\sqrt{\pi}2^s(j+1)!}\Gamma\left(s+\frac{3}{2}\right)\left(s+\frac{3}{2}\right)_j\sin\left(\pi\left(\frac{j}{2}+\frac{1}{4}+\frac{s}{2}\right)\right)x^{j+\frac{1}{2}},\end{aligned} \tag{4.16}$$

where we multiplied the numerator and denominator by $\Gamma\left(\frac{3}{4}+\frac{1}{2}j+\frac{1}{2}s\right)$ in the last step and then used (2.2) and (2.4). Thus, by (4.15) and (4.16),

$$\begin{aligned}\frac{1}{2\pi i}\int_{(\lambda)} F(z,s,x)\,dz &= \frac{1}{2\pi i}\int_{(X)} F(z,s,x)\,dz \\ &\quad + \sum_{0\leq k\leq\frac{1}{2}\left(-\frac{1}{2}X-1\right)} R_{-2(2k+1)}(F) + \sum_{0\leq k\leq\frac{1}{2}(-X-1)} R_{-(2k+1)}(F).\end{aligned} \tag{4.17}$$

From (4.8),

$$|F(z-4,s,x)| = |x|^2\left(1+O_s\left(\frac{1}{|z|}\right)\right)|F(z,s,x)|,$$

and hence

$$|F(z-4\ell,s,x)| = |x|^{2\ell}\left(1+O_s\left(\frac{1}{|z|}\right)\right)^{\ell}|F(z,s,x)|, \tag{4.18}$$

for any positive integer ℓ and $\mathrm{Re}\, z<0$. Therefore, from (4.18),

$$\begin{aligned}\left|\int_{(X-4k)} F(z,s,x)\,dz\right| &= \left|\int_{(X)} F(z,s,x)x^{2\ell}\left(1+O_s\left(\frac{1}{|z|}\right)\right)^{\ell} dz\right|\\ &= |x|^{2\ell}\left(1+O_s\left(\frac{1}{|X|}\right)\right)^{\ell}\left|\int_{(X)} F(z,s,x)\,dz\right|.\end{aligned}$$

Since $x<1$, we can find an $X<\lambda$, with $|X|$ sufficiently large, so that

$$x^2\left(1+O_s\left(\frac{1}{|X|}\right)\right)<1. \tag{4.19}$$

With the given choice of X and the fact that $\left|\int_{(X)} F(z,s,x)\,dz\right|$ is finite, upon letting $\ell\to\infty$ and using (4.19), we find that

$$\lim_{\ell\to\infty}\int_{X-4\ell-i\infty}^{X-4\ell+i\infty} F(z,s,x)\,dz=0. \tag{4.20}$$

Thus if we shift the line of integration (X) to $-\infty$ through the sequence of vertical lines $\{(X-4k)\}_{k=1}^{\infty}$, from (4.17) and (4.20), we arrive at

$$\frac{1}{2\pi i}\int_{(\lambda)} F(z,s,x)\,dz=\sum_{k=0}^{\infty}R_{-2(2k+1)}(F)+\sum_{j=0}^{\infty}R_{-(2j+1)}(F). \tag{4.21}$$

Since $x\leq 1$, using (4.15), we find that

$$\begin{aligned}\sum_{k=0}^{\infty}R_{-2(2k+1)}(F) &= \frac{\Gamma(s)\cos\left(\frac{1}{2}\pi s\right)}{2^{s-1}\pi x}\sum_{k=0}^{\infty}\frac{(s)_{2(k+1)}}{(1/2)_{2(k+1)}}(ix)^{2(k+1)}\\ &= \frac{\Gamma(s)\cos\left(\frac{1}{2}\pi s\right)}{2^{s-1}\pi x}\left\{{}_3F_2\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2},1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-x^2\right)-1\right\},\end{aligned} \tag{4.22}$$

where for $x=1$, we additionally require that $\sigma<\frac{1}{2}$ in order to ensure the conditional convergence of the ${}_3F_2$ [3, p. 62].

From (4.16),

$$\begin{aligned}\sum_{j=0}^{\infty}R_{-(2j+1)}(F) &= \frac{\Gamma\left(s+\frac{3}{2}\right)}{2^s\sqrt{\pi}}\sum_{j=0}^{\infty}\sin\left(\pi\left(\frac{j}{2}+\frac{1}{4}+\frac{s}{2}\right)\right)\frac{\left(s+\frac{3}{2}\right)_j}{(j+1)!}x^{(j+\frac{1}{2})} \qquad (4.23)\\ &= \frac{\Gamma\left(s+\frac{3}{2}\right)}{2^s\sqrt{\pi}}\left\{\sqrt{x}\sin\left(\pi\left(\frac{1}{4}+\frac{s}{2}\right)\right)\sum_{j=0}^{\infty}\frac{\left(s+\frac{3}{2}\right)_{2j}}{(2j+1)!}(ix)^{2j}\right.\\ &\qquad\left.+x^{3/2}\cos\left(\pi\left(\frac{1}{4}+\frac{s}{2}\right)\right)\sum_{j=0}^{\infty}\frac{\left(s+\frac{3}{2}\right)_{2j+1}}{(2j+2)!}(ix)^{2j}\right\}\\ &= \frac{i\Gamma\left(s+\frac{1}{2}\right)}{2^{s+1}\sqrt{\pi x}}\left[\sin\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left\{(1+ix)^{-(s+\frac{1}{2})}-(1-ix)^{-(s+\frac{1}{2})}\right\}\right.\\ &\qquad\left.+i\cos\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left\{(1+ix)^{-(s+\frac{1}{2})}+(1-ix)^{-(s+\frac{1}{2})}-2\right\}\right],\end{aligned}$$

where in the last step we used the identities

$$\sum_{j=0}^{\infty}\frac{(a)_{2j}x^{2j}}{(2j+1)!}=\frac{(1+x)^{1-a}-(1-x)^{1-a}}{2x(1-a)},$$

$$\sum_{j=0}^{\infty}\frac{(a)_{2j+1}x^{2j+1}}{(2j+2)!}=\frac{-\left((1+x)^{1-a}+(1-x)^{1-a}-2\right)}{2x(1-a)},$$

valid for $|x|<1$. Combining (4.21), (4.22), and (4.23), we deduce that

$$\begin{aligned}\frac{1}{2\pi i}\int_{(\lambda)}F(z,s,x)\,dz&=\frac{\cos\left(\frac{\pi s}{2}\right)\Gamma(s)}{2^{s-1}\pi x}\left\{{}_3F_2\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2},1\\ \frac14,\frac34\end{matrix};-x^2\right)-1\right\}\\&+\frac{i\Gamma\left(s+\frac12\right)}{2^{s+1}\sqrt{\pi x}}\left[\sin\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left\{(1+ix)^{-(s+\frac12)}-(1-ix)^{-(s+\frac12)}\right\}\right.\\&\left.+\,i\cos\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left\{(1+ix)^{-(s+\frac12)}+(1-ix)^{-(s+\frac12)}-2\right\}\right].\end{aligned}$$

Using (4.2), we see that this proves (1.13). This completes the proof of Lemma 1.2.

If x is an integer in Theorem 1.1, then the term corresponding to it on the right-hand side of (1.10) can be included either in the first (finite) sum or in the second (infinite) sum. This follows from the fact that the integral $I(s,x)$ in the lemma above is continuous at $x=1$. Though elementary, we warn readers that it is fairly tedious to verify this by showing that the right-hand sides of (1.12) and (1.13) are equal when $x=1$, and requires the following transformation between ${}_3F_2$ hypergeometric functions, which is actually the special case $q=2$ of a general connection formula between ${}_pF_q$'s [66, p. 410, formula **16.8.8**].

Theorem 4.1. *For $a_1-a_2, a_1-a_3, a_2-a_3\notin\mathbb{Z}$, and $z\notin(0,1)$,*

$$\begin{aligned}{}_3F_2(a_1,a_2,a_3;b_1,b_2;z)&=\frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\left(\frac{\Gamma(a_1)\Gamma(a_2-a_1)\Gamma(a_3-a_1)}{\Gamma(b_1-a_1)\Gamma(b_2-a_1)}(-z)^{-a_1}\right.\\&\times{}_3F_2\left(a_1,a_1-b_1+1,a_1-b_2+1;a_1-a_2+1,a_1-a_3+1;\frac1z\right)\\&+\frac{\Gamma(a_2)\Gamma(a_1-a_2)\Gamma(a_3-a_2)}{\Gamma(b_1-a_2)\Gamma(b_2-a_2)}(-z)^{-a_2}\\&\times{}_3F_2\left(a_2,a_2-b_1+1,a_2-b_2+1;-a_1+a_2+1,a_2-a_3+1;\frac1z\right)\\&+\frac{\Gamma(a_3)\Gamma(a_1-a_3)\Gamma(a_2-a_3)}{\Gamma(b_1-a_3)\Gamma(b_2-a_3)}(-z)^{-a_3}\\&\left.\times{}_3F_2\left(a_3,a_3-b_1+1,a_3-b_2+1;-a_1+a_3+1,-a_2+a_3+1;\frac1z\right)\right).\end{aligned}\tag{4.24}$$

5. Coalescence

In the proofs of Theorems 1.1 and 1.3 using contour integration, the convergence of the series of residues of the corresponding functions necessitates the consideration of two sums – one over $n<x$ and the other over $n\geq x$. However, for some special values of s, namely $s=2m+\frac12$, where m is a non-negative integer, the two sums over $n<x$ and $n\geq x$ coalesce into a single infinite sum. This section contains corollaries of these theorems when s takes these special values.

Theorem 5.1. *Let $x \notin \mathbb{Z}$. Then, for any non-negative integer m,*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\sigma_{2m+\frac{1}{2}}(n)}{\sqrt{n}} e^{-2\pi\sqrt{2nx}} \sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right) \\
&= \frac{\zeta\left(\frac{1}{2} - 2m\right)}{2\pi\sqrt{x}} - \frac{(2m)!\zeta\left(-\frac{1}{2} - 2m\right)}{\sqrt{2}(2\pi x)^{2m+1}} + \frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(-2m) \\
&\quad + \frac{\sqrt{x}}{\pi^{2m+\frac{1}{2}}} \sum_{n=1}^{\infty} \frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \Bigg[- \frac{(2m)!}{\sqrt{\pi}} \left(\frac{n}{2x}\right)^{2m+\frac{3}{2}} \\
&\quad \times \left\{ \left(1 + \frac{in}{x}\right)^{-(2m+1)} + \left(1 - \frac{in}{x}\right)^{-(2m+1)} \right\} \\
&\quad + \frac{(-1)^m n}{2^{2m}\pi x} \Gamma\left(2m + \frac{1}{2}\right) {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1 \\ \frac{1}{4} - m, \frac{3}{4} - m \end{matrix} ; -\frac{n^2}{x^2} \right) \Bigg].
\end{aligned} \tag{5.1}$$

Proof. Let $s = 2m + \frac{1}{2}$, $m \geq 0$, in Theorem 1.1. To examine the summands in the sum over $n < x$, observe first that $1/\Gamma\left(\frac{1}{4} - \frac{1}{2}s\right) = 0$. Since $a = 1$, the second expression in the summands is given by

$$\begin{aligned}
&- \frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{a\Gamma\left(s + \frac{1}{2}\right) \cot\left(\frac{\pi s}{2}\right)}{2^{s+1}\sqrt{\pi}} \left(\frac{n}{x}\right)^{s+1} \left\{ \left(1 + \frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)} + \left(1 - \frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)} \right\} \\
&= - \frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{(2m)!}{\sqrt{\pi}} \left(\frac{n}{2x}\right)^{2m+\frac{3}{2}} \frac{\left(1 - \frac{in}{x}\right)^{2m+1} + \left(1 + \frac{in}{x}\right)^{2m+1}}{(1 + n^2/x^2)^{2m+1}} \\
&= - \frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{(2m)!}{\sqrt{\pi}} \frac{n^{2m+\frac{3}{2}} x^{2m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}(x^2 + n^2)^{2m+1}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k} \left(\frac{n}{x}\right)^{2k}.
\end{aligned} \tag{5.2}$$

The third expressions in the summands become

$$\begin{aligned}
&\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{n2^{-s}}{x \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s)} {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1 \\ \frac{1-s}{2}, 1 - \frac{s}{2} \end{matrix} ; -\frac{n^2}{x^2} \right) \\
&= \frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{(-1)^m n 2^{-2m}}{x\Gamma\left(\frac{1}{2} - 2m\right)} {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1 \\ \frac{1}{4} - m, \frac{3}{4} - m \end{matrix} ; -\frac{n^2}{x^2} \right).
\end{aligned} \tag{5.3}$$

Hence, by (5.2) and (5.3), the summands over $n < x$ are given by

$$\begin{aligned}
&\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \Bigg\{ - \frac{(2m)!}{\sqrt{\pi}} \frac{n^{2m+\frac{3}{2}} x^{2m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}(x^2 + n^2)^{2m+1}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k} \left(\frac{n}{x}\right)^{2k} \\
&\qquad + \frac{(-1)^m n 2^{-2m}}{x\Gamma\left(\frac{1}{2} - 2m\right)} {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1 \\ \frac{1}{4} - m, \frac{3}{4} - m \end{matrix} ; -\frac{n^2}{x^2} \right) \Bigg\}.
\end{aligned} \tag{5.4}$$

For the summands over $n > x$, observe that the third expression is equal to zero, since $\cos\left(\frac{1}{4}\pi + \frac{1}{2}\pi\left(2m + \frac{1}{2}\right)\right) = 0$. The first expression becomes

$$- \frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{n\Gamma(s)\cos\left(\frac{1}{2}\pi s\right)}{2^{s-1}\pi x} \left\{ {}_3F_2\left(\begin{matrix} \frac{s}{2}, \frac{1+s}{2}, 1 \\ \frac{1}{4}, \frac{3}{4} \end{matrix} ; -\frac{x^2}{n^2} \right) - 1 \right\}$$

$$= \frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{(-1)^{m+1} n 2^{-2m}}{x\Gamma\left(\frac{1}{2}-2m\right)} \left\{ {}_3F_2\left(\begin{matrix} \frac{1}{4}+m, \frac{3}{4}+m, 1 \\ \frac{1}{4}, \frac{3}{4} \end{matrix} ; -\frac{x^2}{n^2} \right) - 1 \right\}, \tag{5.5}$$

where we used (2.3) with $s = 2m$. The second expressions of the summands become

$$\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{i(-1)^{m+1}\sqrt{n}(2m)!}{2^{2m+\frac{3}{2}}\sqrt{\pi x}} \frac{(1-ix/n)^{2m+1} - (1+ix/n)^{2m+1}}{(1+x^2/n^2)^{2m+1}}. \tag{5.6}$$

Note that

$$\left(1-\frac{ix}{n}\right)^{2m+1} - \left(1+\frac{ix}{n}\right)^{2m+1} = -\sum_{k=0}^{2m+1} \binom{2m+1}{k} \left(\frac{ix}{n}\right)^k \left(1+(-1)^{2m+1-k}\right).$$

These summands are non-zero only when k is odd, and so if we let $2j = 2m+1-k$, we see that

$$\left(1-\frac{ix}{n}\right)^{2m+1} - \left(1+\frac{ix}{n}\right)^{2m+1} = 2i(-1)^{m+1}\left(\frac{x}{n}\right)^{2m+1} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j} \left(\frac{n}{x}\right)^{2j}.$$

Thus, after simplification, the second expressions (5.6) equal

$$-\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{(2m)!}{\sqrt{\pi}} \frac{n^{2m+\frac{3}{2}} x^{2m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}(x^2+n^2)^{2m+1}} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j} \left(\frac{n}{x}\right)^{2j}. \tag{5.7}$$

Thus, by (5.5) and (5.7), the summands over $n > x$ equal

$$\begin{aligned} &\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \Bigg[-\frac{(2m)!}{\sqrt{\pi}} \frac{n^{2m+\frac{3}{2}} x^{2m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}(x^2+n^2)^{2m+1}} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j} \left(\frac{n}{x}\right)^{2j} \\ &\quad + \frac{(-1)^{m+1} n 2^{-2m}}{x\Gamma\left(\frac{1}{2}-2m\right)} \left\{ {}_3F_2\left(\begin{matrix} \frac{1}{4}+m, \frac{3}{4}+m, 1 \\ \frac{1}{4}, \frac{3}{4} \end{matrix} ; -\frac{x^2}{n^2} \right) - 1 \right\} \Bigg]. \end{aligned} \tag{5.8}$$

From (5.4) and (5.8), it is clear that we want to prove that

$${}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1 \\ \frac{1}{4}-m, \frac{3}{4}-m \end{matrix} ; -\frac{n^2}{x^2} \right) + {}_3F_2\left(\begin{matrix} \frac{1}{4}+m, \frac{3}{4}+m, 1 \\ \frac{1}{4}, \frac{3}{4} \end{matrix} ; -\frac{x^2}{n^2} \right) = 1, \tag{5.9}$$

for $x > 0$ and $n \in \mathbb{N}$. To that end, use (4.24) with $a_1 = \frac{1}{4}, a_2 = \frac{3}{4}, a_3 = 1, b_1 = \frac{1}{4} - m, b_2 = \frac{3}{4} - m$, and $z = -n^2/x^2$. This gives, for all $x, n > 0$,

$${}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1 \\ \frac{1}{4}-m, \frac{3}{4}-m \end{matrix} ; -\frac{n^2}{x^2} \right) = \frac{(4m+3)(4m+1)x^2}{3n^2} {}_3F_2\left(\begin{matrix} \frac{7}{4}+m, \frac{5}{4}+m, 1 \\ \frac{7}{4}, \frac{5}{4} \end{matrix} ; -\frac{x^2}{n^2} \right). \tag{5.10}$$

Now for $n > x$, we can use the series representation (1.8) for ${}_3F_2$ on the right-hand side to obtain

$$\begin{aligned} {}_3F_2\left(\begin{matrix} \frac{7}{4}+m, \frac{5}{4}+m, 1 \\ \frac{7}{4}, \frac{5}{4} \end{matrix} ; -\frac{x^2}{n^2} \right) &= 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{7}{4}+m\right)_k \left(\frac{5}{4}+m\right)_k (1)_k}{\left(\frac{7}{4}\right)_k \left(\frac{5}{4}\right)_k k!} \left(-\frac{x^2}{n^2}\right)^k \\ &= 1 - \frac{3n^2}{(4m+3)(4m+1)x^2} \sum_{k=1}^{\infty} \frac{\left(\frac{3}{4}+m\right)_{k+1} \left(\frac{1}{4}+m\right)_{k+1} (1)_{k+1}}{\left(\frac{3}{4}\right)_{k+1} \left(\frac{1}{4}\right)_{k+1} (k+1)!} \left(-\frac{x^2}{n^2}\right)^{k+1} \\ &= \frac{-3n^2}{(4m+3)(4m+1)x^2} \left\{ {}_3F_2\left(\begin{matrix} \frac{1}{4}+m, \frac{3}{4}+m, 1 \\ \frac{1}{4}, \frac{3}{4} \end{matrix} ; -\frac{x^2}{n^2} \right) - 1 \right\}. \end{aligned} \tag{5.11}$$

Combining (5.10) and (5.11), we obtain (5.9) for $n > x$.

Now set $a_1 = \frac{1}{4} + m$, $a_2 = \frac{3}{4} + m$, $a_3 = 1$, $b_1 = \frac{1}{4}$, $b_2 = \frac{3}{4}$, and $z = -x^2/n^2$ in (4.24) and use, for $n < x$, the series representation for the ${}_3F_2$ on the right-hand side of the resulting identity to arrive at (5.9) for $n < x$. This shows that (5.9) holds for all $x > 0$ and $n \in \mathbb{N}$.

Hence, the summands in the sums over $n < x$ and $n > x$ in Theorem 1.1 are the same when $s = 2m + \frac{1}{2}$. Now slightly rewrite (5.4) to finish the proof of Theorem 5.1. □

Similarly, when $s = 2m + \frac{1}{2}$ in Theorem 1.3, we obtain the following.

Theorem 5.2. *For any non-negative integer m,*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\sigma_{2m+\frac{1}{2}}(n)}{\sqrt{n}} e^{-2\pi\sqrt{2nx}} \sin\left(\frac{\pi}{4} - 2\pi\sqrt{2nx}\right) \\
&= \left(2\pi\sqrt{x} + \frac{(2m)!}{\sqrt{2}(2\pi x)^{2m+1}}\right) \zeta\left(-\frac{1}{2} - 2m\right) + \frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(-2m) \\
&\quad + \frac{\sqrt{\pi}(2m)!}{(2\pi)^{2m+\frac{3}{2}}} \sum_{n=1}^{\infty} \sigma_{2m+\frac{1}{2}}(n) \left\{(x-in)^{-(2m+1)} + (x+in)^{-(2m+1)}\right\}. \qquad (5.12)
\end{aligned}$$

Notice the resemblance of the series on the right-hand side of (5.12) with the divergent series in Ramanujan's incorrect "identity" (1.6). Since the series on the right side above has a + sign between the two binomial expressions in the summands, the order of n in the summand is at least $-\frac{3}{2} + \epsilon$, for each $\epsilon > 0$, unlike $-\frac{1}{2} + \epsilon$ in Ramanujan's series, because of which the latter is divergent.

When $m \geq 1$, we can omit the term $\frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(-2m)$ from both (5.1) and (5.12) since $\zeta(-2m) = 0$.

In Theorem 5.1, we assume $x \notin \mathbb{Z}$, whereas there is no such restriction in Theorem 5.2, because Theorems 1.1 and 5.1 involve ${}_3F_2$'s that are conditionally convergent, with the restriction $\sigma < \frac{1}{2}$ when x is an integer. Thus, the condition $\sigma \geq \frac{1}{2}$ implies that $x \notin \mathbb{Z}$, which is the case when $s = 2m + \frac{1}{2}$ for $m \geq 0$. However, ${}_3F_2$'s do not appear in Theorem 1.3, and so the restriction on x (other than the requirement $x > 0$) is not needed.

Adding (5.1) and (5.12) and simplifying gives the next theorem.

Theorem 5.3. *For $x \notin \mathbb{Z}$,*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\sigma_{2m+\frac{1}{2}}(n)}{\sqrt{n}} e^{-2\pi\sqrt{2nx}} \cos\left(2\pi\sqrt{2nx}\right) \\
&= \frac{1}{2\pi\sqrt{2x}}\zeta\left(\frac{1}{2} - 2m\right) + \pi\sqrt{2x}\zeta\left(\frac{-1}{2} - 2m\right) + \zeta\left(\frac{1}{2}\right)\zeta(-2m) \\
&\quad + \frac{(-1)^m}{\pi\sqrt{x}(2\pi)^{2m+\frac{1}{2}}}\Gamma\left(2m + \frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{1}{2}}} {}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{3}{4}, 1 \\ \frac{1}{4} - m, \frac{3}{4} - m\end{matrix}; -\frac{n^2}{x^2}\right). \qquad (5.13)
\end{aligned}$$

Subtracting (5.1) from (5.12) and simplifying leads to the next result.

Theorem 5.4. *For $x \notin \mathbb{Z}$,*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\sigma_{2m+\frac{1}{2}}(n)}{\sqrt{n}} e^{-2\pi\sqrt{2nx}} \sin\left(2\pi\sqrt{2nx}\right) \\
&= \frac{\zeta\left(\frac{1}{2} - 2m\right)}{2\pi\sqrt{2x}} - \sqrt{2}\left(\pi\sqrt{x} + \frac{(2m)!}{\sqrt{2}(2\pi x)^{2m+1}}\right)\zeta\left(\frac{-1}{2} - 2m\right)
\end{aligned}$$

$$+\frac{\sqrt{x}}{\sqrt{2}\pi^{2m+\frac{1}{2}}}\sum_{n=1}^{\infty}\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}}\Bigg[-2\frac{(2m)!}{\sqrt{\pi}}\left(\frac{n}{2x}\right)^{2m+\frac{3}{2}}$$
$$\times\left\{\left(1+\frac{in}{x}\right)^{-(2m+1)}+\left(1-\frac{in}{x}\right)^{-(2m+1)}\right\}$$
$$+\frac{(-1)^m n}{2^{2m}\pi x}\Gamma\left(2m+\frac{1}{2}\right){}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1}{4}-m,\frac{3}{4}-m\end{matrix};-\frac{n^2}{x^2}\right)\Bigg]. \tag{5.14}$$

In Theorem 5.1, as well as in (5.13) and (5.14), we should be careful while interpreting the ${}_3F_2$-function. For example, if $n<x$, then it can be expanded as a series. Otherwise, for $n>x$, the ${}_3F_2$-function represents the analytic continuation of the series. Of course, when $n>x$, one can replace the ${}_3F_2$-function by

$$-\left\{{}_3F_2\left(\begin{matrix}\frac{1}{4}+m,\frac{3}{4}+m,1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{x^2}{n^2}\right)-1\right\},$$

as can be seen from (5.9), and then use the series expansion of this other ${}_3F_2$-function.

5.1. **The Case $m=0$.** When $m=0$ in Theorem 5.1, we obtain the following corollary.

Corollary 5.5. *Let $x\notin\mathbb{Z}$ and $x>0$. Then,*

$$\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right) \tag{5.15}$$
$$=\frac{1}{2}\left\{\left(\frac{1}{\pi\sqrt{x}}-\frac{1}{\sqrt{2}}\right)\zeta\left(\frac{1}{2}\right)-\frac{1}{\pi x\sqrt{2}}\zeta\left(-\frac{1}{2}\right)\right\}+\frac{x}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}\frac{(\sqrt{2x}-\sqrt{n})}{x^2+n^2}.$$

Proof. The corollary follows readily from Theorem 5.1. We only need to observe that when $n<x$,

$${}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{n^2}{x^2}\right)=\frac{x^2}{x^2+n^2},$$

and when $n>x$,

$${}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{n^2}{x^2}\right)=-\left\{{}_3F_2\left(\begin{matrix}\frac{1}{4}+m,\frac{3}{4}+m,1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{x^2}{n^2}\right)-1\right\}$$
$$=-\left(\frac{1}{1+x^2/n^2}-1\right)=\frac{x^2}{x^2+n^2}$$

to complete our proof. □

Similarly, when $m=0$ in Theorem 5.2, we derive the following corollary.

Corollary 5.6. *For $x>0$,*

$$\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}-2\pi\sqrt{2nx}\right)$$
$$=\left(2\pi\sqrt{x}+\frac{1}{2\sqrt{2}\pi x}\right)\zeta\left(-\frac{1}{2}\right)-\frac{1}{2\sqrt{2}}\zeta\left(\frac{1}{2}\right)+\frac{x}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{x^2+n^2}. \tag{5.16}$$

We now show that the two previous corollaries can also be obtained by evaluating special cases of the infinite series

$$2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{1}{2}s}\left(e^{\pi is/4}K_s\left(4\pi e^{\pi i/4}\sqrt{nx}\right)\mp e^{-\pi is/4}K_s\left(4\pi e^{-\pi i/4}\sqrt{nx}\right)\right). \tag{5.17}$$

Second Proof of Corollary 5.5. Use the remarks following (7.2) and then replace x by $xe^{\pi i/2}$ and by $xe^{-\pi i/2}$ in (7.1), and then subtract the resulting two identities to obtain, in particular for $x>0$,

$$\begin{aligned}
&2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\left(e^{\pi is/4}K_s\left(4\pi e^{\pi i/4}\sqrt{nx}\right)-e^{-\pi is/4}K_s\left(4\pi e^{-\pi i/4}\sqrt{nx}\right)\right)\\
&=-\frac{ix^{s/2-1}}{2\pi}\cot\left(\frac{\pi s}{2}\right)\zeta(s)-\frac{i(2\pi)^{-s-1}}{\pi x^{1+s/2}}\Gamma(s+1)\zeta(s+1)-\frac{ix^{s/2}}{2}\tan\left(\frac{\pi s}{2}\right)\zeta(s+1)\\
&\quad+\frac{i\pi x}{6}\frac{\zeta(2-s)}{\sin\left(\frac{1}{2}\pi s\right)}-\frac{ix^{3-s/2}}{\pi\sin\left(\frac{1}{2}\pi s\right)}\sum_{n=1}^{\infty}\frac{\sigma_{-s}(n)}{x^2+n^2}\left(n^{s-2}+x^{s-2}\cos\left(\frac{\pi s}{2}\right)\right).
\end{aligned} \tag{5.18}$$

Now let $s=-\frac{1}{2}$ in (5.18). Using (1.17) and (1.18), we see that the left-hand side simplifies to

$$\begin{aligned}
&\frac{1}{\sqrt{2}x^{1/4}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{1/4}}\left(e^{-\pi i/4-4\pi e^{\pi i/4}\sqrt{nx}}-e^{\pi i/4-4\pi e^{-\pi i/4}\sqrt{nx}}\right)\\
&=-\frac{i\sqrt{2}}{x^{1/4}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right).
\end{aligned} \tag{5.19}$$

The right-hand side of (5.18) becomes

$$\begin{aligned}
&\frac{i}{2\pi x^{5/4}}\zeta\left(-\frac{1}{2}\right)-\frac{i}{\sqrt{2}\pi x^{3/4}}\zeta\left(\frac{1}{2}\right)+\frac{i}{2x^{1/4}}\zeta\left(\frac{1}{2}\right)\\
&\quad-\frac{i\pi x^{5/4}}{3\sqrt{2}}\zeta\left(\frac{5}{2}\right)+\frac{ix^{3/4}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{x^2+n^2}+\frac{i\sqrt{2}x^{13/4}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}(x^2+n^2)}.
\end{aligned} \tag{5.20}$$

Thus, from (5.19) and (5.20), we deduce that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right)\\
&=\frac{1}{2}\left\{\left(\frac{1}{\pi\sqrt{x}}-\frac{1}{\sqrt{2}}\right)\zeta\left(\frac{1}{2}\right)-\frac{1}{\pi x\sqrt{2}}\zeta\left(\frac{-1}{2}\right)\right\}+\frac{\pi x^{3/2}}{6}\zeta\left(\frac{5}{2}\right)\\
&\quad-\frac{x}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{x^2+n^2}-\frac{x^{7/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}(x^2+n^2)}.
\end{aligned} \tag{5.21}$$

From (5.15) and (5.21), it is clear that we want to prove that

$$\frac{\pi x^{3/2}}{6}\zeta\left(\frac{5}{2}\right)-\frac{x^{7/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}(x^2+n^2)}=\frac{x^{3/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}(x^2+n^2)}. \tag{5.22}$$

To that end, observe that

$$\frac{x^{7/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}(x^2+n^2)}+\frac{x^{3/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{x^2+n^2}=\frac{x^{3/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}}.$$

Finally, from (3.7) and the fact that $\zeta(2)=\pi^2/6$, we find that

$$\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}}=\frac{\pi^2}{6}\zeta\left(\frac{5}{2}\right). \tag{5.23}$$

This proves (5.22) and hence completes an alternative proof of (5.15). □

Similarly, if we let $s=-\frac{1}{2}$ in (7.3), then we obtain (5.16) upon simplification. Adding (5.15) and (5.16), we obtain the following result.

Theorem 5.7. *Let $x\notin\mathbb{Z}$. Then,*

$$\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\cos\left(2\pi\sqrt{2nx}\right)$$
$$=\left(\frac{1}{2\pi\sqrt{2x}}-\frac{1}{2}\right)\zeta\left(\frac{1}{2}\right)+\pi\sqrt{2x}\zeta\left(-\frac{1}{2}\right)+\frac{x^{3/2}}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}(x^2+n^2)}.$$

Subtracting (5.15) from (5.16) gives the next result.

Theorem 5.8. *Let $x\notin\mathbb{Z}$. Then,*

$$\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(2\pi\sqrt{2nx}\right)$$
$$=\frac{1}{2\pi\sqrt{2x}}\zeta\left(\frac{1}{2}\right)-\left(\frac{1}{2\pi x}+\pi\sqrt{2x}\right)\zeta\left(-\frac{1}{2}\right)+\frac{x}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}\frac{(\sqrt{x}-\sqrt{2n})}{x^2+n^2}.$$

6. Connection with the Voronoï Summation Formula

A celebrated formula of Voronoï [82] for $\sum_{n\le x}d(n)$ is given by

$$\sideset{}{'}\sum_{n\le x}d(n)=x(\log x+(2\gamma-1))+\frac{1}{4}$$
$$+\sqrt{x}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\left(-Y_1(4\pi\sqrt{nx})-\frac{2}{\pi}K_1(4\pi\sqrt{nx})\right), \tag{6.1}$$

where $Y_\nu(x)$ denotes the Bessel function of order ν of the second kind, and $K_\nu(x)$ denotes the modified Bessel function of order ν. Thus, the error term $\Delta(x)$ in the Dirichlet divisor problem (1.1) admits the infinite series representation

$$\Delta(x)=\sqrt{x}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\left(-Y_1(4\pi\sqrt{nx})-\frac{2}{\pi}K_1(4\pi\sqrt{nx})\right).$$

In [82], Voronoï also gave a more general form of (6.1), namely,

$$\sum_{\alpha<n<\beta}d(n)f(n)=\int_{\alpha}^{\beta}(2\gamma+\log t)f(t)\,dt$$

$$+ 2\pi \sum_{n=1}^{\infty} d(n) \int_{\alpha}^{\beta} f(t) \left(\frac{2}{\pi} K_0(4\pi\sqrt{nt}) - Y_0(4\pi\sqrt{nt}) \right) dt, \tag{6.2}$$

where $f(t)$ is a function of bounded variation in (α, β) and $0 < \alpha < \beta$. A. L. Dixon and W. L. Ferrar [31] gave a proof of (6.2) under the more restrictive condition that f has a bounded second differential coefficient in (α, β). J. R. Wilton [85] proved (6.2) under less restrictive conditions. In his proof, he assumed $f(t)$ has compact support on $[\alpha, \beta]$ and $V_{\alpha}^{\beta-\epsilon} f(t) \to V_{\alpha}^{\beta-0} f(t)$ as ϵ tends to 0. Here $V_{\alpha}^{\beta} f(t)$ denotes the total variation of $f(t)$ over (α, β). In 1929, Koshliakov [53] gave a very short proof of (6.2) for $0 < \alpha < \beta$, $\alpha, \beta \notin \mathbb{Z}$, for f analytic inside a closed contour strictly containing the interval $[\alpha, \beta]$. Koshliakov's proof in [53] is based on the series $\varphi(x)$, defined in (1.20), and its representation

$$\varphi(x) = -\gamma - \frac{1}{2} \log x - \frac{1}{4\pi x} + \frac{x}{\pi} \sum_{n=1}^{\infty} \frac{d(n)}{x^2 + n^2}.$$

See also [8, 9] for Voronoï-type summation formulas for a large class of arithmetical functions generated by Dirichlet series satisfying a functional equation involving the Gamma function. For Voronoï-type summation formulas involving an exponential factor, see [51]. The Voronoï summation formula has been found to be useful in physics too; for example, S. Egger and F. Steiner [36, 34] showed that it plays the role of an exact trace formula for a Schrödinger operator on a certain non-compact quantum graph. They also gave a short proof of the Voronoï summation formula in [35].

The extension of (6.2) for $\alpha = 0$ is somewhat more difficult, since one needs to impose a further condition on $f(t)$. When $f''(t)$ is bounded in (δ, α) and $t^{3/4} f''(t)$ is integrable over $(0, \delta)$ for $0 < \delta < \alpha$, Dixon and Ferrar [31] proved that

$$\sum_{0<n<\beta} d(n) f(n) = \frac{f(0+)}{4} + \int_0^{\beta} (2\gamma + \log t) f(t)\, dt$$
$$+ 2\pi \sum_{n=1}^{\infty} d(n) \int_0^{\beta} f(t) \left(\frac{2}{\pi} K_0(4\pi\sqrt{nt}) - Y_0(4\pi\sqrt{nt}) \right) dt. \tag{6.3}$$

Wilton [85] obtained (6.3) under the assumption that $\log x V_{0+}^{x} f(t)$ tends to 0 as $x \to 0+$. D. A. Hejhal [48] gave a proof of (6.3) for $\beta \to \infty$ under the assumption that f is twice continuously differentiable and possesses compact support. For other proofs of the Voronoï summation formula, the reader is referred to papers by T. Meurman [62] and A. Ivić [50].

Consider the following Voronoï summation formula in an extended form due to A. Oppenheim [67], and in the version given by A. Laurinčikas [59]. For $x > 0, x \notin \mathbb{Z}$, and $-\frac{1}{2} < \sigma < \frac{1}{2}$,

$$\sum_{n<x} \sigma_{-s}(n) = \zeta(1+s)x + \frac{\zeta(1-s)}{1-s} x^{1-s} - \frac{1}{2}\zeta(s) + \frac{x}{2\sin\left(\frac{1}{2}\pi s\right)} \sum_{n=1}^{\infty} \sigma_s(n) \tag{6.4}$$
$$\times \left(\sqrt{nx}\right)^{-1-s} \left(J_{s-1}(4\pi\sqrt{nx}) + J_{1-s}(4\pi\sqrt{nx}) - \frac{2}{\pi} \sin(\pi s) K_{1-s}(4\pi\sqrt{nx}) \right),$$

so that, by (1.3), $\Delta_{-s}(x)$ is represented by the expression involving the series on the right-hand side of (6.4). (Note that Laurinčikas proved (6.4) for $0 < s < \frac{1}{2}$. However, one can extend it to $-\frac{1}{2} < \sigma < \frac{1}{2}$.) Wilton [86] proved the same result in a more general setting by considering the 'integrated function', that is, the Riesz sum

$$\frac{1}{\Gamma(\lambda+1)}\sum_{n\leq x}{}^{\prime}\sigma_{-s}(n)(x-n)^{\lambda}.$$

Laurinčikas [59] gave a different proof of (6.4) many years later.

We will now explain the connection of Ramanujan's series

$$\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right)$$

and its companion with the extended form of the Voronoï summation formula.

As mentioned by Hardy [45], [46, pp. 268–292], if we use the asymptotic formulas (2.17) and (2.18) for $Y_1(4\pi\sqrt{nx})$ and $K_1(4\pi\sqrt{nx})$, respectively, in (6.1), we find that

$$\Delta(x)=\frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}\cos\left(4\pi\sqrt{nx}-\frac{\pi}{4}\right)+R(x), \tag{6.5}$$

where $R(x)$ is a series absolutely and uniformly convergent for all positive values of x. The first series on the left side of (6.5) is convergent for all real values of x, and uniformly convergent throughout any compact interval not containing an integer. At each integer x, it has a finite discontinuity.

If we replace the Bessel functions in (6.4) by their asymptotic expansions, namely (2.16) and (2.18), similar to what Hardy did, then the most important part of the error term $\Delta_{-s}(x)$ is given by

$$\frac{x^{\frac{1}{4}-\frac{1}{2}s}\cot\left(\frac{1}{2}\pi s\right)}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{\frac{s}{2}+\frac{3}{4}}}\cos\left(4\pi\sqrt{nx}-\frac{\pi}{4}\right).$$

This series, though similar to the one in (6.5) or in (1.2), is different from Ramanujan's series (1.21) in that the exponential factor, namely $e^{-2\pi\sqrt{2nx}}$, is not present.

A generalization of (1.20), namely,

$$\varphi(x,s):=2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{1}{2}s}\left(e^{\pi is/4}K_s\left(4\pi e^{\pi i/4}\sqrt{nx}\right)+e^{-\pi is/4}K_s\left(4\pi e^{-\pi i/4}\sqrt{nx}\right)\right), \tag{6.6}$$

was studied in [30]. Note that $\varphi(x,0)=\varphi(x)$, and that $\varphi(x)$ was used by Koshliakov [53] in his short proof of (6.2).

Replacing the Bessel functions in (6.6) by their asymptotic expansions from (2.18), we find that the main terms are given by

$$\begin{aligned}&\frac{\sqrt{2}}{x^{1/4}}\cos\left(\frac{\pi}{4}\left(s+\frac{1}{2}\right)\right)\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s/2+1/4}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}-2\pi\sqrt{2nx}\right)\\&+\frac{\sqrt{2}}{x^{1/4}}\sin\left(\frac{\pi}{4}\left(s+\frac{1}{2}\right)\right)\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s/2+1/4}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right).\end{aligned} \tag{6.7}$$

In our extensive study, the forms of the series in (6.7) are the closest that we could find that resemble the series in Ramanujan's original claim (1.6), or in our Theorem 1.1, or the companion series

$$\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}-2\pi\sqrt{2nx}\right).$$

Note that the only place where they differ is in the power of n. Similar remarks can be made about (1.19) and (6.7).

Series similar to these arise in the mean square estimates of $\int_1^x \Delta_{-s}(t)^2\, dt$ by Meurman [63, equations (3.7), (3.8)]. (An excellent survey on recent progress on divisor problems and mean square theorems has been written by K.–M. Tsang [81].) Similar series have also arisen in the work of H. Cramér [27], and in the recent work of S. Bettin and J. B. Conrey [19, p. 220–223]. Thus it seems that the two series in (6.7) are more closely connected to the generalized Dirichlet divisor problem than are Ramanujan's series and its companion. We have found identities, similar to those in Theorems 1.1 and 1.3, for each of the series in (6.7). However, we refrain ourselves from stating them as they are similar to the ones already proved.

Remark. It is interesting to note here that at the bottom of page 368 in [71], one finds the following note in Hardy's handwriting: "Idea. You can replace the Bessel functions of the Voronoï identity by circular functions, at the price of complicating the 'sum'. Interesting idea, but probably of no value for the study of the divisor problem." In view of the applications of such series mentioned in the above paragraph, it appears that Hardy's judgement was incorrect.

The series in (6.6) can be used to derive an extended form of the Voronoï summation formula (6.2) in the form contained in the following theorem. This proof generalizes the technique enunciated by Koshliakov in [53].

Theorem 6.1. *Let $0<\alpha<\beta$ and $\alpha,\beta\notin\mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha,\beta]$. Assume that $-\frac{1}{2}<\sigma<\frac{1}{2}$. Then,*

$$\begin{aligned}\sum_{\alpha<j<\beta}\sigma_{-s}(j)f(j) &= \int_\alpha^\beta(\zeta(1+s)+t^{-s}\zeta(1-s))f(t)\,dt\\ &\quad+2\pi\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{1}{2}s}\int_\alpha^\beta t^{-\frac{1}{2}s}f(t)\bigg\{\left(\frac{2}{\pi}K_s(4\pi\sqrt{nt})-Y_s(4\pi\sqrt{nt})\right)\\ &\qquad\times\cos\left(\frac{\pi s}{2}\right)-J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right)\bigg\}\,dt. \qquad (6.8)\end{aligned}$$

We wish to extend (6.8) to allow $\alpha=0$ so as to obtain (6.4) as a special case of Theorem 6.1. To do this, we need to impose some additional restrictions on f. As an intermediate result, we state the following theorem which generalizes Theorem 3 in [85].

Theorem 6.2. *Let $0<\alpha<\frac{1}{2}$, $-\frac{1}{2}<\sigma<\frac{1}{2}$, and $0<\theta<\min\left(1,\frac{1+2\sigma}{1-2\sigma}\right)$. Let $N\in\mathbb{N}$ such that $N^\theta\alpha>1$. If f is twice differentiable as a function of t, and is of bounded variation in $(0,\alpha)$, then as $N\to\infty$,*

$$\begin{aligned}&f(0+)\frac{\zeta(-s)}{2}-\int_0^\alpha f(t)(\zeta(1-s)+t^s\zeta(1+s))\,dt\\ &+2\pi{\sum_{n=1}^{N}}'\frac{\sigma_s(n)}{n^{s/2}}\int_0^\alpha f(t)t^{\frac{1}{2}s}\bigg\{J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right)\\ &\qquad+\left(Y_s(4\pi\sqrt{nt})-\frac{2}{\pi}K_s(4\pi\sqrt{nt})\right)\cos\left(\frac{\pi s}{2}\right)\bigg\}\,dt\end{aligned}$$

$$\ll \begin{cases} (2\gamma+\log N)(V_0^{N^{-\theta}}f(t)+N^{(\theta-1)/4}(|f(\alpha)|+V_0^{\alpha}f(t))), & \text{if } s=0,\\ V_0^{N^{-\theta}}f(t)+(N^{(1-\theta)(2\sigma-1)/4}+N^{(\theta(1-2\sigma)-(2\sigma+1))/4}) \\ \qquad\qquad\qquad\qquad \times(|f(\alpha)|+V_0^{\alpha}f(t)), & \text{if } s\neq 0.\end{cases}$$

Additionally, if we assume the limits

$$\lim_{x\to 0+} V_0^x f(t)=0, \ \textit{if } s\neq 0 \qquad \textit{and} \qquad \lim_{x\to 0+}\log x\, V_0^x f(t)=0, \ \textit{if } s=0, \tag{6.9}$$

then

$$\begin{aligned} f(0+)\frac{\zeta(-s)}{2} &- \int_0^{\alpha} f(t)(\zeta(1-s)+t^s\zeta(1+s))\,dt \\ &+2\pi\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s/2}}\int_0^{\alpha} f(t)t^{\frac{s}{2}}\left\{J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right)\right. \\ &\qquad\left.+\left(Y_s(4\pi\sqrt{nt})-\frac{2}{\pi}K_s(4\pi\sqrt{nt})\right)\cos\left(\frac{\pi s}{2}\right)\right\}dt=0. \end{aligned} \tag{6.10}$$

Clearly, for $0<\alpha<\frac{1}{2}$, we have

$$\sum_{0<j\leq\alpha}{}' \sigma_{-s}(j)f(j)=0. \tag{6.11}$$

Also, if we substitute for $Y_s(4\pi\sqrt{nt})$ via (2.15) and employ (1.18), we find that the kernel in (6.10), namely,

$$J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right)+\left(Y_s(4\pi\sqrt{nt})-\frac{2}{\pi}K_s(4\pi\sqrt{nt})\right)\cos\left(\frac{\pi s}{2}\right)$$

is invariant under the replacement of s by $-s$. Therefore replacing s by $-s$ in (6.10), then replacing zero on the right-hand side of (6.10) by $-\sum_{0<j\leq\alpha}\sigma_{-s}(j)f(j)$ using (6.11), and then finally subtracting the resulting equation so obtained from (6.8), we arrive at the following result.

Theorem 6.3. *Let $0<\alpha<\frac{1}{2}, \alpha<\beta$ and $\beta\notin\mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha,\beta]$, and of bounded variation in $0<t<\alpha$. Furthermore, if f satisfies the limit conditions in* (6.9)*, and $-\frac{1}{2}<\sigma<\frac{1}{2}$, then*

$$\begin{aligned} \sum_{0<j<\beta}\sigma_{-s}(j)f(j) &= -f(0+)\frac{\zeta(s)}{2}+\int_0^{\beta}(\zeta(1+s)+t^{-s}\zeta(1-s))f(t)\,dt \\ &\quad+2\pi\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{1}{2}s}\int_0^{\beta}t^{-\frac{1}{2}s}f(t)\left\{\left(\frac{2}{\pi}K_s(4\pi\sqrt{nt})-Y_s(4\pi\sqrt{nt})\right)\right. \\ &\qquad\left.\times\cos\left(\frac{\pi s}{2}\right)-J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right)\right\}dt. \end{aligned}$$

6.1. **Oppenheim's Formula** (6.4) **as a Special Case of Theorem 6.3.** Letting $\lambda=-s+1$, $\mu=s$, and $x=4\pi\sqrt{nt}$ in [69, p. 37, equation (1.8.1.1)], [69, p. 42, equation (1.9.1.1)]

[3] and [69, p. 47, equation (1.12.1.2)], and then simplifying, we see that

$$\int t^{-\frac{1}{2}s}\left\{ \left(\frac{2}{\pi}K_s(4\pi\sqrt{nt}) - Y_s(4\pi\sqrt{nt})\right)\cos\left(\frac{\pi s}{2}\right) - J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right) \right\} dt$$
$$= \frac{t^{(1-s)/2}}{4\pi\sqrt{n}\sin\left(\frac{1}{2}\pi s\right)}\left(J_{s-1}(4\pi\sqrt{nt}) + J_{1-s}(4\pi\sqrt{nt}) - \frac{2}{\pi}\sin(\pi s)K_{1-s}(4\pi\sqrt{nt})\right). \tag{6.12}$$

Let $f(t) \equiv 1$ and $\beta = x \notin \mathbb{Z}$ in Theorem 6.3. Then,

$$\sum_{j<x} \sigma_{-s}(j) = -\frac{1}{2}\zeta(s) + \int_0^x (\zeta(1+s) + t^{-s}\zeta(1-s))\,dt$$
$$+ 2\pi\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{1}{2}s}\int_0^x t^{-\frac{1}{2}s}\left\{ \left(\frac{2}{\pi}K_s(4\pi\sqrt{nt}) - Y_s(4\pi\sqrt{nt})\right)\right.$$
$$\left.\times\cos\left(\frac{\pi s}{2}\right) - J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right)\right\} dt. \tag{6.13}$$

Note that

$$\int (\zeta(1+s) + t^{-s}\zeta(1-s))\,dt = t\zeta(1+s) + \frac{t^{1-s}}{1-s}\zeta(1-s). \tag{6.14}$$

Since $-\frac{1}{2} < \sigma < \frac{1}{2}$ and the right-hand sides of (6.12) and (6.14) vanish as t tends to 0, from (6.12), (6.13), and (6.14), we obtain (6.4).

Remark. The analysis above also shows that for $\alpha > 0, \alpha \notin \mathbb{Z}$,

$$\sum_{\alpha<j<x}\sigma_{-s}(j) = x\zeta(1+s) + \frac{x^{1-s}}{1-s}\zeta(1-s) - \alpha\zeta(1+s) - \frac{\alpha^{1-s}}{1-s}\zeta(1-s)$$
$$+ \frac{1}{2\sin\left(\frac{1}{2}\pi s\right)}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{(s+1)/2}}\left\{x^{(1-s)/2}\left(J_{s-1}(4\pi\sqrt{nx}) + J_{1-s}(4\pi\sqrt{nx})\right.\right.$$
$$\left. - \frac{2}{\pi}\sin(\pi s)K_{1-s}(4\pi\sqrt{nx})\right)$$
$$\left. - \alpha^{(1-s)/2}\left(J_{s-1}(4\pi\sqrt{n\alpha}) + J_{1-s}(4\pi\sqrt{n\alpha}) - \frac{2}{\pi}\sin(\pi s)K_{1-s}(4\pi\sqrt{n\alpha})\right)\right\}. \tag{6.15}$$

---

[3]This formula, as is stated, contains many misprints. The correct version should read

$$\int_{x_1}^{x_2} y^{\lambda}Y_{\nu}(y)\,dy = \begin{Bmatrix}-1\\1\end{Bmatrix}\frac{\cos(\nu\pi)\Gamma(-\nu)x^{\lambda+\nu+1}}{2^{\nu}\pi(\lambda+\nu+1)}\,{}_1F_2\left(\frac{\lambda+\nu+1}{2};1+\nu,\frac{\lambda+\nu+3}{2};-\frac{x^2}{4}\right)$$
$$+\begin{Bmatrix}-1\\1\end{Bmatrix}\frac{2^{\nu}\Gamma(\nu)x^{\lambda-\nu+1}}{\pi(\lambda-\nu+1)}\,{}_1F_2\left(\frac{\lambda-\nu+1}{2};1-\nu,\frac{\lambda-\nu+3}{2};-\frac{x^2}{4}\right)$$
$$-\begin{Bmatrix}0\\1\end{Bmatrix}\frac{2^{\lambda}}{\pi}\cos\left(\frac{(\lambda-\nu+1)\pi}{2}\right)\Gamma\left(\frac{\lambda+\nu+1}{2}\right)\Gamma\left(\frac{\lambda-\nu+1}{2}\right).$$
$$\left[\begin{Bmatrix}x_1=0, x_2=x;\ \ \mathrm{Re}(\lambda) > |\mathrm{Re}(\nu)|-1\\ x_1=x, x_2=\infty;\ \ \mathrm{Re}(\lambda) < \frac{1}{2}\end{Bmatrix}\right].$$

From (6.4) and (6.15), we conclude that, for $-\frac{1}{2} < \sigma < \frac{1}{2}$,

$$\lim_{\alpha\to 0+} \frac{\alpha^{(1-s)/2}}{\sin\left(\frac{1}{2}\pi s\right)} \sum_{n=1}^{\infty} \frac{\sigma_s(n)}{n^{(s+1)/2}} \Bigg(J_{s-1}(4\pi\sqrt{n\alpha}) + J_{1-s}(4\pi\sqrt{n\alpha}) - \frac{2}{\pi}\sin(\pi s) K_{1-s}(4\pi\sqrt{n\alpha}) \Bigg) = \zeta(s),$$

which is likely to be difficult to prove directly.

7. Proof of Theorem 6.1

We begin with a result due to H. Cohen [24, Theorem 3.4].

Theorem 7.1. *Let $x > 0$ and $s \notin \mathbb{Z}$, where $\sigma \geq 0$* [4]*. Then, for any integer k such that $k \geq \lfloor (\sigma+1)/2 \rfloor$,*

$$8\pi x^{s/2} \sum_{n=1}^{\infty} \sigma_{-s}(n) n^{s/2} K_s(4\pi\sqrt{nx}) = A(s,x)\zeta(s) + B(s,x)\zeta(s+1) + \frac{2}{\sin(\pi s/2)} \left(\sum_{1\leq j\leq k} \zeta(2j)\zeta(2j-s)x^{2j-1} + x^{2k+1} \sum_{n=1}^{\infty} \sigma_{-s}(n) \frac{n^{s-2k} - x^{s-2k}}{n^2 - x^2} \right), \tag{7.1}$$

where

$$\begin{aligned} A(s,x) &= \frac{x^{s-1}}{\sin(\pi s/2)} - (2\pi)^{1-s}\Gamma(s), \\ B(s,x) &= \frac{2}{x}(2\pi)^{-s-1}\Gamma(s+1) - \frac{\pi x^s}{\cos(\pi s/2)}. \end{aligned} \tag{7.2}$$

By analytic continuation, the identity in Theorem 7.1 is valid not only for $x > 0$ but for $-\pi < \arg x < \pi$. Take $k = 1$ in (7.1). The condition $\lfloor (\sigma+1)/2 \rfloor \leq 1$ implies that $0 \leq \sigma < 3$. We consider $0 \leq \sigma < \frac{1}{2}$. Note that Koshliakov [53] proved the case $s = 0$, and the theorem follows for the remaining values of σ, i.e., for $-\frac{1}{2} < \sigma < 0$, by the invariance noted in the previous footnote.

Replace x by iz in (7.1) for $-\pi < \arg z < \frac{1}{2}\pi$, and then by $-iz$ for $-\frac{1}{2}\pi < \arg z < \pi$. Now add the resulting two identities and simplify, so that for $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$,

$$\Lambda(z,s) = \Phi(z,s), \tag{7.3}$$

where

$$\Lambda(z,s) := z^{-s/2}\varphi(z,s), \tag{7.4}$$

with $\varphi(x,s)$ defined in (6.6), and

$$\Phi(z,s) := -(2\pi z)^{-s}\Gamma(s)\zeta(s) + \frac{\zeta(s)}{2\pi z} - \frac{1}{2}\zeta(1+s) + \frac{z}{\pi}\sum_{n=1}^{\infty} \frac{\sigma_{-s}(n)}{z^2+n^2}. \tag{7.5}$$

As a function of z, $\Phi(z,s)$ is analytic in the entire complex plane except on the negative real axis and at $z = in, n \in \mathbb{Z}$. Hence, $\Phi(iz,s)$ is analytic in the entire complex plane except on the positive imaginary axis and at $z \in \mathbb{Z}$. Similarly, $\Phi(-iz,s)$ is analytic in the entire complex plane except on the negative imaginary axis and at $z = n \in \mathbb{Z}$. This implies that

[4]As mentioned in [24], the condition $\sigma \geq 0$ is not restrictive since, because of (1.18), the left side of the identity in this theorem is invariant under the replacement of s by $-s$.

$\Phi(iz,s)+\Phi(-iz,s)$ is analytic in both the left and right half-planes, except possibly when z is an integer. However, it is easy to see that

$$\lim_{z\to\pm n}(z\mp n)\Phi(iz,s)=\frac{1}{2\pi i}\sigma_{-s}(n)\ \text{ and }\ \lim_{z\to\pm n}(z\mp n)\Phi(-iz,s)=-\frac{1}{2\pi i}\sigma_{-s}(n),$$

so that

$$\lim_{z\to\pm n}(z\mp n)\left(\Phi(iz,s)+\Phi(-iz,s)\right)=0.$$

In particular, this implies that $\Phi(iz,s)+\Phi(-iz,s)$ is analytic in the entire right half-plane.

Now observe that for z inside an interval (u,v) on the positive real line not containing any integer, we have, using the definition (7.5),

$$\Phi(iz,s)+\Phi(-iz,s)=-2(2\pi z)^{-s}\Gamma(s)\zeta(s)\cos\left(\tfrac{1}{2}\pi s\right)-\zeta(1+s). \tag{7.6}$$

Since both $\Phi(iz,s)+\Phi(-iz,s)$ and $-2(2\pi z)^{-s}\Gamma(s)\zeta(s)\cos\left(\frac{1}{2}\pi s\right)-\zeta(1+s)$ are analytic in the right half-plane as functions of z, by analytic continuation, the identity (7.6) holds for any z in the right half-plane. Finally, using the functional equation (2.6) for $\zeta(s)$, we can simplify (7.6) to deduce that, for $-\frac{1}{2}\pi<\arg z<\frac{1}{2}\pi$,

$$\Phi(iz,s)+\Phi(-iz,s)=-z^{-s}\zeta(1-s)-\zeta(1+s). \tag{7.7}$$

Next, let f be an analytic function of z within a closed contour intersecting the real axis in α and β, where $0<\alpha<\beta$, $m-1<\alpha<m$, $n<\beta<n+1$, and $m,n\in\mathbb{Z}$. Let γ_1 and γ_2 denote the portions of the contour in the upper and lower half-planes, respectively, so that the notations $\alpha\gamma_1\beta$ and $\alpha\gamma_2\beta$, for example, denote paths from α to β in the upper and lower half-planes, respectively. By the residue theorem,

$$\frac{1}{2\pi i}\int_{\alpha\gamma_2\beta\gamma_1\alpha}f(z)\Phi(iz,s)\,dz=\sum_{\alpha<j<\beta}R_j(f(z)\Phi(iz,s)).$$

Since $f(z)\Phi(iz,s)$ has a simple pole at each integer j, $\alpha<j<\beta$, with residue $\frac{1}{2\pi i}\sigma_s(j)f(j)$, we find that

$$\begin{aligned}\sum_{\alpha<j<\beta}\sigma_{-s}(j)f(j)&=\int_{\alpha\gamma_2\beta}f(z)\Phi(iz,s)\,dz-\int_{\alpha\gamma_1\beta}f(z)\Phi(iz,s)\,dz\\&=\int_{\alpha\gamma_2\beta}f(z)\Phi(iz,s)\,dz-\int_{\alpha\gamma_1\beta}f(z)\left(-\Phi(-iz,s)-z^{-s}\zeta(1-s)-\zeta(1+s)\right)dz\\&=\int_{\alpha\gamma_2\beta}f(z)\Phi(iz,s)\,dz+\int_{\alpha\gamma_1\beta}f(z)\Phi(-iz,s)\,dz+\int_{\alpha\gamma_1\beta}f(z)\left(z^{-s}\zeta(1-s)+\zeta(1+s)\right)dz,\end{aligned}$$

where in the penultimate step, we used (7.7). Using the residue theorem again, we readily see that

$$\int_{\alpha\gamma_1\beta}f(z)\left(z^{-s}\zeta(1-s)+\zeta(1+s)\right)dz=\int_\alpha^\beta f(t)\left(\zeta(1+s)+t^{-s}\zeta(1-s)\right)dt.$$

Since $\Lambda(z,s)=\Phi(z,s)$ for $-\frac{1}{2}\pi<\arg z<\frac{1}{2}\pi$, it is easy to see that $\Lambda(iz,s)=\Phi(iz,s)$, for $-\pi<\arg z<0$, and $\Lambda(-iz,s)=\Phi(-iz,s)$, for $0<\arg z<\pi$. Thus,

$$\begin{aligned}\sum_{\alpha<j<\beta}\sigma_{-s}(j)f(j)=&\int_{\alpha\gamma_2\beta}f(z)\Lambda(iz,s)\,dz+\int_{\alpha\gamma_1\beta}f(z)\Lambda(-iz,s)\,dz\\&+\int_\alpha^\beta f(t)\left(\zeta(1+s)+t^{-s}\zeta(1-s)\right)dt.\end{aligned} \tag{7.8}$$

Using the asymptotic expansion (2.18), we see that the series

$$\Lambda(iz,s)=2(iz)^{-\frac{s}{2}}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{1}{2}s}\left(e^{i\pi s/4}K_s\left(4\pi e^{i\pi/4}\sqrt{inz}\right)+e^{-i\pi s/4}K_s\left(4\pi e^{-i\pi/4}\sqrt{inz}\right)\right)$$

is uniformly convergent in compact subintervals of $-\pi<\arg z<0$, and the series

$$\Lambda(-iz,s)=2(-iz)^{-\frac{1}{2}s}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{1}{2}s}\left(e^{i\pi s/4}K_s\left(4\pi e^{i\pi/4}\sqrt{-inz}\right)\right.$$
$$\left.+e^{-i\pi s/4}K_s\left(4\pi e^{-i\pi/4}\sqrt{-inz}\right)\right)$$

is uniformly convergent in compact subsets of $0<\arg z<\pi$. Thus, interchanging the order of summation and integration in (7.8), we deduce that

$$\sum_{\alpha<j<\beta}\sigma_{-s}(j)f(j)=2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{1}{2}s}\int_{\alpha\gamma_2\beta}f(z)(iz)^{-\frac{1}{2}s}\left(e^{i\pi s/4}K_s\left(4\pi e^{i\pi/4}\sqrt{inz}\right)\right.$$
$$\left.+e^{-i\pi s/4}K_s\left(4\pi e^{-i\pi/4}\sqrt{inz}\right)\right)\,dz$$
$$+2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{1}{2}s}\int_{\alpha\gamma_1\beta}f(z)(-iz)^{-\frac{1}{2}s}\left(e^{i\pi s/4}K_s\left(4\pi e^{i\pi/4}\sqrt{-inz}\right)\right.$$
$$\left.+e^{-i\pi s/4}K_s\left(4\pi e^{-i\pi/4}\sqrt{-inz}\right)\right)\,dz$$
$$+\int_{\alpha}^{\beta}f(t)\left(\zeta(1+s)+t^{-s}\zeta(1-s)\right)\,dt.$$

Employing the residue theorem again, this time for each of the integrals inside the two sums, and simplifying, we find that

$$\sum_{\alpha<j<\beta}\sigma_{-s}(j)f(j)=2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{1}{2}s}$$
$$\times\int_{\alpha}^{\beta}t^{-\frac{1}{2}s}f(t)\Bigg(K_s\left(4\pi i\sqrt{nt}\right)+K_s\left(-4\pi i\sqrt{nt}\right)+2\cos\left(\frac{\pi s}{2}\right)K_s\left(4\pi\sqrt{nt}\right)\Bigg)\,dt$$
$$+\int_{\alpha}^{\beta}f(t)\left(\zeta(1+s)+t^{-s}\zeta(1-s)\right)\,dt. \tag{7.9}$$

Note that for $-\pi<\arg z\leq\frac{1}{2}\pi$, the modified Bessel function $K_\nu(z)$ is related to the Hankel function $H_\nu^{(1)}(z)$ by [41, p. 911, formula **8.407.1**]

$$K_\nu(z)=\frac{\pi i}{2}e^{\nu\pi i/2}H_\nu^{(1)}(iz), \tag{7.10}$$

where the Hankel function is defined by [41, p. 911, formula **8.405.1**]

$$H_\nu^{(1)}(z):=J_\nu(z)+iY_\nu(z). \tag{7.11}$$

Employing the relations (7.10) and (7.11), we have, for $x>0$,

$$K_s(ix)+K_s(-ix)=\frac{\pi i}{2}e^{i\pi s/2}\left(H_s^{(1)}(-x)+H_s^{(1)}(x)\right)$$
$$=\frac{\pi i}{2}e^{i\pi s/2}\left\{\left(J_s(x)+J_s(-x)\right)+i\left(Y_s(x)+Y_s(-x)\right)\right\}. \tag{7.12}$$

For $m \in \mathbb{Z}$ [41, p. 927, formulas **8.476.1, 8.476.2**]

$$J_\nu(e^{m\pi i}z) = e^{m\nu\pi i}J_\nu(z), \tag{7.13}$$

$$Y_\nu(e^{m\pi i}z) = e^{-m\nu\pi i}Y_\nu(z) + 2i\sin(m\nu\pi)\cot(\nu\pi)\,J_\nu(z). \tag{7.14}$$

Using the relations (7.13) and (7.14) with $m = 1$, we can simplify (7.12) and put it in the form

$$\begin{aligned} &K_s(ix) + K_s(-ix) \\ &= \frac{\pi i}{2}e^{i\pi s/2}\left\{\left(J_s(x) + e^{i\pi s}J_s(x)\right) + i\left(Y_s(x) + e^{-i\pi s}Y_s(x) + 2i\cos(\pi s)\,J_s(x)\right)\right\} \\ &= \frac{\pi i}{2}e^{i\pi s/2}\left\{\left(1 - e^{-i\pi s}\right)J_s(x) + i\left(1 + e^{-i\pi s}\right)Y_s(x)\right\} \\ &= -\pi\left(J_s(x)\sin\left(\frac{\pi s}{2}\right) + Y_s(x)\cos\left(\frac{\pi s}{2}\right)\right). \end{aligned} \tag{7.15}$$

Now replace x by $4\pi\sqrt{nt}$ in (7.15) and substitute in (7.9) to obtain (6.8) after simplification. This completes the proof.

8. Proof of Theorem 6.2

For any integer λ, define

$$G_{\lambda+s}(z) := -J_{\lambda+s}(z)\sin\left(\frac{\pi s}{2}\right) - \left(Y_{\lambda+s}(z) - (-1)^\lambda\frac{2}{\pi}K_{\lambda+s}(z)\right)\cos\left(\frac{\pi s}{2}\right) \tag{8.1}$$

and

$$F_{\lambda+s}(z) := -J_{\lambda+s}(z)\sin\left(\frac{\pi s}{2}\right) - \left(Y_{\lambda+s}(z) + (-1)^\lambda\frac{2}{\pi}K_{\lambda+s}(z)\right)\cos\left(\frac{\pi s}{2}\right). \tag{8.2}$$

Remark. Throughout this section, we keep s fixed such that $-\frac{1}{2} < \sigma < \frac{1}{2}$. So while interpreting $F_{s+\lambda}(z)$ or $G_{s+\lambda}(z)$, care should be taken to not conceive them as functions obtained after replacing s by $s+\lambda$ in $F_s(z)$ or $G_s(z)$, but instead as those where s remains fixed and only λ varies.

From [83, pp. 66, 79] we have

$$\frac{d}{dz}\left\{z^\nu J_\nu(z)\right\} = z^\nu J_{\nu-1}(z), \tag{8.3}$$

$$\frac{d}{dz}\left\{z^\nu K_\nu(z)\right\} = -z^\nu K_{\nu-1}(z), \tag{8.4}$$

$$\frac{d}{dz}\left\{z^\nu Y_\nu(z)\right\} = z^\nu Y_{\nu-1}(z). \tag{8.5}$$

Using (8.3), (8.4), and (8.5) we deduce that

$$\frac{d}{dt}\left\{\left(\frac{t}{u}\right)^{(s+\lambda)/2}G_{s+\lambda}(4\pi\sqrt{tu})\right\} = 2\pi\left(\frac{t}{u}\right)^{(s+\lambda-1)/2}G_{s+\lambda-1}(4\pi\sqrt{tu}), \tag{8.6}$$

for $u > 0$. Similarly,

$$\frac{d}{dt}\left\{\left(\frac{t}{u}\right)^{(s+\lambda)/2}F_{s+\lambda}(4\pi\sqrt{tu})\right\} = 2\pi\left(\frac{t}{u}\right)^{(s+\lambda-1)/2}F_{s+\lambda-1}(4\pi\sqrt{tu}), \tag{8.7}$$

for $u > 0$.

From (1.3) and (6.4), recall the definition

$$\Delta_{-s}(x) = \frac{x}{2\sin\left(\frac{1}{2}\pi s\right)} \sum_{n=1}^{\infty} \sigma_s(n) \left(\sqrt{nx}\right)^{-1-s} \times \left(J_{s-1}(4\pi\sqrt{nx}) + J_{1-s}(4\pi\sqrt{nx}) - \frac{2}{\pi}\sin(\pi s) K_{1-s}(4\pi\sqrt{nx}) \right),$$

for $-\frac{1}{2} < \sigma < \frac{1}{2}$ and $x > 0$. If we replace s by $-s$ in the equation above and use (2.15), we find by a straightforward computation that

$$\Delta_s(x) = \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{(s+1)/2} \sigma_s(n) G_{s+1}(4\pi\sqrt{nx}), \tag{8.8}$$

for $-\frac{1}{2} < \sigma < \frac{1}{2}$ and $x > 0$. Fix $x > 0$. By the asymptotic expansions of Bessel functions (2.16), (2.17), and (2.18), there exists a sufficiently large integer N_0 such that

$$G_\nu(4\pi\sqrt{nx}) \ll_\nu \frac{1}{(nx)^{1/4}} \quad \text{and} \quad F_\nu(4\pi\sqrt{nx}) \ll_\nu \frac{1}{(nx)^{1/4}}, \tag{8.9}$$

for all $n > N_0$. Hence, for $-\frac{1}{2} < \sigma < \frac{1}{2}$ and $x > 0$,

$$\sum_{n>N_0} \left(\frac{x}{n}\right)^{\lambda+\frac{1}{2}s} \sigma_s(n) G_{s+2\lambda}(4\pi\sqrt{nx}) \ll x^{\lambda+(2\sigma-1)/4} \sum_{n>N_0} \frac{\sigma_\sigma(n)}{n^{\lambda+(1+2\sigma)/4}} \ll x^{\lambda+(2\sigma-1)/4},$$

provided that $2\lambda > |\sigma| + \frac{3}{2}$. Therefore, for $\lambda \geq 1$, $-\frac{1}{2} < \sigma < \frac{1}{2}$, and $x > 0$, the series

$$\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{\lambda+\frac{1}{2}s} \sigma_s(n) G_{s+2\lambda}(4\pi\sqrt{nx})$$

is absolutely convergent. Similarly, for $\lambda \geq 1$, $-\frac{1}{2} < \sigma < \frac{1}{2}$, and $x > 0$, the series

$$\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{\lambda+\frac{s}{2}} \sigma_s(n) F_{s+2\lambda}(4\pi\sqrt{nx})$$

is absolutely convergent. Denote

$$D_s(x) := {\sum_{n\leq x}}' \sigma_s(n) \tag{8.10}$$

and

$$\Phi_s(x) := x\zeta(1-s) + \frac{x^{1+s}}{1+s}\zeta(1+s) - \frac{1}{2}\zeta(-s). \tag{8.11}$$

Therefore, from (6.4), we write

$$D_s(x) = \Phi_s(x) + \Delta_s(x) \tag{8.12}$$

for $-\frac{1}{2} < \sigma < \frac{1}{2}$.

The following lemmas are key ingredients in the proof of Theorem 6.2. They are special cases of two results in [86]. We note, however, that the definitions of G and F in [86] are different from those in (8.1) and (8.2) that we use.

Lemma 8.1. *If $x > 0$, $N > 0$, and $-\frac{1}{2} < \sigma < \frac{1}{2}$, then*

$$\begin{aligned}\Delta_s(x) = &\sum_{n=1}^{N}{}' \left(\frac{x}{n}\right)^{(s+1)/2} \sigma_s(n) G_{s+1}(4\pi\sqrt{nx}) - \left(\frac{x}{N}\right)^{(s+1)/2} G_{s+1}(4\pi\sqrt{Nx})\Delta_s(N)\\ &+ \frac{N^s\zeta(1+s)+\zeta(1-s)}{2\pi}\left(\frac{x}{N}\right)^{s/2} F_s(4\pi\sqrt{Nx})\\ &+ \frac{s\zeta(1+s)}{2\pi}\int_N^\infty \left(\frac{x}{t}\right)^{s/2} F_s(4\pi\sqrt{xt})t^{s-1}\,dt\\ &+ 2\pi\sum_{n=1}^{\infty}\sigma_s(n)\int_N^\infty \left(\frac{x}{t}\right)^{(s+2)/2} F_{s+2}(4\pi\sqrt{xt})\left(\frac{t}{n}\right)^{(s+1)/2} G_{s+1}(4\pi\sqrt{nt})\,dt.\end{aligned} \tag{8.13}$$

Proof. Take $\lambda = 0$, $\kappa = 1$, and $\theta = 1$ in Theorem 2 of [86, p. 404], and make use of the notations (1.21) and (3.13) given in it. □

We wish to invert the order of summation and integration in the last expression on the right-hand side of (8.13). In order to justify that, we need the following lemma.

Lemma 8.2. *If $N > A$, $Nx > A$, $-\frac{1}{2} < \sigma < \frac{1}{2}$, and*

$$I_s(x,n;N) := 2\pi\int_N^\infty \left(\frac{x}{t}\right)^{(s+2)/2} F_{s+2}(4\pi\sqrt{xt})\left(\frac{t}{n}\right)^{(s+1)/2} G_{s+1}(4\pi\sqrt{nt})\,dt,$$

then

$$\sum_{n=1}^{\infty}\sigma_s(n)I_s(x,n;N) = C_s(x,N) + O\left(\frac{x^{1+\epsilon}}{\sqrt{N}}\right),$$

for every $\epsilon > 0$, where

$$\begin{aligned}&C_s(x,N) = 0, \quad \text{if} \quad x < \tfrac{1}{2} \quad \text{or} \quad x \in \mathbb{N},\\ &C_s(x,N) = \frac{1}{\pi}\left(\frac{x}{y}\right)^{(2s+5)/4}\sigma_s(y)\int_{4\pi\sqrt{N}|\sqrt{y}-\sqrt{x}|}^\infty \frac{\sin(t\,\mathrm{sgn}(y-x))}{t}\,dt,\\ &\qquad \text{if} \quad x \neq y = \lfloor x + \tfrac{1}{2}\rfloor \geq 1.\end{aligned}$$

Proof. This is the special case $\lambda = 0, \kappa = 1$ of Lemma 6 of [86, p. 412]. □

Proof of Theorem 6.2. By Lemma 8.2, we see that the last expression on the right-hand side of (8.13) tends to 0 as $N \to \infty$. Hence, by interchanging the summation and integration in this expression, we deduce that

$$\begin{aligned}\Delta_s(x) = &\sum_{n=1}^{N}{}' \left(\frac{x}{n}\right)^{(s+1)/2} \sigma_s(n) G_{s+1}(4\pi\sqrt{nx})\\ &+ \frac{N^s\zeta(1+s)+\zeta(1-s)}{2\pi}\left(\frac{x}{N}\right)^{s/2} F_s(4\pi\sqrt{Nx})\\ &- \left(\frac{x}{N}\right)^{(s+1)/2} G_{s+1}(4\pi\sqrt{Nx})\Delta_s(N)\\ &+ \frac{s\zeta(1+s)}{2\pi}\int_N^\infty \left(\frac{x}{t}\right)^{s/2} F_s(4\pi\sqrt{xt})t^{s-1}\,dt\\ &+ 2\pi\int_N^\infty \left(\frac{x}{t}\right)^{(s+2)/2} F_{s+2}(4\pi\sqrt{xt})\Delta_s(t)\,dt.\end{aligned} \tag{8.14}$$

Let $a \geq 0$ and $b \geq 0$. From (8.10),

$$\sum_{a\leq n\leq b} f(n)\sigma_s(n) = \int_a^b f(t)\,dD_s(t), \tag{8.15}$$

where we write the sum as a Lebesgue-Stieltjes integral.

For $a = 0$ and $b = \alpha < \frac{1}{2}$, the left-hand side of (8.15) equals 0. Therefore, from (8.6), (8.7), (8.11), (8.12), (8.14), (8.15), and (8.8),

$$\begin{aligned}
-\int_0^\alpha f(t)(\zeta(1-s)+t^s\zeta(1+s))\,dt &= \int_0^\alpha f(t)\,d\Delta_s(t)\\
= 2\pi \sum_{n=1}^{N}{}' \frac{\sigma_s(n)}{n^{s/2}} &\int_0^\alpha t^{s/2}G_s(4\pi\sqrt{nt})f(t)\,dt\\
&+ \frac{N^s\zeta(1+s)+\zeta(1-s)}{N^{(s-1)/2}}\int_0^\alpha t^{(s-1)/2}F_{s-1}(4\pi\sqrt{Nt})f(t)\,dt\\
&- \frac{2\pi}{N^{s/2}}\Delta_s(N)\int_0^\alpha t^{s/2}G_s(4\pi\sqrt{Nt})f(t)\,dt\\
&+ \frac{s\zeta(1+s)}{2\pi}\int_0^\alpha f(t)\frac{d}{dt}\left(\int_N^\infty \left(\frac{t}{u}\right)^{s/2}F_s(4\pi\sqrt{tu})u^{s-1}\,du\right)dt\\
&+ 2\pi\int_0^\alpha f(t)\frac{d}{dt}\left(\int_N^\infty \left(\frac{t}{u}\right)^{(s+2)/2}F_{s+2}(4\pi\sqrt{tu})\Delta_s(u)\,du\right)dt.
\end{aligned} \tag{8.16}$$

Using (8.7) twice, we see that

$$\begin{aligned}
\frac{d}{dt}\left(\int_N^\infty \left(\frac{t}{u}\right)^{s/2}F_s(4\pi\sqrt{tu})u^{s-1}\,du\right) &= 2\pi\int_N^\infty (tu)^{(s-1)/2}F_{s-1}(4\pi\sqrt{tu})\,du\\
&= t^{s/2-1}u^{s/2}F_s(4\pi\sqrt{tu})\Big|_N^\infty\\
&= -t^{s/2-1}N^{s/2}F_s(4\pi\sqrt{tN}),
\end{aligned} \tag{8.17}$$

where in the last step we use (2.16)–(2.18), and the fact that $\sigma < \frac{1}{2}$. The interchange of differentiation and integration above is justified from (8.9). Denote

$$I_s(t,N) := 2\pi\int_N^\infty \left(\frac{t}{u}\right)^{(s+2)/2}F_{s+2}(4\pi\sqrt{tu})\Delta_s(u)\,du. \tag{8.18}$$

Performing an integration by parts on the last expression on the right-hand side of (8.16) and using (8.17) and (8.18), we find that

$$\begin{aligned}
-\int_0^\alpha f(t)(\zeta(1-s)+t^s\zeta(1+s))\,dt &- 2\pi\sum_{n=1}^{N}{}'\frac{\sigma_s(n)}{n^{s/2}}\int_0^\alpha t^{s/2}G_s(4\pi\sqrt{nt})f(t)\,dt\\
= \frac{N^s\zeta(1+s)+\zeta(1-s)}{N^{(s-1)/2}} &\int_0^\alpha t^{(s-1)/2}F_{s-1}(4\pi\sqrt{Nt})f(t)\,dt\\
&- \frac{2\pi}{N^{s/2}}\Delta_s(N)\int_0^\alpha t^{s/2}G_s(4\pi\sqrt{Nt})f(t)\,dt\\
&- \frac{s\zeta(1+s)N^{s/2}}{2\pi}\int_0^\alpha f(t)t^{s/2-1}F_s(4\pi\sqrt{Nt})\,dt
\end{aligned}$$

$$+ f(\alpha)I_s(\alpha,N) - \int_0^\alpha I_s(t,N)f'(t)\,dt, \tag{8.19}$$

where in the last step we made use of the fact that for $-\frac{1}{2}<\sigma<\frac{1}{2}$,

$$\lim_{t\to 0} t^{(s+2)/2}F_{s+2}(4\pi\sqrt{tu})=0.$$

Here again the limit can be moved inside the integral because of (8.9).

Since $\alpha<\frac{1}{2}$, by Lemma 8.2, $I_s(t,N)\ll N^{-1/2}$, for all $0<t\leq\alpha$. Also by hypothesis, f is differentiable, so

$$V_0^\alpha f(t)=\int_0^\alpha |f'(t)|\,dt,$$

where $V_0^\alpha f(t)$ is the total variation of f on the interval $(0,\alpha)$. Therefore the last two terms on the right-hand side of (8.19) are of the form

$$O(N^{-1/2}(|f(\alpha)|+V_0^\alpha f(t))). \tag{8.20}$$

Recall the bound $\Delta_s(N)\ll N^{\frac{1}{2}(1+\sigma)}$ [86, Lemma 7]. From (8.6) and (8.9),

$$\begin{aligned}
&\frac{2\pi}{N^{s/2}}\Delta_s(N)\int_0^\alpha t^{s/2}G_s(4\pi\sqrt{Nt})\,dt\\
&=\Delta_s(N)\left(\frac{\alpha}{N}\right)^{(s+1)/2}G_{s+1}(4\pi\sqrt{N\alpha})\\
&\ll \alpha^{\frac{\sigma}{2}}\left(\frac{\alpha}{N}\right)^{\frac{1}{4}}.
\end{aligned} \tag{8.21}$$

Here we also made use of the fact that

$$\lim_{t\to 0} t^{(s+1)/2}G_{s+1}(4\pi\sqrt{Nt})=0.$$

Again, from (8.6) and (8.9),

$$\begin{aligned}
&\frac{N^s\zeta(1+s)+\zeta(1-s)}{N^{(s-1)/2}}\int_0^\alpha t^{(s-1)/2}F_{s-1}(4\pi\sqrt{Nt})\,dt\\
&=\frac{N^s\zeta(1+s)+\zeta(1-s)}{2\pi N^{s/2}}\alpha^{s/2}F_s(4\pi\sqrt{N\alpha})\\
&\ll\begin{cases}(2\gamma+\log N)(\alpha N)^{-1/4}, & \text{if } s=0,\\ (\alpha N)^{(2\sigma-1)/4}+\alpha^{(2\sigma-1)/4}N^{(-2\sigma-1)/4}, & \text{if } s\neq 0,\end{cases}
\end{aligned} \tag{8.22}$$

since $\lim_{t\to 0}t^{s/2}F_s(4\pi\sqrt{Nt})=0$. Finally,

$$\begin{aligned}
&\frac{s\zeta(1+s)N^{s/2}}{2\pi}\int_0^\alpha t^{s/2-1}F_s(4\pi\sqrt{Nt})\,dt\\
&=\frac{s\zeta(1+s)N^{s/2}}{2\pi}\left(\int_0^\infty-\int_\alpha^\infty\right)t^{s/2-1}F_s(4\pi\sqrt{Nt})\,dt\\
&=: I_1-I_2.
\end{aligned}$$

Using the functional equation of $\zeta(s)$, namely (2.6), and the formula [86, p. 409, equation 4.65], we find that

$$I_1=\frac{s\zeta(1+s)N^{s/2}}{2\pi}\int_0^\infty t^{s/2-1}F_s(4\pi\sqrt{Nt})\,dt$$

$$= -(2\pi)^{-s-1}\sin(\pi s/2)\Gamma(s+1)\zeta(1+s) = \frac{\zeta(-s)}{2}.$$

Using (8.9), we deduce that

$$I_2 = \frac{s\zeta(1+s)N^{s/2}}{2\pi}\int_{\alpha}^{\infty} t^{s/2-1}F_s(4\pi\sqrt{Nt})\,dt \ll (\alpha N)^{(2\sigma-1)/4}, \tag{8.23}$$

since $-\frac{1}{2} < \sigma < \frac{1}{2}$. Using (8.20)–(8.23) in (8.19), we find that

$$\begin{aligned} &f(0+)\frac{\zeta(-s)}{2} - \int_0^{\alpha} f(t)(\zeta(1-s)+t^s\zeta(1+s))\,dt \\ &\quad - 2\pi\sum_{n=1}^{N}{}' \frac{\sigma_s(n)}{n^{s/2}}\int_0^{\alpha} t^{s/2}G_s(4\pi\sqrt{nt})f(t)\,dt \\ &= \frac{N^s\zeta(1+s)+\zeta(1-s)}{N^{(s-1)/2}}\int_0^{\alpha} t^{(s-1)/2}F_{s-1}(4\pi\sqrt{Nt})(f(t)-f(0+))\,dt \\ &\quad - \frac{2\pi}{N^{s/2}}\Delta_s(N)\int_0^{\alpha} t^{s/2}G_s(4\pi\sqrt{Nt})(f(t)-f(0+))\,dt \\ &\quad - \frac{s\zeta(1+s)}{2\pi}\int_0^{\alpha}(f(t)-f(0+))t^{s/2-1}N^{s/2}F_s(4\pi\sqrt{Nt})\,dt \\ &\quad + O((\alpha N)^{(2\sigma-1)/4} + \alpha^{(2\sigma-1)/4}N^{(-2\sigma-1)/4}) + O((2\gamma+\log N)(\alpha N)^{-1/4}). \end{aligned} \tag{8.24}$$

By the second mean value theorem for integrals in the form given in [85, p. 31],

$$\left|\int_a^b f(t)\phi(t)\,dt - f(b)\int_a^b \phi(t)\,dt\right| \le V_a^b f(t) \max_{a\le c<d\le b}\left|\int_c^d \phi(t)\,dt\right|, \tag{8.25}$$

where ϕ is integrable on $[a,b]$.

Recall that $N^{\theta}\alpha > 1$ for some $0 < \theta < \min\left(1, \frac{1+2\sigma}{1-2\sigma}\right)$. Dividing the interval $(0,\alpha)$ into two sub-intervals $(0, N^{-\theta})$ and $(N^{-\theta}, \alpha)$, applying (8.25), and using an argument like that in (8.22), we see that

$$\begin{aligned} &\frac{N^s\zeta(1+s)+\zeta(1-s)}{N^{(s-1)/2}}\int_{N^{-\theta}}^{\alpha} t^{(s-1)/2}F_{s-1}(4\pi\sqrt{Nt})(f(t)-f(0+))\,dt \\ &\ll \begin{cases} (2\gamma+\log N)N^{(\theta-1)/4}V_{N^{-\theta}}^{\alpha}f(t), & \text{if } s=0, \\ (N^{(1-\theta)(2\sigma-1)/4} + N^{(\theta(1-2\sigma)-(2\sigma+1))/4})V_{N^{-\theta}}^{\alpha}f(t), & \text{if } s\neq 0, \end{cases} \end{aligned} \tag{8.26}$$

and

$$\begin{aligned} &\frac{N^s\zeta(1+s)+\zeta(1-s)}{N^{(s-1)/2}}\int_0^{N^{-\theta}} t^{(s-1)/2}F_{s-1}(4\pi\sqrt{Nt})(f(t)-f(0+))\,dt \\ &\ll \begin{cases} (2\gamma+\log N)V_0^{N^{-\theta}}f(t), & \text{if } s=0, \\ V_0^{N^{-\theta}}f(t), & \text{if } s\neq 0. \end{cases} \end{aligned}$$

By (8.25) and arguments similar to those in (8.21) and (8.23),

$$\frac{2\pi}{N^{s/2}}\Delta_s(N)\int_{N^{-\theta}}^{\alpha} t^{s/2}G_s(4\pi\sqrt{Nt})(f(t)-f(0+))\,dt \ll \alpha^{\frac{\sigma}{2}}\left(\frac{\alpha}{N}\right)^{\frac{1}{4}} V_{N^{-\theta}}^{\alpha}f(t),$$

$$\frac{2\pi}{N^{s/2}}\Delta_s(N)\int_0^{N^{-\theta}} t^{s/2}G_s(4\pi\sqrt{Nt})(f(t)-f(0+))\,dt \ll N^{\frac{-2\theta\sigma-1-\theta}{4}}V_0^{N^{-\theta}}f(t),$$

$$\frac{s\zeta(1+s)}{2\pi}\int_{N^{-\theta}}^{\alpha}(f(t)-f(0+))t^{s/2-1}N^{s/2}F_s(4\pi\sqrt{Nt})\,dt \ll N^{\frac{(1-\theta)(2\sigma-1)}{4}}V_{N^{-\theta}}^{\alpha}f(t),$$

and

$$\frac{s\zeta(1+s)}{2\pi}\int_0^{N^{-\theta}}(f(t)-f(0+))t^{s/2-1}N^{s/2}F_s(4\pi\sqrt{Nt})\,dt \ll V_0^{N^{-\theta}}f(t). \tag{8.27}$$

Combining (8.26)–(8.27) together with (8.24), we obtain

$$\begin{aligned}
&f(0+)\frac{\zeta(-s)}{2}-\int_0^{\alpha}f(t)(\zeta(1-s)+t^s\zeta(1+s))\,dt\\
&\qquad -2\pi{\sum_{n=1}^{N}}'\frac{\sigma_s(n)}{n^{s/2}}\int_0^{\alpha}t^{s/2}G_s(4\pi\sqrt{nt})f(t)\,dt\\
&\ll \begin{cases}(2\gamma+\log N)(V_0^{N^{-\theta}}f(t)+N^{(\theta-1)/4}V_{N^{-\theta}}^{\alpha}f(t)), & \text{if } s=0,\\ V_0^{N^{-\theta}}f(t)+(N^{(1-\theta)(2\sigma-1)/4}+N^{(\theta(1-2\sigma)-(2\sigma+1))/4)})V_{N^{-\theta}}^{\alpha}f(t), & \text{if } s\neq 0,\end{cases}\\
&\ll \begin{cases}(2\gamma+\log N)(V_0^{N^{-\theta}}f(t)+N^{(\theta-1)/4}(|f(\alpha)|+V_0^{\alpha}f(t))), & \text{if } s=0,\\ V_0^{N^{-\theta}}f(t)+(N^{(1-\theta)(2\sigma-1)/4}+N^{(\theta(1-2\sigma)-(2\sigma+1))/4)})\\ \qquad\qquad\times(|f(\alpha)|+V_0^{\alpha}f(t)), & \text{if } s\neq 0.\end{cases}
\end{aligned}$$

Furthermore, if $\log x\, V_0^x f(t)\to 0$ as $x\to 0+$ when $s=0$, and if $V_0^x f(t)\to 0$ as $x\to 0+$ when $s\neq 0$, then the assumption $0<\theta<\min\left(1,\frac{1+2\sigma}{1-2\sigma}\right)$ implies that

$$\begin{aligned}
f(0+)\frac{\zeta(-s)}{2}&-\int_0^{\alpha}f(t)(\zeta(1-s)+t^s\zeta(1+s))\,dt\\
&-2\pi\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s/2}}\int_0^{\alpha}t^{s/2}G_s(4\pi\sqrt{nt})f(t)\,dt=0.
\end{aligned}$$

This completes the proof of Theorem 6.2. □

9. An Interpretation of Ramanujan's Divergent Series

As mentioned in the introduction, the series on the left-hand side of (1.6) is divergent for all real values of s, since $\sigma_s(n)\geq n^s$. However, as we show below, there is a valid interpretation of this series using the theory of analytic continuation.

Throughout this section, we assume $x>0$, $\sigma>0$, and $\text{Re } w>1$. Define a function $F(s,x,w)$ by

$$F(s,x,w):=\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{w-\frac{1}{2}}}\left((x-in)^{-s-\frac{1}{2}}-(x+in)^{-s-\frac{1}{2}}\right). \tag{9.1}$$

Ramanujan's divergent series corresponds to letting $w=\frac{1}{2}$ in (9.1). Note that

$$(x-in)^{-s-\frac{1}{2}}-(x+in)^{-s-\frac{1}{2}}=\frac{2i\sin\left(\left(s+\frac{1}{2}\right)\tan^{-1}(n/x)\right)}{(x^2+n^2)^{\frac{s}{2}+\frac{1}{4}}}.$$

Since for $\sigma > -\frac{3}{2}$ and $n > 0$ [41, p. 524, formula **3.944**, no. 5]

$$\int_0^\infty e^{-xt}t^{s-\frac{1}{2}}\sin(nt)\,dt = \Gamma\left(s+\frac{1}{2}\right)\frac{\sin\left(\left(s+\frac{1}{2}\right)\tan^{-1}(n/x)\right)}{(x^2+n^2)^{\frac{s}{2}+\frac{1}{4}}}, \tag{9.2}$$

we deduce from (9.1)–(9.2) that

$$F(s,x,w) = \frac{2i}{\Gamma\left(s+\frac{1}{2}\right)}\sum_{n=1}^\infty \frac{\sigma_s(n)}{n^{w-\frac{1}{2}}}\int_0^\infty e^{-xt}t^{s-\frac{1}{2}}\sin nt\,dt.$$

From [64, p. 42, formula (5.1)], for $-1 < c = \text{Re } z < 1$,

$$\sin(nt) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(z)\sin\left(\frac{\pi z}{2}\right)(nt)^{-z}\,dz.$$

Hence,

$$\begin{aligned} F(s,x,w) &= \frac{1}{\pi\Gamma\left(s+\frac{1}{2}\right)}\int_0^\infty e^{-xt}t^{s-\frac{1}{2}}\sum_{n=1}^\infty\frac{\sigma_s(n)}{n^{w-\frac{1}{2}}}\int_{c-i\infty}^{c+i\infty}\Gamma(z)\sin\left(\frac{\pi z}{2}\right)(nt)^{-z}\,dz\,dt \\ &= \frac{1}{\pi\Gamma\left(s+\frac{1}{2}\right)}\int_0^\infty e^{-xt}t^{s-\frac{1}{2}}\int_{c-i\infty}^{c+i\infty}t^{-z}\Gamma(z)\sin\left(\frac{\pi z}{2}\right)\left(\sum_{n=1}^\infty\frac{\sigma_s(n)}{n^{w+z-\frac{1}{2}}}\right)dz\,dt, \end{aligned} \tag{9.3}$$

where the interchange of the order of summation and integration in both instances is justified by absolute convergence. Now if Re $z > \frac{3}{2}-$ Re w and Re $z > \frac{3}{2}-$ Re $w + \sigma$, from (3.7), we see that

$$\sum_{n=1}^\infty\frac{\sigma_s(n)}{n^{w+z-\frac{1}{2}}} = \zeta\left(w+z-\frac{1}{2}\right)\zeta\left(w+z-s-\frac{1}{2}\right).$$

Substituting this in (9.3), we find that

$$\begin{aligned} F(s,x,w) &= \frac{1}{\pi\Gamma\left(s+\frac{1}{2}\right)}\int_0^\infty e^{-xt}t^{s-\frac{1}{2}}\int_{c-i\infty}^{c+i\infty}t^{-z}\Gamma(z)\sin\left(\frac{\pi z}{2}\right) \\ &\qquad\times\zeta\left(w+z-\frac{1}{2}\right)\zeta\left(w+z-s-\frac{1}{2}\right)dz\,dt \\ &= \frac{1}{\pi\Gamma\left(s+\frac{1}{2}\right)}\int_{c-i\infty}^{c+i\infty}\Gamma(z)\sin\left(\frac{\pi z}{2}\right)\zeta\left(w+z-\frac{1}{2}\right)\zeta\left(w+z-s-\frac{1}{2}\right) \\ &\qquad\times\int_0^\infty e^{-xt}t^{s-z-\frac{1}{2}}\,dt\,dz, \end{aligned} \tag{9.4}$$

with the interchange of the order of integration again being easily justifiable. For Re $z < \sigma + \frac{1}{2}$, we have

$$\int_0^\infty e^{-xt}t^{s-z-\frac{1}{2}}\,dt = \frac{\Gamma\left(s-z+\frac{1}{2}\right)}{x^{s-z+\frac{1}{2}}}.$$

Substituting this in (9.4), we obtain the integral representation

$$F(s,x,w) = \frac{x^{-s-\frac{1}{2}}}{\pi\Gamma\left(s+\frac{1}{2}\right)}\int_{c-i\infty}^{c+i\infty}\Gamma(z)\sin\left(\frac{\pi z}{2}\right)\zeta\left(w+z-\frac{1}{2}\right)$$

$$\times \zeta\left(w+z-s-\frac{1}{2}\right)\Gamma\left(s-z+\frac{1}{2}\right)x^z\,dz. \tag{9.5}$$

Note that if we shift the line of integration Re $z=c$ to Re $z=d$ such that $d=\frac{3}{2}+\sigma-\eta$ with $\eta>0$, we encounter a simple pole of the integrand due to $\Gamma\left(s-z+\frac{1}{2}\right)$. Employing the residue theorem and noting that, from (2.5) and (3.15), the integrals over the horizontal segments tend to zero as the height of the rectangular contour tends to ∞, we have

$$\begin{aligned} F(s,x,w) &= \frac{x^{-s-\frac{1}{2}}}{\pi\Gamma\left(s+\frac{1}{2}\right)}\int_{d-i\infty}^{d+i\infty}\Gamma(z)\sin\left(\frac{\pi z}{2}\right)\zeta\left(w+z-\frac{1}{2}\right) \\ &\qquad\times \zeta\left(w+z-s-\frac{1}{2}\right)\Gamma\left(s-z+\frac{1}{2}\right)x^z\,dz \\ &\quad -\frac{2ix^{-s-\frac{1}{2}}}{\Gamma\left(s+\frac{1}{2}\right)}\Gamma\left(s+\frac{1}{2}\right)\sin\left(\frac{\pi}{2}\left(s+\frac{1}{2}\right)\right)\zeta(w+s)\zeta(w)x^{s+\frac{1}{2}}. \end{aligned} \tag{9.6}$$

Note that the residue in equation (9.6) is analytic in w except for simple poles at 1 and $1-s$. Consider the integrand in (9.6). The zeta functions $\zeta\left(w+z-\frac{1}{2}\right)$ and $\zeta\left(w+z-s-\frac{1}{2}\right)$ have simple poles at $w=\frac{3}{2}-z$ and $w=\frac{3}{2}+s-z$, respectively. However, since Re $z=\frac{3}{2}+\sigma-\eta$ and $\sigma>0$, the integrand is analytic as a function of w as long as Re $w>\eta$. By a well-known theorem [77, p. 30, Theorem 2.3], the integral is also analytic in w for Re $w>\eta$. Thus, the right-hand side of (9.6) is analytic in w, which allows us to analytically continue $F(s,x,w)$ as a function of w to the region Re $w>\eta$, and hence to Re $w>0$, since η is any arbitrary positive number.

As remarked in the beginning of this section, letting $w=\frac{1}{2}$ in (9.1) yields Ramanujan's divergent series. However, the analytic continuation of $F(s,x,w)$ to Re $w>0$ allows us to substitute $w=\frac{1}{2}$ in (9.6) and thereby give a valid interpretation of Ramanujan's divergent series. The only exception to this is when $s=\frac{1}{2}$, since then $w=\frac{1}{2}=1-s$ is a pole of the right-hand side of (9.6), as discussed above.

If we further shift the line of integration in (9.5) from Re $z=\frac{3}{2}+\sigma-\eta$ to Re $z=\frac{5}{2}+\sigma-\eta$, and likewise to $+\infty$, we obtain a meromorphic continuation of $F(s,x,w)$, as a function of w, to the whole complex plane.

10. Generalization of the Ramanujan–Wigert Identity

The identity (1.16) found by Ramanujan, and then extended by Wigert, has the following one variable generalization in the same spirit as Theorem 1.1.

Theorem 10.1. *Let $\psi_s(n):=\sum_{j^2|n}j^s$. Let $\alpha>0$ and $\beta>0$ be such that $\alpha\beta=4\pi^3$. Recall that ${}_1F_1$ is defined in (1.8). Then, for $\sigma>0$,*

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{\psi_s(n)}{\sqrt{n}}e^{-\sqrt{n\beta}}\sin\left(\frac{\pi}{4}-\sqrt{n\beta}\right) \\ &=\frac{\sqrt{\beta}}{\sqrt{2}}\zeta(-s)+\Gamma(s)\cos\left(\frac{\pi}{4}+\frac{\pi s}{4}\right)\zeta\left(\frac{1+s}{2}\right)(2\beta)^{-\frac{1}{2}s}+\frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(1-s) \\ &\quad+2^{-s-\frac{1}{2}}\pi^{-s-2}\sqrt{\beta}\sum_{n=1}^{\infty}\frac{\psi_{1-s}(n)}{n}\left\{\frac{-\pi^3\sqrt{n}}{\sqrt{\beta}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)}\right. \end{aligned}$$

$$+2^{2s}\pi^{\frac{3}{2}s+2}\left(\frac{n}{\beta}\right)^{(s+1)/2}\Gamma\left(\frac{s}{2}\right){}_1F_1\left(\frac{s}{2};\frac{1}{2};-n\alpha\right)\Bigg\}. \tag{10.1}$$

Note that $\psi_s(n)$ is an extension of Ramanujan's definition of $\psi(n)$ that was defined earlier in (1.15). The proof of Theorem 10.1 is similar to that of Theorem 1.1, and so we will be brief.

Proof. From the definition of $\psi_s(n)$, we note that

$$\sum_{n=1}^{\infty}\frac{\psi_s(n)}{n^{z/2}}=\zeta\left(\frac{z}{2}\right)\zeta(z-s)$$

for Re $z>2$ and Re $z>1+\sigma$. Let

$$W(s,\beta):=\sum_{n=1}^{\infty}\frac{\psi_s(n)}{\sqrt{n}}e^{-\sqrt{n\beta}}\sin\left(\frac{\pi}{4}-\sqrt{n\beta}\right).$$

Proceeding as we did in (3.2)–(3.9), we obtain for $c=\text{Re }z>\max\{2,1+\sigma\}$,

$$W(s,\beta)=\frac{1}{2\pi i}\int_{(c)}\Gamma(z-1)\cos\left(\frac{\pi z}{4}\right)\zeta\left(\frac{z}{2}\right)\zeta(z-s)(2\beta)^{(1-z)/2}\,dz.$$

Shifting the line of integration from Re $z=c$ to Re $z=\lambda$, $-1<\lambda<0$, applying the residue theorem, and considering the contributions of the poles at $z=0,1$, and $1+s$, we find that

$$W(s,\beta)=\frac{\sqrt{\beta}}{\sqrt{2}}\zeta(-s)+\Gamma(s)\cos\left(\frac{\pi(1+s)}{4}\right)\zeta\left(\frac{1+s}{2}\right)(2\beta)^{-\frac{1}{2}s}+\frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(1-s)$$
$$+\frac{1}{2\pi i}\int_{(\lambda)}G(z,s,\beta)\,dz, \tag{10.2}$$

where

$$\begin{aligned}G(z,s,\beta)&=\Gamma(z-1)\cos\left(\frac{\pi z}{4}\right)\zeta\left(\frac{z}{2}\right)\zeta(z-s)(2\beta)^{(1-z)/2}\\&=2^{z-s-\frac{1}{2}}\pi^{\frac{3}{2}z-s-2}\Gamma(z-1)\Gamma\left(1-\frac{z}{2}\right)\Gamma(1-z+s)\sin\left(\frac{\pi z}{2}\right)\\&\quad\times\sin\left(\frac{\pi z}{2}-\frac{\pi s}{2}\right)\zeta\left(1-\frac{z}{2}\right)\zeta(1-z+s),\end{aligned} \tag{10.3}$$

and where we used the functional equation of $\zeta(s)$ given in (2.6). Since Re $z<0$ and $\sigma>0$,

$$\zeta\left(1-\frac{z}{2}\right)\zeta(1-z+s)=\sum_{n=1}^{\infty}\frac{\psi_{1-s}(n)}{n^{1-z/2}}. \tag{10.4}$$

Thus, (10.2), (10.3), and (10.4) imply that

$$\begin{aligned}W(s,\beta)&=\frac{\sqrt{\beta}}{\sqrt{2}}\zeta(-s)+\Gamma(s)\cos\left(\frac{\pi(1+s)}{4}\right)\zeta\left(\frac{1+s}{2}\right)(2\beta)^{-\frac{1}{2}s}\\&\quad+\frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(1-s)+2^{-s-\frac{1}{2}}\pi^{-s-2}\sqrt{\beta}\sum_{n=1}^{\infty}\frac{\psi_{1-s}(n)}{n}H\left(s,\frac{\beta}{n}\right),\end{aligned} \tag{10.5}$$

where

$$H\left(s,\frac{\beta}{n}\right)=\frac{1}{2\pi i}\int_{(\lambda)}\Gamma(z-1)\Gamma\left(1-\frac{z}{2}\right)\Gamma(1-z+s)$$

$$\times \sin\left(\frac{\pi z}{2}\right) \sin\left(\frac{\pi z}{2} - \frac{\pi s}{2}\right) \left(\frac{\beta}{4\pi^3 n}\right)^{-z/2} dz$$

$$= \frac{1}{2\pi i} \int_{(\lambda)} \frac{-2^{2z-3}\pi^{(3z+5)/2} (\beta/n)^{-z/2}}{\Gamma\left(\frac{3-z}{2}\right)\Gamma(z-s)} \cos\left(\frac{\pi z}{2}\right) \cos\left(\frac{\pi(z-s)}{2}\right) dz,$$

by routine simplification using (2.1)–(2.4).

To evaluate $H(s, \beta/n)$, we now move the line of integration to $+\infty$ and apply the residue theorem. In this process, we encounter the poles of the integrand at $z = 1$ and at $z = 2k+1+s, k \in \mathbb{N} \cup \{0\}$. The residues at these poles are

$$R_1(H) = \frac{\pi^3\sqrt{n}}{\beta\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)}$$

and

$$R_{2k+1+s}(H) = \frac{(-1)^{k+1}2^{4k+2s}\pi^{3k+4+3s/2}(\beta/n)^{-(k+(s+1)/2)}\Gamma\left(\frac{1}{2}s+k\right)}{\pi^2\Gamma(2k+1)}.$$

With the help of (2.5), it is easy to show that the integrals along the horizontal segments tend to zero as the height of the rectangular contour tends to ∞. Also, arguing similarly as in (4.8)–(4.11), we see that the integral along the shifted vertical line in the limit also equals zero. Thus,

$$H\left(s, \frac{\beta}{n}\right) = -R_1(H) - \sum_{k=0}^{\infty} R_{2k+1+s} \tag{10.6}$$

$$= \frac{-\pi^3\sqrt{n}}{\beta\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)} + 2^{2s}\pi^{\frac{3}{2}s+2}\left(\frac{n}{\beta}\right)^{\frac{s+1}{2}} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{s}{2}+k\right)}{(2k)!}\left(-\frac{16\pi^3 n}{\beta}\right)^k$$

$$= \frac{-\pi^3\sqrt{n}}{\beta\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)} + 2^{2s}\pi^{\frac{3}{2}s+2}\left(\frac{n}{\beta}\right)^{\frac{s+1}{2}} \Gamma\left(\frac{s}{2}\right) {}_1F_1\left(\frac{s}{2}; \frac{1}{2}; -\frac{4\pi^3 n}{\beta}\right),$$

where ${}_1F_1(a; c; z)$ is Kummer's confluent hypergeometric function.

A result similar to the one in Theorem 10.1 could be obtained when we replace the $-$ sign in the sine function by a $+$ sign.

The result now follows from (10.5) and (10.6), and from the fact that $\alpha\beta = 4\pi^3$. □

10.1. **Special Cases of Theorem 10.1.** When $s = 2m+1$, $m \in \mathbb{N} \cup \{0\}$, in Theorem 10.1, we obtain the following result.

Theorem 10.2. *For each non-negative integer m,*

$$\sum_{n=1}^{\infty} \frac{\psi_{2m+1}(n)}{\sqrt{n}} e^{-\sqrt{n\beta}} \sin\left(\frac{\pi}{4} - \sqrt{n\beta}\right)$$

$$= \frac{\sqrt{\beta}}{\sqrt{2}}\zeta(-1-2m) - (2m)!\sin\left(\frac{\pi m}{2}\right)\zeta(1+m)(2\beta)^{-\left(m+\frac{1}{2}\right)} + \frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(-2m)$$

$$+ 2^m\left(\frac{2\pi}{\beta}\right)^{m+\frac{1}{2}} \Gamma\left(m+\frac{1}{2}\right) \sum_{n=1}^{\infty} \psi_{-2m}(n) n^m {}_1F_1\left(m+\frac{1}{2}; \frac{1}{2}; -n\alpha\right). \tag{10.7}$$

When $m = 0$ in (10.7), or equivalently when $s = 1$ in Theorem 10.1, we obtain (1.16). When $s = 2m, m \in \mathbb{N} \cup \{0\}$, in Theorem 10.1, we obtain the following companion result.

Theorem 10.3. *For each non-negative integer* m,

$$\sum_{n=1}^{\infty}\frac{\psi_{2m}(n)}{\sqrt{n}}e^{-\sqrt{n\beta}}\sin\left(\frac{\pi}{4}-\sqrt{n\beta}\right)$$
$$=\frac{\sqrt{\beta}}{\sqrt{2}}\zeta(-2m)-\frac{(2m-1)!}{\sqrt{2}}\sin\left(\frac{\pi m}{2}\right)\zeta\left(\frac{1+2m}{2}\right)(2\beta)^{-m}+\frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(1-2m)$$
$$+2^{\frac{1}{2}-2m}\pi^{-2m}\sum_{n=1}^{\infty}\frac{\psi_{1-2m}(n)}{\sqrt{n}}\Big\{(-1)^{m+1}(2m)!$$
$$+2^{4m-1}\pi^{3m}\left(\frac{n}{\beta}\right)^{m}(m-1)!\,{}_1F_1\left(m;\frac{1}{2};-n\alpha\right)\Big\}.$$

10.2. **A Common Generalization of $\sigma_s(n)$ and $\psi_s(n)$.** Note that if we define $\Omega_s(m,n):=\sum_{j^m|n}j^s$ for $m\in\mathbb{Z}, m\geq 1$, then $\Omega_s(1,n)=\sigma_s(n)$ and $\Omega_s(2,n)=\psi_s(n)$. Thus, the series

$$\sum_{n=1}^{\infty}\frac{\Omega_s(m,n)}{\sqrt{n}}e^{-\sqrt{n\beta}}\sin\left(\frac{\pi}{4}-\sqrt{n\beta}\right) \tag{10.8}$$

corresponds to the one in Theorem 1.3 when $m=1$, and to the one in Theorem 10.1 when $m=2$. It might be interesting to find a representation of (10.8) analogous to those in these theorems for general m.

Attempting to evaluate (10.8) by first writing the sum as a line integral, then shifting the line of integration to infinity and using the functional equation (2.6) of $\zeta(s)$, we encounter the sum of residues at the poles $2(2k+1+s)/m$ to be a constant (independent of k but depending on m, s, n, and β) times

$$\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{2k+1+s}{m}-\frac{1}{2}\right)}{(2k)!}\left(-\frac{2^{(2m+4)/m}\pi^{(2m+2)/m}n^{2/m}}{\beta^{2/m}}\right)^{k}.$$

When $m=2$, this series essentially reduces to the ${}_1F_1$ present in (10.1). If we let $\sqrt{\beta}=2\pi\sqrt{2x}$ and set $m=1$, we can write the series above as a ${}_3F_2$. However, we need to consider two cases, according as $n<x$ or as $n\geq x$, which should then give Theorem 1.3. But for a general positive integer m, it is doubtful that the series above is summable in closed form.

11. Ramanujan's Entries on Page 335 and Generalizations

We begin this section by stating the two entries on page 335 in Ramanujan's lost notebook [71]. Define

$$F(x)=\begin{cases}\lfloor x\rfloor, & \text{if } x \text{ is not an integer,}\\ x-\frac{1}{2}, & \text{if } x \text{ is an integer.}\end{cases} \tag{11.1}$$

Entry 11.1. *If* $0<\theta<1$ *and* $F(x)$ *is defined by* (11.1), *then*

$$\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\sin(2\pi n\theta)=\pi x\left(\frac{1}{2}-\theta\right)-\frac{1}{4}\cot(\pi\theta)$$
$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}-\frac{J_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\}, \tag{11.2}$$

where $J_\nu(x)$ *denotes the ordinary Bessel function of order* ν.

Entry 11.2. *If $0<\theta<1$ and $F(x)$ is defined by (11.1), then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta)) + \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} + \frac{I_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\}, \quad (11.3)$$

where

$$I_\nu(z) := -Y_\nu(z) + \frac{2}{\pi}\cos(\pi\nu)K_\nu(z), \quad (11.4)$$

where $Y_\nu(x)$ denotes the Bessel function of the second kind of order ν, and $K_\nu(x)$ denotes the modified Bessel function of order ν.

Entries 11.1 and 11.2 were established by Berndt, S. Kim, and Zaharescu under different conditions on the summation variables m, n in [14, 16, 18]. An expository account of their work along with a survey of the circle and divisor problems can be found in [17]. See also the book [4, Chapter 2] by Andrews and the first author.

It is easy to see from (11.1) that the left-hand sides of (11.2) and (11.3) are finite. When $x \to 0+$, Entries (11.2) and (11.3) give the following interesting limit evaluations:

$$\lim_{x\to 0+}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} - \frac{J_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\} = \frac{1}{2}\cot(\pi\theta),$$

and

$$\lim_{x\to 0+}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} + \frac{I_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\} = -\frac{1}{2}.$$

Direct proofs of these limit evaluations appear to be difficult.

As shown in [17, equation (2.8)], when $\theta = \frac{1}{4}$, Entry 11.1 is equivalent to the following famous identity due to Ramanujan and Hardy [45], provided that the double sum in (11.2) is interpreted as $\lim_{N\to\infty}\sum_{m,n\le N}$, rather than as an iterated double sum (see [16, p. 26]):

$$\sum_{0<n\le x}{}' r_2(n) = \pi x - 1 + \sum_{n=1}^{\infty} r_2(n)\left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}).$$

Note that the Bessel functions appearing in (11.3) are the same as those appearing in (6.1). Indeed when $\theta = \frac{1}{2}$, Entry 11.2 is connected with Voronoï's identity for $\sum_{n\le x} d(n)$ as will be shown below. First, following the elementary formula

$$\sum_{n\le x} d(n) = \sum_{n\le x}\sum_{d|n} 1 = \sum_{dj\le x} 1 = \sum_{d\le x}\left[\frac{x}{d}\right],$$

we see that the left-hand side of (11.3), for $\theta = \frac{1}{2}$, can be simplified as

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(\pi n) = \sum_{n\le x}{}'\sum_{d|n}\cos(\pi d).$$

Second, let

$$\ell = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.}\end{cases}$$

Note that

$$\begin{aligned}\sum_{d|n}\cos(\pi d) &= \#\text{ even divisors of } n - \#\text{ odd divisors of } n\\ &= d\left(\frac{n}{2}\right) - \left\{d(n) - \ell d\left(\frac{n}{2}\right)\right\}\\ &= (1+\ell)d\left(\frac{n}{2}\right) - d(n).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(\pi n) &= -\sum_{\substack{n\le x\\ n \text{ odd}}}{}' d(n) + \sum_{\substack{n\le x\\ n \text{ even}}}{}' \left\{2d\left(\frac{n}{2}\right) - d(n)\right\}\\ &= 2\sum_{n\le \frac{1}{2}x}{}' d(n) - \sum_{n\le x}{}' d(n).\end{aligned}$$

Applying the Voronoï summation formula (6.1) to each of the sums above and simplifying, we find that

$$\begin{aligned}\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(\pi n) &= -x\log 2 + \frac{1}{4} - \sqrt{2x}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\left(Y_1(2\pi\sqrt{2nx}) + \frac{2}{\pi}K_1(2\pi\sqrt{2nx})\right)\\ &\quad + \sqrt{x}\sum_{n=1}^{\infty}\frac{d(n/2)}{\sqrt{n/2}}\left(Y_1(2\pi\sqrt{2nx}) + \frac{2}{\pi}K_1(2\pi\sqrt{2nx})\right)\\ &= -x\log 2 + \frac{1}{4} - \sqrt{2x}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\Bigg(\sum_{\substack{d|k\\ d \text{ odd}}} 1\Bigg)\left(Y_1(2\pi\sqrt{2kx}) + \frac{2}{\pi}K_1(2\pi\sqrt{2kx})\right).\end{aligned}$$

Letting $k = m(2n+1)$ and interpreting the double sum as $\lim_{N\to\infty}\sum_{m,n\le N}$, we deduce that

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(\pi n) = -x\log 2 + \frac{1}{4} + \sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{I_1(4\pi\sqrt{m(n+\frac{1}{2})x})}{\sqrt{m(n+\frac{1}{2})}}, \tag{11.5}$$

where $I_1(z)$ is defined by (11.4). Then (11.5) is exactly Entry 11.2 with $\theta = \frac{1}{2}$.

It should be mentioned here that Dixon and Ferrar [32] established, for $a, b > 0$, the identity

$$a^{\mu/2}\sum_{n=0}^{\infty}\frac{r_2(n)}{(n+b)^{\mu/2}}K_{\mu}(2\pi\sqrt{a(n+b)}) = b^{(1-\mu)/2}\sum_{n=0}^{\infty}\frac{r_2(n)}{(n+a)^{(1-\mu)/2}}K_{1-\mu}(2\pi\sqrt{b(n+a)}). \tag{11.6}$$

Generalizations have been given by Berndt [7, p. 343, Theorem 9.1] and F. Oberhettinger and K. Soni [65, p. 24]. Using Jacobi's identity

$$r_2(n) = 4\sum_{\substack{d|n\\ d\,\text{odd}}}(-1)^{(d-1)/2},$$

we can recast (11.6) as an identity between double series

$$a^{\mu/2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{K_{\mu}\left(4\pi\sqrt{a\left(\left(n+\frac{1}{4}\right)m+\frac{b}{4}\right)}\right)}{((4n+1)m+b)^{\mu/2}}-\frac{K_{\mu}\left(4\pi\sqrt{a\left(\left(n+\frac{3}{4}\right)m+\frac{b}{4}\right)}\right)}{((4n+3)m+b)^{\mu/2}}\right\}$$
$$=b^{(1-\mu)/2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{K_{1-\mu}\left(4\pi\sqrt{b\left(\left(n+\frac{1}{4}\right)m+\frac{a}{4}\right)}\right)}{((4n+1)m+a)^{(1-\mu)/2}}-\frac{K_{1-\mu}\left(4\pi\sqrt{b\left(\left(n+\frac{3}{4}\right)m+\frac{a}{4}\right)}\right)}{((4n+3)m+a)^{(1-\mu)/2}}\right\}.$$

In this section, we establish one-variable generalizations of Entries 11.1 and 11.2, where the double sums here are also interpreted as $\lim_{N\to\infty}\sum_{m,n\leq N}$, instead of as iterated double sums. It is an open problem to determine if the series can be replaced by iterated double series. As in Entries 11.1 and 11.2, the series on the left-hand sides of Theorems 11.3 and 11.4 are finite.

Theorem 11.3. *Let $\zeta(s,a)$ denote the Hurwitz zeta function. Let $0<\theta<1$. Then, for $|\sigma|<\frac{1}{2}$,*

$$\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\sin(2\pi n\theta)}{n^s}=-x\frac{\sin(\pi s/2)\Gamma(-s)}{(2\pi)^{-s}}(\zeta(-s,\theta)-\zeta(-s,1-\theta)) \tag{11.7}$$
$$-\frac{\cos(\pi s/2)\Gamma(1-s)}{2(2\pi)^{1-s}}(\zeta(1-s,\theta)-\zeta(1-s,1-\theta))+\frac{x}{2}\sin\left(\frac{\pi s}{2}\right)$$
$$\times\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{M_{1-s}\left(4\pi\sqrt{mx\,(n+\theta)}\right)}{(mx)^{(1+s)/2}(n+\theta)^{(1-s)/2}}-\frac{M_{1-s}\left(4\pi\sqrt{mx\,(n+1-\theta)}\right)}{(mx)^{(1+s)/2}(n+1-\theta)^{(1-s)/2}}\right\},$$

where

$$M_{\nu}(x)=\frac{2}{\pi}K_{\nu}(x)+\frac{1}{\sin(\pi\nu)}(J_{\nu}(x)-J_{-\nu}(x))=\frac{2}{\pi}K_{\nu}(x)+Y_{\nu}(x)+J_{\nu}(x)\tan\left(\frac{\pi\nu}{2}\right). \tag{11.8}$$

We show that Entry 11.1 is identical with Theorem 11.3 when $s=0$. First observe that [5, p. 264, Theorem 12.13]

$$\zeta(0,\theta)=\frac{1}{2}-\theta \tag{11.9}$$

and

$$\lim_{s\to0}(\zeta(1-s,\theta)-\zeta(1-s,1-\theta))=\psi(1-\theta)-\psi(\theta)=\pi\cot(\pi\theta),$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ denotes the digamma function. Since, by (2.14), $J_{-1}(x)=-J_1(x)$,

$$\lim_{s\to0}\sin(\pi s/2)M_{1-s}(x)=J_1(x). \tag{11.10}$$

Now taking the limit as $s\to0$ on both sides of (11.7) and using (11.9)–(11.10), we obtain Entry 11.1.

Theorem 11.4. *Let $0<\theta<1$. Then, for $|\sigma|<\frac{1}{2}$,*

$$\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\cos(2\pi n\theta)}{n^s}=x\frac{\cos(\pi s/2)\Gamma(-s)}{(2\pi)^{-s}}(\zeta(-s,\theta)+\zeta(-s,1-\theta)) \tag{11.11}$$
$$-\frac{\sin(\frac{1}{2}\pi s)\Gamma(1-s)}{2(2\pi)^{1-s}}(\zeta(1-s,\theta)+\zeta(1-s,1-\theta))-\frac{x}{2}\cos\left(\frac{\pi s}{2}\right)$$

$$\times \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{H_{1-s}\left(4\pi\sqrt{mx\,(n+\theta)}\right)}{(mx)^{\frac{1+s}{2}}(n+\theta)^{\frac{1-s}{2}}}+\frac{H_{1-s}\left(4\pi\sqrt{mx\,(n+1-\theta)}\right)}{(mx)^{\frac{1+s}{2}}(n+1-\theta)^{\frac{1-s}{2}}}\right\},$$

where

$$H_\nu(x)=\frac{2}{\pi}K_\nu(x)-\frac{1}{\sin(\pi\nu)}(J_\nu(x)+J_{-\nu}(x))=\frac{2}{\pi}K_\nu(x)+Y_\nu(x)-J_\nu(x)\cot\left(\frac{\pi\nu}{2}\right). \quad (11.12)$$

We demonstrate that Entry 11.2 can be obtained from Theorem 11.4 as the particular case $s=0$. First,

$$\begin{aligned}\lim_{s\to 0}\Gamma(-s)(\zeta(-s,\theta)+\zeta(-s,1-\theta)) &= \lim_{s\to 0}(-s)\Gamma(-s)\frac{(\zeta(-s,\theta)+\zeta(-s,1-\theta))}{-s}\\ &=\zeta'(0,\theta)+\zeta'(0,1-\theta)\\ &=-\log(2\sin(\pi\theta)), \end{aligned} \quad (11.13)$$

where we used the fact that $\zeta'(0,\theta)=\log(\Gamma(\theta))-\frac{1}{2}\log(2\pi)$ [11]. Second, since $s=1$ is a simple pole of $\zeta(s,\theta)$ with residue 1, then

$$\begin{aligned}&\lim_{s\to 0}\sin(\pi s/2)(\zeta(1-s,\theta)+\zeta(1-s,1-\theta))\\ &\qquad=\lim_{s\to 0}\frac{\sin(\pi s/2)}{s}s(\zeta(1-s,\theta)+\zeta(1-s,1-\theta))=-\pi.\end{aligned}$$

Third, by (2.15),

$$\lim_{s\to 0}\frac{1}{2\sin(\pi s/2)}(J_{1-s}(x)+J_{s-1}(x))=-Y_1(x). \quad (11.14)$$

Taking the limit as $s\to 0$ in (11.11) while using (11.13)–(11.14), we obtain Entry 11.2.

12. Further Preliminary Results

Let us define the generalized twisted divisor sum by

$$\sigma_s(\chi,n):=\sum_{d|n}\chi(d)d^s, \quad (12.1)$$

which, for Re $z>\max\{1,1+\sigma\}$, has the generating function

$$\zeta(z)L(z-s,\chi)=\sum_{n=1}^{\infty}\frac{\sigma_s(\chi,n)}{n^z}.$$

The following lemma from the papers of Voronoï [82] and Oppenheim [67] is instrumental in proving our main theorems.

Lemma 12.1. *If $x>0$, $x\notin\mathbb{Z}$, and $-\frac{1}{2}<\sigma<\frac{1}{2}$, then*

$$\sideset{}{'}\sum_{n\le x}\sigma_{-s}(n)=-\cos(\tfrac{1}{2}\pi s)\sum_{n=1}^{\infty}\sigma_{-s}(n)\left(\frac{x}{n}\right)^{(1-s)/2}H_{1-s}\left(4\pi\sqrt{nx}\right)+xZ(s,x)-\frac{1}{2}\zeta(s), \quad (12.2)$$

where $H_\nu(x)$ is defined in (11.12), *and where*

$$Z(s,x)=\begin{cases}\zeta(1+s)+\dfrac{\zeta(1-s)}{1-s}x^{-s}, & \text{if } s\neq 0,\\ \log x+2\gamma-1, & \text{if } s=0,\end{cases} \quad (12.3)$$

is analytic for all s.

We show that (12.2) reduces to Voronoï's formula (6.1) when $s = 0$. From the definition (11.12) of H_ν and (11.14), we find that

$$H_1(4\pi\sqrt{nx}) = Y_1(4\pi\sqrt{nx}) + \frac{2}{\pi}K_1(4\pi\sqrt{nx}) = -I_1(4\pi\sqrt{nx}).$$

We now show that

$$\lim_{s\to 0} Z(s,x) = \log x + 2\gamma - 1 = Z(0,x). \tag{12.4}$$

Recall that the Laurent series expansion of $\zeta(s)$ near the pole $s = 1$ is given by

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \frac{(-1)^n \gamma_n (s-1)^n}{n!},$$

where γ_n, $n \geq 1$, are the Stieltjes constants defined by [10]

$$\gamma_n = \lim_{N\to\infty} \left(\sum_{k=1}^{N} \frac{\log^n k}{k} - \frac{\log^{n+1} N}{n+1} \right). \tag{12.5}$$

Thus, by (12.3), for $s > 0$,

$$Z(s,x) = \frac{s-1+x^{-s}}{s(s-1)} + \gamma\left(1 - \frac{x^{-s}}{s-1}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n \gamma_n s^n}{n!} + \frac{x^{-s}}{1-s} \sum_{n=1}^{\infty} \frac{\gamma_n s^n}{n!}.$$

Hence,

$$\lim_{s\to 0} Z(s,x) = \lim_{s\to 0} \frac{s-1+x^{-s}}{s(s-1)} + 2\gamma = -\lim_{s\to 0}(1 - \log x x^{-s}) + 2\gamma = \log x + 2\gamma - 1,$$

which proves (12.4).

Lemma 12.2. *Let $F(x)$ be defined by (11.1). For each character χ modulo q, where q is prime, define the Gauss sum*

$$\tau(\chi) = \sum_{n \,(\mathrm{mod}\, q)} \chi(n) e^{2\pi i n/q}. \tag{12.6}$$

If $0 < a < q$ and $(a,q) = 1$, then, for any complex number s,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin\left(\frac{2\pi n a}{q}\right) n^s = -iq^s \sum_{\substack{d|q \\ d>1}} \frac{1}{d^s \phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi(a)\tau(\bar{\chi}) {\sum_{1\leq n\leq dx/q}}' \sigma_s(\chi, n),$$

where $\phi(n)$ denotes Euler's ϕ-function.

Proof. First, we see that

$$\sum_{n\leq x}{}' \sigma_s(n) = \sum_{n\leq x}{}' \sum_{d|n} d^s = \sum_{d\leq x} d^s \sum_{m=1}^{\lfloor x/d \rfloor}{}' 1 = \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) n^s. \tag{12.7}$$

Similarly, for any Dirichlet character χ modulo q,

$$\sum_{n\leq x}{}' \sigma_s(\chi, n) = \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \chi(n) n^s, \tag{12.8}$$

where $\sigma_s(\chi,n)$ is defined in (12.1). We have

$$\begin{aligned}\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi na}{q}\right)n^s&=\sum_{n=1}^{\infty}\sum_{d|q}\sum_{(n,q)=q/d}F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi na}{q}\right)n^s\\&=\sum_{d|q}\sum_{\substack{m=1\\(m,d)=1}}^{\infty}F\left(\frac{dx}{qm}\right)\sin\left(\frac{2\pi ma}{d}\right)\left(\frac{qm}{d}\right)^s\\&=\sum_{\substack{d|q\\d>1}}\sum_{\substack{m=1\\(m,d)=1}}^{\infty}F\left(\frac{dx}{qm}\right)\sin\left(\frac{2\pi ma}{d}\right)\left(\frac{qm}{d}\right)^s.\end{aligned}$$

Now using the fact [15, p. 72, Lemma 2.5]

$$\sin\left(\frac{2\pi ma}{d}\right)=\frac{1}{i\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi\text{ odd}}}\chi(a)\tau(\bar{\chi})\chi(m), \tag{12.9}$$

we find that

$$\begin{aligned}&\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi na}{q}\right)n^s\\&=\sum_{\substack{d|q\\d>1}}\frac{1}{i\phi(d)}\sum_{\substack{m=1\\(m,d)=1}}^{\infty}F\left(\frac{dx}{qm}\right)\left(\frac{qm}{d}\right)^s\sum_{\substack{\chi \bmod d\\ \chi\text{ odd}}}\tau(\bar{\chi})\chi(m)\chi(a)\\&=-iq^s\sum_{\substack{d|q\\d>1}}\frac{1}{d^s\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi\text{ odd}}}\tau(\bar{\chi})\chi(a)\sideset{}{'}\sum_{n\le dx/q}\sigma_s(\chi,n),\end{aligned}$$

as can be seen from (12.8). This completes the proof of Lemma 12.2. ∎

Lemma 12.3. *If $0<a<q$ and $(a,q)=1$, then, for any complex number s,*

$$\begin{aligned}&\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\cos\left(\frac{2\pi na}{q}\right)n^s\\&\qquad=q^s\sideset{}{'}\sum_{1\le n\le x/q}\sigma_s(n)+q^s\sum_{\substack{d|q\\d>1}}\frac{1}{d^s\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi\text{ even}}}\chi(a)\tau(\bar{\chi})\sideset{}{'}\sum_{1\le n\le dx/q}\sigma_s(\chi,n).\end{aligned}$$

Proof. We have

$$\begin{aligned}\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\cos\left(\frac{2\pi na}{q}\right)n^s&=\sum_{n=1}^{\infty}\sum_{d|q}\sum_{(n,q)=q/d}F\left(\frac{x}{n}\right)\cos\left(\frac{2\pi na}{q}\right)n^s\\&=\sum_{d|q}\sum_{\substack{m=1\\(m,d)=1}}^{\infty}F\left(\frac{dx}{qm}\right)\cos\left(\frac{2\pi ma}{d}\right)\left(\frac{qm}{d}\right)^s\\&=\sum_{m=1}^{\infty}F\left(\frac{x}{qm}\right)(qm)^s+\sum_{\substack{d|q\\d>1}}\sum_{\substack{m=1\\(m,d)=1}}^{\infty}F\left(\frac{dx}{qm}\right)\cos\left(\frac{2\pi na}{d}\right)\left(\frac{qm}{d}\right)^s.\end{aligned}$$

Invoking (12.7) and (12.9) above, we find that

$$\begin{aligned}\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos\left(\frac{2\pi na}{q}\right) n^s &= q^s {\sum_{n\leq x/q}}' \sigma_s(n)\\ &\quad + q^s \sum_{\substack{d|q\\ d>1}} \frac{1}{d^s\phi(d)} \sum_{\substack{m=1\\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) m^s \sum_{\substack{\chi \bmod d\\ \chi \text{ even}}} \tau(\bar{\chi})\chi(a)\chi(m)\\ &= q^s {\sum_{n\leq x/q}}' \sigma_s(n) + q^s \sum_{\substack{d|q\\ d>1}} \frac{1}{d^s\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi \text{ even}}} \tau(\bar{\chi})\chi(a) {\sum_{n\leq dx/q}}' \sigma_s(\chi, n).\end{aligned}$$

Thus, we have finished the proof of Lemma 12.3. □

We need a lemma from [21, p. 5, Lemma 1].

Lemma 12.4. *Let σ_a denote the abscissa of absolute convergence for*

$$\phi(s) := \sum_{n=1}^{\infty} a_n \lambda_n^{-s}.$$

Then for $k \geq 0$, $\sigma > 0$, and $\sigma > \sigma_a$,

$$\frac{1}{\Gamma(k+1)} {\sum_{\lambda_n \leq x}}' a_n (x-\lambda_n)^k = \frac{1}{2\pi i}\int_{(\sigma)} \frac{\Gamma(s)\phi(s)x^{s+k}}{\Gamma(s+k+1)}\, ds,$$

where the prime $\prime$ on the summation sign indicates that if $k = 0$ and $x = \lambda_m$ for some positive integer m, then we count only $\frac{1}{2}a_m$.

We recall the following version of the Phragmén-Lindelöf theorem [61, p. 109].

Lemma 12.5. *Let f be holomorphic in a strip S given by $a < \sigma < b$, $|t| > \eta > 0$, and continuous on the boundary. If for some constant $\theta < 1$,*

$$f(s) \ll \exp(e^{\theta\pi|s|/(b-a)}),$$

uniformly in S, $f(a+it) = o(1)$, and $f(b+it) = o(1)$ as $|t| \to \infty$, then $f(\sigma+it) = o(1)$ uniformly in S as $|t| \to \infty$.

We also need two lemmas, proven by K. Chandrasekharan and R. Narasimhan [21, Corollaries 1 and 2, p. 11] (see also [6, Lemmas 12 and 13]), which are based on results of A. Zygmund [88] for equi-convergent series. We recall that two series

$$\sum_{j=-\infty}^{\infty} a_j(x) \quad \text{and} \quad \sum_{j=-\infty}^{\infty} b_j(x)$$

are uniformly equi-convergent on an interval if

$$\sum_{j=-n}^{n} [a_j(x) - b_j(x)]$$

converges uniformly on that interval as $n \to \infty$ [6, Definition 5].

Lemma 12.6. *Let a_n be a positive strictly increasing sequence of numbers tending to ∞, and suppose that $a_n = a_{-n}$. Suppose that J is a closed interval contained in an interval I of length 2π. Assume that*

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty.$$

Then, if g is a function with period 2π which equals

$$\sum_{n=-\infty}^{\infty} c_n e^{ia_n x}$$

on I, the Fourier series of g converges uniformly on J.

Lemma 12.7. *With the same notation as Lemma 12.6, assume that*

$$\sup_{0\leq h\leq 1}\left|\sum_{k<a_n<k+h} c_n\right| = o(1),$$

as $k \to \infty$, and

$$\sum_{n=-\infty}^{\infty} \frac{|c_n|}{a_n} < \infty.$$

Let $A(x)$ be a $\mathcal{C}^\infty$ function with compact support on I, which is equal to 1 on J. Furthermore, let $B(x)$ be a $\mathcal{C}^\infty$ function. Then, the series

$$B(x)\sum_{n=-\infty}^{\infty} c_n e^{ia_n x}$$

is uniformly equi-convergent on J with the differentiated series of the Fourier series of a function with period 2π, which equals

$$A(x)\sum_{n=-\infty}^{\infty} c(n) W_n(x)$$

on I, where $W_n(x)$ is an antiderivative of $B(x)e^{ia_n x}$.

Let the Fourier series of any function f defined, say, in the interval $(-\pi, \pi)$, be

$$S[f] := \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

The following result of Zygmund [89, Theorem 6.6, p. 53] expresses the Riemann-Lebesgue localization principle.

Lemma 12.8. *If two functions f_1 and f_2 are equal in an interval I, then $S[f_1]$ and $S[f_2]$ are uniformly equi-convergent in any interval I' interior to I.*

For each integer λ define

$$\tilde{G}_{\lambda+s}(z) := J_{\lambda+s}(z)\cos\left(\frac{\pi s}{2}\right) - \left(Y_{\lambda+s}(z) - (-1)^\lambda \frac{2}{\pi} K_{\lambda+s}(z)\right)\sin\left(\frac{\pi s}{2}\right). \tag{12.10}$$

By (8.3), (8.5), and (8.4),

$$\frac{d}{dx}\left(\frac{x}{u}\right)^{(1+k-s)/2}\sigma_s(n)\tilde{G}_{1+k-s}(4\pi\sqrt{xu}) = 2\pi\left(\frac{x}{u}\right)^{(k-s)/2}\sigma_s(n)\tilde{G}_{k-s}(4\pi\sqrt{xu}). \tag{12.11}$$

Let us consider the Dirichlet series $\sum_{n=1}^{\infty} a_n \mu_n^{-s}$ with abscissa of absolute convergence σ_a and $0 < \mu_1 < \mu_2 < \cdots < \mu_n \to \infty$. For $y > 0$ and $\nu = \lambda + s$, define

$$\tilde{F}_\nu(y) := \sum_{n=1}^{\infty} a_n \left(\frac{qy^2}{\mu_n}\right)^{\nu/2} \tilde{G}_\nu\left(4\pi y\sqrt{\frac{\mu_n}{q}}\right)$$

and

$$F_\nu(y) := \sum_{n=1}^{\infty} a_n \left(\frac{qy^2}{\mu_n}\right)^{\nu/2} G_\nu\left(4\pi y\sqrt{\frac{\mu_n}{q}}\right),$$

where $G_{\lambda+s}(z)$ is defined in (8.1). Suppose that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{\mu_n^{\frac{1}{2}\nu+\frac{3}{4}}} < \infty \tag{12.12}$$

and

$$\sup_{0\leq h\leq 1}\left|\sum_{m^2<\mu_n\leq(m+h)^2} \frac{a_n}{\mu_n^{\frac{1}{2}\nu+\frac{1}{4}}}\right| = o(1), \tag{12.13}$$

as $m \to \infty$.

The following lemma is similar to Theorem II in [21] and Lemma 14 in [6].

Lemma 12.9. *The function $2y\tilde{F}_\nu(y)$ is uniformly equi-convergent on any interval J of length less than 1 with the differentiated series of the Fourier series of a function with period 1, which on I equals $A(y)\tilde{F}_{\nu+1}(y)$, where I is of length 1 and contains J. Moreover, $\tilde{F}_\nu(y)$ is a continuous function.*

Proof. We examine the function

$$\begin{aligned} f(y) := 2q^{\nu/2}y^{1+\nu}\sum_{n=1}^{\infty}\left(\frac{a_n}{\mu_n}\right)^{\nu/2}&\left\{\tilde{G}_\nu\left(4\pi y\sqrt{\frac{\mu_n}{q}}\right)\right. \\ -\frac{q^{1/4}}{\pi\mu_n^{1/4}(2y)^{1/2}}&\left(\cos\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0+\sin\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0'\right) \\ -\frac{q^{3/4}}{2\pi^2\mu_n^{3/4}y^{3/2}}&\left.\left(\sin\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_1+\cos\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_1'\right)\right\}, \end{aligned} \tag{12.14}$$

where d_0, d_0', d_1, and d_1' are constants. Since $y > 0$, then by the definition (12.10), (2.16), (2.17), (2.18), and (12.12), the function $f(y)$ in (12.14) is a continuously differentiable function. Let g be a function with period 1 which equals f on I. Since f is continuously differentiable, the Fourier series of g is uniformly convergent on J. By the hypothesis (12.12), (12.13), and Lemma 12.7, the series

$$\begin{aligned} 2q^{\nu/2}y^{1+\nu}\sum_{n=1}^{\infty}\left(\frac{a_n}{\mu_n}\right)^{\nu/2}&\frac{q^{1/4}}{\pi\mu_n^{1/4}(2y)^{1/2}} \\ \times&\left(\cos\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0+\sin\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0'\right) \end{aligned}$$

is uniformly equi-convergent on J with the derived series of the Fourier series of a function that is of period 1 and equals on I,

$$\begin{aligned} A(y) \sum_{n=1}^{\infty} \left(\frac{a_n}{\mu_n}\right)^{\nu/2} \int_{\alpha}^{y} 2q^{\nu/2} t^{1+\nu} \frac{q^{1/4}}{\pi \mu_n^{1/4} (2t)^{1/2}} \\ \times \left(\cos\left(4\pi t \sqrt{\frac{\mu_n}{q}} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) d_0 + \sin\left(4\pi t \sqrt{\frac{\mu_n}{q}} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) d_0' \right) dt, \end{aligned} \tag{12.15}$$

for some $\alpha > 0$. Using Lemma 12.6, we can prove a result similar to that of (12.15) for the series

$$\begin{aligned} 2q^{\nu/2} y^{1+\nu} \sum_{n=1}^{\infty} \left(\frac{a_n}{\mu_n}\right)^{\nu/2} \frac{q^{3/4}}{2\pi^2 \mu_n^{3/4} (y)^{3/2}} \\ \times \left(\cos\left(4\pi y \sqrt{\frac{\mu_n}{q}} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) d_0 + \sin\left(4\pi y \sqrt{\frac{\mu_n}{q}} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) d_0' \right). \end{aligned}$$

Hence, the series

$$2y \sum_{n=1}^{\infty} a_n \left(\frac{qy^2}{\mu_n}\right)^{\nu/2} \tilde{G}_\nu\left(4\pi y \sqrt{\frac{\mu_n}{q}}\right)$$

is uniformly equi-convergent on J with the derived series of the Fourier series of a function that is of period 1 and equals on I,

$$\begin{aligned} A(y) \sum_{n=1}^{\infty} a_n \int_0^y 2t \left(\frac{qt^2}{\mu_n}\right)^{\nu/2} \tilde{G}_\nu\left(4\pi t \sqrt{\frac{\mu_n}{q}}\right) dt \\ = \frac{A(y)}{2\pi} \sum_{n=1}^{\infty} a_n \left(\frac{qy^2}{\mu_n}\right)^{(\nu+1)/2} \tilde{G}_{\nu+1}\left(4\pi y \sqrt{\frac{\mu_n}{q}}\right). \end{aligned}$$

In the last step we use (12.11). This completes the proof of the lemma.

□

The following lemma is proved by the same kind of argument.

Lemma 12.10. *The function $2yF_\nu(y)$ is uniformly equi-convergent on any interval J of length less than* 1 *with the differentiated series of the Fourier series of a function with period* 1*, which on I equals $A(y)F_{\nu+1}(y)$, where I is of length* 1 *and contains J. Moreover, $F_\nu(y)$ is a continuous function.*

13. Proof of Theorem 11.3

We prove the theorem under the assumption that the double series on the right-hand sides of (11.7) and (11.11) are summed symmetrically, i.e., the product mn of the indices of summation tends to ∞. Under this assumption, we prove that the double series in (11.7) and (11.11) are uniformly convergent with respect to θ on any compact subinterval of $(0, 1)$. By continuity, it is sufficient to prove the theorem for all primes q and all fractions $\theta = a/q$, where $0 < a < q$. Therefore for these values of θ, Theorem 11.3 is equivalent to the following theorem.

Theorem 13.1. *Recall that M_ν is defined in* (11.8)*. Let q be a prime and $0 < a < q$. Let*

$$L_s(a, q, x) = -\frac{x}{2} \sin\left(\frac{\pi s}{2}\right) \tag{13.1}$$

$$\times \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{M_{1-s}\left(4\pi\sqrt{mx\,(n+a/q)}\right)}{(mx)^{(1+s)/2}(n+a/q)^{(1-s)/2}}-\frac{M_{1-s}\left(4\pi\sqrt{mx\,(n+1-a/q)}\right)}{(mx)^{(1+s)/2}(n+1-a/q)^{(1-s)/2}}\right\},$$

where $M_s(z)$ is defined in (11.8)*. Then, for $|\sigma|<\frac{1}{2}$,*

$$L_s(a,q,x)+\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\sin(2\pi na/q)}{n^s}=-x\frac{\sin(\pi s/2)\Gamma(-s)}{(2\pi)^{-s}}\left(\zeta(-s,\frac{a}{q})-\zeta\left(-s,1-\frac{a}{q}\right)\right)$$
$$-\frac{\cos(\pi s/2)\Gamma(1-s)}{2(2\pi)^{1-s}}\left(\zeta\left(1-s,\frac{a}{q}\right)-\zeta\left(1-s,1-\frac{a}{q}\right)\right),$$

where $\zeta(s,a)$ denotes the Hurwitz zeta function.

First we need the following theorem.

Theorem 13.2. *If χ is a non-principal odd primitive character modulo q, $x>0$, $|\sigma|<1/2$, and k is a non-negative integer, then*

$$\frac{1}{\Gamma(k+1)}\sideset{}{'}\sum_{n\le x}\sigma_{-s}(\chi,n)(x-n)^k$$
$$=\frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)}-\frac{x^kL(s,\chi)}{2\Gamma(k+1)}+2\sum_{n=1}^{\lfloor\frac{k+1}{2}\rfloor}\frac{(-1)^{n-1}x^{k-2n+1}}{\Gamma(k-2n+2)}\frac{\zeta(2n)}{(2\pi)^{2n}}L(1-2n+s,\chi)$$
$$+\frac{i}{\tau(\bar{\chi})(2\pi)^k}\sum_{n=1}^{\infty}\sigma_{-s}(\bar{\chi},n)\left(\frac{qx}{n}\right)^{(1-s+k)/2}\tilde{G}_{1-s+k}\left(4\pi\sqrt{\frac{nx}{q}}\right),$$

where $\tilde{G}_{\lambda-s}(z)$ is defined in (12.10)*. The series on the right-hand side converges uniformly on any interval for $x>0$, where the left-hand side is continuous. The convergence is bounded on any interval $0<x_1\le x\le x_2<\infty$ when $k=0$.*

Proof. From (12.1) and Lemma 12.4, for a fixed $x>0$, we see that

$$\frac{1}{\Gamma(k+1)}\sideset{}{'}\sum_{n\le x}\sigma_{-s}(\chi,n)(x-n)^k=\frac{1}{2\pi i}\int_{(c)}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}dw, \tag{13.2}$$

where $\max\{1,1-\sigma,\sigma\}<c$ and $k\ge 0$. Consider the positively oriented rectangular contour R with vertices $[c\pm iT,1-c\pm iT]$. Observe that the integrand on the right-hand side of (13.2) has poles at $w=1$ and $w=0$ inside the contour R. By the residue theorem,

$$\frac{1}{2\pi i}\int_R\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}dw \tag{13.3}$$
$$=R_1\left(\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\right)+R_0\left(\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\right),$$

where we recall that $R_a(f(w))$ denotes the residue of the function $f(w)$ at the pole $w=a$. Straightforward computations show that

$$R_0\left(\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\right)=\frac{\zeta(0)L(s,\chi)x^{1+k}}{\Gamma(k+1)} \tag{13.4}$$

and

$$R_1\left(\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\right)=\frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)}. \tag{13.5}$$

We show that the contribution from the integrals along the horizontal sides $(\sigma \pm iT, 1-c \leq \sigma \leq c)$ on the left-hand side of (13.3) tends to zero as $|t| \to \infty$. We prove this fact by showing that

$$\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} = o(1),$$

as $|\text{Im } w| \to \infty$, uniformly for $1-c \leq \text{ Re } w \leq c$. The functional equation for $L(s,\chi)$ for an odd primitive Dirichlet character χ is given by [28, p. 69]

$$\left(\frac{\pi}{q}\right)^{-(1+s)/2}\Gamma\left(\frac{1+s}{2}\right)L(s,\chi) = \frac{i\tau(\chi)}{\sqrt{q}}\left(\frac{\pi}{q}\right)^{-(2-s)/2}\Gamma\left(\frac{2-s}{2}\right)L(1-s,\overline{\chi}), \tag{13.6}$$

where $\tau(\chi)$ is the Gauss sum defined in (12.6). Combining the functional equation (2.6) of $\zeta(w)$ and the functional equation (13.6) of $L(w+s,\chi)$ for odd primitive χ, we deduce the functional equation

$$\zeta(w)L(w+s,\chi) = \frac{i\pi^{2w+s-1}}{\tau(\bar{\chi})q^{w+s-1}}\eta(w,s)\zeta(1-w)L(1-w-s,\bar{\chi}), \tag{13.7}$$

where

$$\eta(w,s) = \frac{\Gamma\left(\frac{1}{2}(1-w)\right)\Gamma\left(\frac{1}{2}(2-w-s)\right)}{\Gamma\left(\frac{1}{2}w\right)\Gamma\left(\frac{1}{2}(1+w+s)\right)}.$$

Since $c > \max\{1, 1-\sigma, \sigma\}$,

$$\zeta(c+it)L(c+it+s,\chi) = O(1),$$

as $|t| \to \infty$. Using (2.5), we see that

$$\frac{\Gamma(w)}{\Gamma(w+k+1)} = O(|\text{Im } w|^{-1-k}), \tag{13.8}$$

uniformly in $1-c \leq \text{ Re } w \leq c$, as $|\text{Im } w| \to \infty$. Therefore, for $w = c+it$,

$$\zeta(w)L(w,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} = o(1), \tag{13.9}$$

as $|t| \to \infty$. Again, using Stirling's formula (2.5) for the Gamma function and the relation (13.7), we find that, for $w = 1-c+it$,

$$\begin{aligned}
&\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)},\\
&= \frac{i\pi^{2w+s-1}}{\tau(\bar{\chi})q^{w+s-1}}\eta(w,s)\zeta(1-w)L(1-w-s,\bar{\chi})\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\\
&= O_{q,s}(t^{2c-\sigma-k-2})\\
&= o(1),
\end{aligned} \tag{13.10}$$

as $|t| \to \infty$, provided that $k > 2c-\sigma-2$. From (13.8) and [28, pp. 79, 82, equations (2),(15)],

$$\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} \ll_q \exp\left(C|w|\log|w|\right), \tag{13.11}$$

for some constant C and $|\text{Im } w| \to \infty$. Since the function on the left-hand side of (13.11) is holomorphic for $|\text{Im } w| > \eta' > 0$, then, by using (13.9), (13.10), (13.11), and Lemma 12.5, we deduce that

$$\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}=o(1),$$

uniformly for $1-c\leq$ Re $w\leq c$ and $|\text{Im } w|\to\infty$. Therefore,

$$\int_{c\pm iT}^{1-c\pm iT}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}dw=o(1), \tag{13.12}$$

as $T\to\infty$. Using the evaluation $\zeta(0)=-\frac{1}{2}$ and combining (13.2), (13.3), (13.4), (13.5), and (13.12), we deduce that

$$\begin{aligned}\frac{1}{\Gamma(k+1)}\sideset{}{'}\sum_{n\leq x}\sigma_{-s}(\chi,n)(x-n)^k&=\frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)}-\frac{L(s,\chi)x^k}{2\Gamma(k+1)}\\&\quad+\frac{1}{2\pi i}\int_{(1-c)}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}dw, \end{aligned}\tag{13.13}$$

provided that $k\geq 0$ and $k>2c-\sigma-2$. Define

$$I(y):=\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}y^w\,dw. \tag{13.14}$$

Using the functional equation (13.7) in the integrand on the right-hand side of (13.13) and inverting the order of summation and integration, we find that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(1-c)}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\,dw\\&=\frac{ix^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\zeta(1-w)L(1-w-s,\bar{\chi})\left(\frac{\pi^2x}{q}\right)^w\,dw\\&=\frac{ix^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\left(\frac{\pi^2x}{q}\right)^w\sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi},n)}{n^{1-w}}\,dw\\&=\frac{ix^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi},n)}{n^{1+k}}\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\left(\frac{\pi^2nx}{q}\right)^w\,dw\\&=\frac{ix^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi},n)}{n}I\left(\frac{\pi^2nx}{q}\right),\end{aligned}\tag{13.15}$$

provided that $k>2c-\sigma-1$. We compute the integral $I(y)$ by using the residue calculus, shifting the line of integration to the right, and letting $c\to-\infty$.

Let k be a positive integer and $\sigma\neq 0$. From (13.14), we can write

$$I(y):=\frac{1}{2\pi i}\int_{(1-c)}F(w)\,dw,$$

where

$$F(w):=\frac{\Gamma(w)\Gamma\left(\frac{1}{2}(1-w)\right)\Gamma\left(\frac{1}{2}(2-w-s)\right)y^w}{\Gamma(1+k+w)\Gamma\left(\frac{1}{2}w\right)\Gamma\left(\frac{1}{2}(1+w+s)\right)}.$$

Note that the poles of the function $F(w)$ on the right side of the line $1-c+it$, $-\infty<t<\infty$, are at $w=2m+1$ and $w=2m+2-s$ for $m=0,1,2,\dots$. Thus,

$$R_{2m+1}(F(w))=(-1)^{m+1}\frac{2\Gamma(2m+1)\Gamma\left(-m-\frac{1}{2}(s-1)\right)y^{2m+1}}{m!\Gamma(2+k+2m)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(1+m+\frac{1}{2}(s)\right)}$$

and

$$R_{2m+2-s}(F(w))=(-1)^{m+1}\frac{2\Gamma(2m+2-s)\Gamma\left(-m+\frac{1}{2}(s-1)\right)y^{2m+2-s}}{m!\Gamma(3+k+2m-s)\Gamma\left(m+\frac{1}{2}(2-s)\right)\Gamma\left(m+\frac{3}{2}\right)}.$$

With the aid of the duplication formula (2.4) and the reflection formula (2.2) for $\Gamma(s)$, we find that

$$R_{2m+1}(F(w))=-\frac{2^{s-1}}{\cos(\pi s/2)}\frac{(2\sqrt{y})^{4m+2}}{(2m+k+1)!\Gamma(2m+s+1)} \tag{13.16}$$

and

$$R_{2m+2-s}(F(w))=\frac{2(2y)^{2-s}}{\cos(\pi s/2)}\frac{(2\sqrt{y})^{4m}}{(2m+1)!\Gamma(2m+k+3-s)}. \tag{13.17}$$

Now from [83, pp. 77–78], we recall that the modified Bessel function $I_\nu(z)$ is defined by

$$I_\nu(z):=\sum_{m=0}^{\infty}\frac{(z/2)^{2m+\nu}}{m!\Gamma(m+1+\nu)}, \tag{13.18}$$

and that $K_\nu(z)$ can be represented as

$$K_\nu(z)=\frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\pi\nu)}. \tag{13.19}$$

(We emphasize that the definition of $I_\nu(z)$ given in (13.18) should not be confused with the definition of $I_\nu(z)$ given by Ramanujan in (11.4).) Therefore, from (2.14), (13.18), and (13.16), for k even,

$$\begin{aligned}\sum_{m=0}^{\infty}R_{2m+1}(F(w))&=-\frac{2^{s-1-2k}y^{-k}}{\cos(\pi s/2)}\sum_{m=0}^{\infty}\frac{(2\sqrt{y})^{4m+2k+2}}{(2m+k+1)!\Gamma(2m+1+s)}\\&=-\frac{2^{s-1-2k}y^{-k}}{\cos(\pi s/2)}\left\{\sum_{m=0}^{\infty}\frac{(2\sqrt{y})^{4m+2k+2}}{(2m+1)!\Gamma(2m+1+s-k)}\right.\\&\qquad\left.-\sum_{m=1}^{k/2}\frac{(2\sqrt{y})^{4m-2}}{(2m-1)!\Gamma(2m-1+s-k)}\right\}\\&=-\frac{2^{-1-k}y^{(1-s-k)/2}}{\cos(\pi s/2)}(I_{-1+s-k}(4\sqrt{y})-J_{-1+s-k}(4\sqrt{y}))\\&\qquad+\frac{2^{s-1-2k}y^{-k}}{\cos(\pi s/2)}\sum_{m=1}^{k/2}\frac{(2\sqrt{y})^{4m-2}}{(2m-1)!\Gamma(2m-1+s-k)}\\&=-\frac{2^{-1-k}y^{(1-s-k)/2}}{\cos(\pi s/2)}(I_{-1+s-k}(4\sqrt{y})-J_{-1+s-k}(4\sqrt{y}))\\&\qquad+\frac{2^{s+1}}{\cos(\pi s/2)}\sum_{m=1}^{k/2}\frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.\end{aligned} \tag{13.20}$$

Similarly, for k odd

$$\sum_{m=0}^{\infty} R_{2m+1}(F(w)) = -\frac{2^{-1-k}y^{(1-s-k)/2}}{\cos(\pi s/2)}(I_{-1+s-k}(4\sqrt{y}) + J_{-1+s-k}(4\sqrt{y})) + \frac{2^{s+1}}{\cos(\pi s/2)} \sum_{m=1}^{(k+1)/2} \frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}. \tag{13.21}$$

From (13.17), (2.14), and (13.18), we find that

$$\sum_{m=0}^{\infty} R_{2m+1-s}(F(w)) = \frac{2^{-1-k}y^{(1-s-k)/2}}{\cos(\pi s/2)}(-J_{1-s+k}(4\sqrt{y}) + I_{1-s+k}(4\sqrt{y})). \tag{13.22}$$

Invoking (13.19) in the sum of (13.20), (13.21), and (13.22), we deduce that

$$\sum_{m=0}^{\infty} (R_{2m+1}(F(w)) + R_{2m+1-s}(F(w))) = -\frac{\sin(\pi s/2)}{2^k y^{(-1+s+k)/2}} \times \left(\frac{J_{1-s+k}(4\sqrt{y}) + (-1)^{k+1}J_{-1+s-k}(4\sqrt{y})}{\sin \pi s} - (-1)^{k+1}\frac{2}{\pi}K_{1-s+k}(4\sqrt{y}) \right) + \frac{2^{s+1}}{\cos(\pi s/2)} \sum_{m=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.$$

Consider the positively oriented contour $\mathcal{R}_N$ formed by the points $\{1-c-iT, 2N+\frac{3}{2}-iT, 2N+\frac{3}{2}+iT, 1-c+iT\}$, where $T>0$ and N is a positive integer. By the residue theorem,

$$\frac{1}{2\pi i}\int_{\mathcal{R}_N} F(w)\,dw = \sum_{0\le k\le N} R_{2k+1}(F(w)) + \sum_{0\le k\le N} R_{2k+1-s}(F(w)). \tag{13.23}$$

Recall Stirling's formula in the form [28, p. 73, equation (5)]

$$\Gamma(s) = \sqrt{2\pi}e^{-s}s^{s-1/2}e^{f(s)},$$

for $-\pi < \arg s < \pi$ and $f(s) = O(1/|s|)$, as $|s| \to \infty$. Therefore, for fixed $T>0$ and $\sigma \to \infty$,

$$\Gamma(s) = O\left(e^{-\sigma+(\sigma-1/2)\log\sigma}\right). \tag{13.24}$$

Hence, for the integral over the right side of the rectangular contour $\mathcal{R}_N$,

$$\int_{2N+3/2-iT}^{2N+3/2+iT} F(w)\,dw \ll_{T,s} y^{2N+3/2}e^{4N-(4N+2+k+\sigma)\log N} = o(1), \tag{13.25}$$

as $N \to \infty$. Using Stirling's formula (2.5) to estimate the integrals over the horizontal sides of $\mathcal{R}_N$, we find that

$$\int_{1-c\pm iT}^{\infty\pm iT} F(w)\,dw \ll_s \int_{1-c}^{\infty} y^{\sigma}T^{-2\beta-\sigma-k}\,d\sigma \ll_{s,y} \frac{y^{1-c}}{T^{2c-\sigma-k-2}\log T} = o(1), \tag{13.26}$$

provided that $k > 2c-\sigma-2$. Using (13.23), (13.25), and (13.26) in (13.14), we deduce that

$$I(y) = \frac{\sin(\pi s/2)}{2^k y^{(-1+s+k)/2}} \left(\frac{J_{1-s+k}(4\sqrt{y}) + (-1)^{k+1}J_{-1+s-k}(4\sqrt{y})}{\sin \pi s} \right. \tag{13.27}$$

$$-(-1)^{k+1}\frac{2}{\pi}K_{1-s+k}(4\sqrt{y})\Bigg) - \frac{2^{s+1}}{\cos(\pi s/2)}\sum_{m=1}^{\lfloor\frac{k+1}{2}\rfloor}\frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.$$

Using the functional equation (13.6), the reflection formula (2.2), and the duplication formula (2.4), for $y=\pi^2 nx/q$, we find that

$$\sum_{n=1}^{\infty}\frac{\sigma_s(\chi,n)}{n}\left\{\frac{2^{s+1}}{\cos(\pi s/2)}\sum_{m=1}^{\lfloor\frac{k+1}{2}\rfloor}\frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}\right\}$$
$$=2i\tau(\bar{\chi})\frac{\pi^{1-s}}{q^{1-s}}\sum_{n=1}^{\lfloor\frac{k+1}{2}\rfloor}(-1)^{n-1}\frac{x^{-2n+1}}{\Gamma(k-2n+2)}\frac{\zeta(2n)}{(2\pi)^{2n}}L(1-2n+s,\chi). \tag{13.28}$$

With the aid of (2.15), we see that

$$\sin(\pi s/2)\left(\frac{J_{1-s+k}(4\sqrt{y})+(-1)^{k+1}J_{-1+s-k}(4\sqrt{y})}{\sin\pi s}-(-1)^{k+1}\frac{2}{\pi}K_{1-s+k}(4\sqrt{y})\right)$$
$$=\tilde{G}_{1+k-s}(4\sqrt{y}). \tag{13.29}$$

Combining (13.13), (13.15), (13.27), and (13.28), we see that

$$\frac{1}{\Gamma(k+1)}\sum_{n\le x}{}^{'}\sigma_{-s}(\chi,n)(x-n)^k=\frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)}-\frac{L(s,\chi)x^k}{2\Gamma(k+1)} \tag{13.30}$$
$$+2\sum_{n=1}^{\lfloor\frac{k+1}{2}\rfloor}(-1)^{n-1}\frac{x^{k-2n+1}}{\Gamma(k-2n+2)}\frac{\zeta(2n)}{(2\pi)^{2n}}L(1-2n+s,\chi)$$
$$+\frac{i}{\tau(\bar{\chi})(2\pi)^k}\sum_{n=1}^{\infty}\sigma_{-s}(\bar{\chi},n)\left(\frac{xq}{n}\right)^{\frac{1-s+k}{2}}\tilde{G}_{1-s+k}\left(4\pi\sqrt{\frac{nx}{q}}\right),$$

provided that $k\ge 0$, $\sigma\ne 0$, and $k>2c-\sigma-1$.

For $x>0$ fixed, by the asymptotic expansions for Bessel functions (2.16), (2.17), and (2.18), there exists a sufficiently large integer N_0 such that

$$\tilde{G}_{1+k-s}(4\pi\sqrt{\frac{nx}{q}})\ll_q\frac{1}{(nx)^{1/4}},$$

for all $n>N_0$. Hence, for $x>0$,

$$\sum_{n>N_0}\left(\frac{qx}{n}\right)^{(1+k-s)/2}\sigma_s(n)\tilde{G}_{1+k-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)\ll_q x^{(2k-2\sigma-1)/4}\sum_{n>N_0}\frac{\sigma_\sigma(n)}{n^{(2k-2\sigma+3)/4}}$$
$$\ll_q x^{(2k-2\sigma-1)/4},$$

provided that $k>|\sigma|+\frac{1}{2}$. Therefore, for $k>|\sigma|+\frac{1}{2}$ and $x>0$, the series

$$\sum_{n=1}^{\infty}\left(\frac{qx}{n}\right)^{(1+k-s)/2}\sigma_s(n)\tilde{G}_{1+k-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)$$

is absolutely and uniformly convergent for $0<x_1\le x\le x_2<\infty$. Thus, by differentiating a suitable number of times with the aid of (12.11), we find that (13.30) may be then upheld for $k>|\sigma|+\frac{1}{2}$. Since $|\sigma|<\frac{1}{2}$, the series on the left-hand side of (13.30) is continuous for $k>|\sigma|+\frac{1}{2}$. Conversely, we can see that the series on the left-hand side of (13.30) is continuous when $k > 0$, which implies that $|\sigma| < \frac{1}{2}$. Thus, the identity (13.30) is valid for $k > |\sigma| + \frac{1}{2}$ and $\sigma \neq 0$. Since the series on the right-hand side of (13.30) is absolutely and uniformly convergent for $0 < x_1 \leq x \leq x_2 < \infty$, we can take the limit as $s \to 0$ on both sides of (13.30) for $|\sigma| < \frac{1}{2}$ and $k > |\sigma| + \frac{1}{2}$. Hence, the identity (13.30) is valid for $k > |\sigma| + \frac{1}{2}$ with $|\sigma| < \frac{1}{2}$.

Suppose that the identity

$$\frac{1}{\Gamma(k+1)} \sideset{}{'}\sum_{n \leq x} \sigma_{-s}(\chi, n)(x-n)^k = \frac{x^{k+1} L(1+s,\chi)}{\Gamma(k+2)} - \frac{L(s,\chi)x^k}{2\Gamma(k+1)}$$
$$+ 2 \sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^{n-1} \frac{x^{k-2n+1}}{\Gamma(k-2n+2)} \frac{\zeta(2n)}{(2\pi)^{2n}} L(1-2n+s,\chi)$$
$$+ \frac{i}{\tau(\bar{\chi})(2\pi)^k} \sum_{n=1}^{\infty} \sigma_{-s}(\bar{\chi}, n) \left(\frac{xq}{n}\right)^{\frac{1-s+k}{2}} \tilde{G}_{1-s+k}\left(4\pi\sqrt{\frac{nx}{q}}\right), \tag{13.31}$$

is valid for some $k > 0$. Let $\beta > \max\{1, 1-\sigma\}$. Then

$$\sum_{n=1}^{\infty} \frac{|\sigma_s(n)|}{n^{\beta}} < \infty$$

and

$$\sup_{0 \leq h \leq 1} \left| \sum_{m^2 < n \leq (m+h)^2} \frac{\sigma_s(n)}{n^{\beta - 1/2}} \right| = o(1),$$

as $m \to \infty$. Put $x = y^2$ in the identity (13.31), where y lies in an interval J of length less than 1. By Lemma 12.9, $2y$ times the infinite series on the right-hand side of (13.31), with $x = y^2$, is uniformly equi-convergent on J with the differentiated series of the Fourier series of a function with period 1 which equals $A(y)\tilde{F}_{2-s+k}(y)$ on I, provided that $k > |\sigma| - \frac{1}{2}$. But then, $k + 1 > |\sigma| + \frac{1}{2}$. Hence, from (13.30),

$$\frac{i}{\tau(\bar{\chi})(2\pi)^{k+1}} A(y)\tilde{F}_{2-s+k}(y)$$
$$= A(y)\Bigg\{ \sideset{}{'}\sum_{n \leq y^2} \frac{\sigma_{-s}(\chi, n)(y^2-n)^{k+1}}{\Gamma(k+2)} - \frac{y^{2(k+2)} L(1+s,\chi)}{\Gamma(k+3)} + \frac{L(s,\chi)y^{2(k+1)}}{2\Gamma(k+2)}$$
$$- 2 \sum_{n=1}^{\lfloor \frac{k+2}{2} \rfloor} (-1)^{n-1} \frac{y^{2(k-2n+2)}}{\Gamma(k-2n+3)} \frac{\zeta(2n)}{(2\pi)^{2n}} L(1-2n+s,\chi) \Bigg\}$$
$$= A(y)\Bigg\{ \int_0^{y^2} \sideset{}{'}\sum_{n \leq t} \frac{\sigma_{-s}(\chi, n)(t-n)^k}{\Gamma(k+1)}\, dt - \frac{y^{2(k+2)} L(1+s,\chi)}{\Gamma(k+3)} + \frac{L(s,\chi)y^{2(k+1)}}{2\Gamma(k+2)}$$
$$- 2 \sum_{n=1}^{\lfloor \frac{k+2}{2} \rfloor} (-1)^{n-1} \frac{y^{2(k-2n+2)}}{\Gamma(k-2n+3)} \frac{\zeta(2n)}{(2\pi)^{2n}} L(1-2n+s,\chi) \Bigg\}$$
$$= A(y)\Bigg\{ \int_0^{y} \sideset{}{'}\sum_{n \leq t^2} \frac{\sigma_{-s}(\chi, n)(t^2-n)^k 2t}{\Gamma(k+1)}\, dt - \frac{y^{2(k+2)} L(1+s,\chi)}{\Gamma(k+3)} + \frac{L(s,\chi)y^{2(k+1)}}{2\Gamma(k+2)}$$

$$-2\sum_{n=1}^{\lfloor\frac{k+2}{2}\rfloor}(-1)^{n-1}\frac{y^{2(k-2n+2)}}{\Gamma(k-2n+3)}\frac{\zeta(2n)}{(2\pi)^{2n}}L(1-2n+s,\chi)\Bigg\}.$$

Note that $A(y)=1$ on J. Therefore, from Lemma 12.8 and the properties of the Fourier series of the function

$$\frac{2y}{\Gamma(k+1)}\sideset{}{'}\sum_{n\le y^2}\sigma_{-s}(\chi,n)(y^2-n)^k$$

in I, we see that the identity (13.30) holds for $k>|\sigma|-\frac{1}{2}$, which completes the proof of Theorem 13.2. ☐

From (11.8) and (13.29), we find that $\sin(\pi s/2)M_{1-s}(z)=\tilde{G}_{1-s}(z)$. The case $k=0$ of Theorem 13.2 gives the following corollary.

Corollary 13.3. *If χ is a non-principal odd primitive character modulo q, $x>0$, and $|\sigma|<1/2$, then*

$$\sideset{}{'}\sum_{n\le x}\sigma_{-s}(\chi,n)=xL(1+s,\chi)-\frac{1}{2}L(s,\chi)$$
$$+\frac{i\sin{(\pi s)/2}}{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\sigma_{-s}(\bar{\chi},n)\left(\frac{qx}{n}\right)^{\frac{1-s}{2}}M_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right),$$

where $M_{1-s}(z)$ is defined in (11.8).

Next, we show that Theorem 13.3 implies Theorem 13.1. We then finish this section and hence finish the proof of Theorem 11.3 by proving that Theorem 11.3 implies Theorem 13.3.

Proof that Theorem 13.3 implies Theorem 13.1. Recall that $L_s(a,q,x)$ and $M_\nu(z)$ are defined in (13.1) and (11.8), respectively. Thus,

$$\begin{aligned}
L_s(a,q,x)&=-\frac{x}{2}\sin\left(\frac{\pi s}{2}\right)\\
&\times\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{M_{1-s}\left(4\pi\sqrt{mx\left(n+\frac{a}{q}\right)}\right)}{(mx)^{(1+s)/2}(n+a/q)^{(1-s)/2}}-\frac{M_{1-s}\left(4\pi\sqrt{mx\left(n+1-\frac{a}{q}\right)}\right)}{(mx)^{(1+s)/2}(n+1-a/q)^{(1-s)/2}}\right\}\\
&=-\frac{x}{2}\sin\left(\frac{\pi s}{2}\right)\\
&\times\sum_{m=1}^{\infty}\left\{\sum_{\substack{n=1\\n\equiv a\bmod q}}^{\infty}\frac{M_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{(1+s)/2}(n/q)^{(1-s)/2}}-\sum_{\substack{n=1\\n\equiv -a\bmod q}}^{\infty}\frac{M_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{(1+s)/2}(n/q)^{(1-s)/2}}\right\}\\
&=-\frac{(qx)^{(1-s)/2}}{2\phi(q)}\sin\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{M_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{m^{(1+s)/2}n^{(1-s)/2}}\sum_{\chi\bmod q}\bar{\chi}(n)(\chi(a)-\chi(-a))\\
&=-\frac{(qx)^{(1-s)/2}}{\phi(q)}\sin\left(\frac{\pi s}{2}\right)\sum_{\substack{\chi\bmod q\\\chi\text{ odd}}}\chi(a)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{\chi}(n)n^s\frac{M_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mn)^{(1+s)/2}}
\end{aligned}$$

$$= -\frac{(qx)^{(1-s)/2}}{\phi(q)} \sin\left(\frac{\pi s}{2}\right) \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a) \sum_{n=1}^{\infty} \sum_{d|n} \bar{\chi}(d) d^s \frac{M_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{(1+s)/2}}$$

$$= -\frac{(qx)^{(1-s)/2}}{\phi(q)} \sin\left(\frac{\pi s}{2}\right) \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a) \sum_{n=1}^{\infty} \sigma_s(\bar{\chi}, n) \frac{M_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{(1+s)/2}}.$$

Now, from Lemma 12.2 and Theorem 13.1,

$$L_s(a,q,x) + \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \frac{\sin(2\pi na/q)}{n^s} = -\frac{ix}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \bmod q \\ \chi \text{ even}}} \chi(a)\tau(\bar{\chi}) L(1+s,\chi) + \frac{i}{2\phi(q)} \sum_{\substack{\chi \neq \chi_0 \bmod q \\ \chi \text{ even}}} \chi(a)\tau(\bar{\chi}) L(s,\chi).$$

Using the functional equation (13.6) of $L(s,\chi)$ for odd primitive characters, we find that

$$\begin{aligned} L_s(a,q,x) + \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \frac{\sin(2\pi na/q)}{n^s} &= \frac{x\pi^{s+1/2}\Gamma\left(\frac{1}{2}(1-s)\right)}{\Gamma\left(\frac{1}{2}(2+s)\right)} \frac{q^{-s}}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a) L(-s,\bar{\chi}) \\ &\quad - \frac{\pi^{s-1/2}\Gamma\left(\frac{1}{2}(2-s)\right)}{2\Gamma\left(\frac{1}{2}(1+s)\right)} \frac{q^{1-s}}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a) L(1-s,\bar{\chi}) \\ &= \frac{x\pi^{s+1/2}\Gamma\left(\frac{1}{2}(1-s)\right)}{2\Gamma\left(\frac{1}{2}(2+s)\right)} \frac{q^{-s}}{\phi(q)} \sum_{\chi \bmod q} (\chi(a) - \chi(q-a)) L(-s,\bar{\chi}) \\ &\quad - \frac{\pi^{s-1/2}\Gamma\left(\frac{1}{2}(2-s)\right)}{4\Gamma\left(\frac{1}{2}(1+s)\right)} \frac{q^{1-s}}{\phi(q)} \sum_{\chi \bmod q} (\chi(a) - \chi(q-a) L(1-s,\bar{\chi}). \end{aligned} \tag{13.32}$$

From [5, p. 249, Chapter 12],

$$L(s,\chi) = q^{-s} \sum_{h=1}^{q} \chi(h) \zeta(s, h/q). \tag{13.33}$$

Multiplying both sides of (13.33) by $\bar{\chi}(a)$ and summing over all characters χ modulo q, we deduce that

$$\zeta(s, a/q) = \frac{q^s}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) L(s,\chi), \tag{13.34}$$

where $\zeta(s,a)$ denotes the Hurwitz zeta function. Using the duplication formula (2.4) and the reflection formula (2.2) for $\Gamma(s)$, we find that

$$\frac{\Gamma\left(\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}(1-s)\right)} = \frac{\cos(\frac{1}{2}\pi s)\Gamma(s)}{2^{s-1}\sqrt{\pi}}. \tag{13.35}$$

Utilizing (13.34) and (13.35) in (13.32), we see that

$$L_s(a,q,x) + \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \frac{\sin(2\pi na/q)}{n^s} = -x \frac{\sin(\pi s/2)\Gamma(-s)}{(2\pi)^{-s}} \left(\zeta\left(-s, \frac{a}{q}\right) - \zeta\left(-s, 1 - \frac{a}{q}\right)\right)$$

$$- \frac{\cos(\pi s/2)\Gamma(1-s)}{2(2\pi)^{1-s}} \left(\zeta\left(1-s, \frac{a}{q}\right) - \zeta\left(1-s, 1-\frac{a}{q}\right)\right),$$

which completes the proof. □

The proof that Theorem 11.3 implies Theorem 13.3 is similar to the proof that Theorem 11.4 implies Theorem 14.3, which we give in the next section.

14. Proof of Theorem 11.4

Arguing as in the previous section, for $0 < a < q$ and q prime, we can show that Theorem 11.4 is equivalent to the following theorem.

Theorem 14.1. *Let q be a prime and $0 < a < q$. Let*

$$G_s(a,q,x) = \frac{x}{2}\cos\left(\frac{\pi s}{2}\right) \tag{14.1}$$
$$\times \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left\{ \frac{H_{1-s}\left(4\pi\sqrt{mx\left(n+\frac{a}{q}\right)}\right)}{(mx)^{(1+s)/2}(n+a/q)^{(1-s)/2}} + \frac{H_{1-s}\left(4\pi\sqrt{mx\left(n+1-\frac{a}{q}\right)}\right)}{(mx)^{(1+s)/2}(n+1-a/q)^{(1-s)/2}} \right\},$$

where $H_\nu(z)$ is defined in (11.12) and where we assume that the product of the summation indices mn tends to infinity. Then, for $|\sigma| < \frac{1}{2}$,

$$G_s(a,q,x) + \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\frac{\cos(2\pi na/q)}{n^s} = x\frac{\cos(\frac{1}{2}\pi s)\Gamma(-s)}{(2\pi)^{-s}}\left(\zeta\left(-s,\frac{a}{q}\right) + \zeta\left(-s, 1-\frac{a}{q}\right)\right)$$
$$- \frac{\sin(\frac{1}{2}\pi s)\Gamma(1-s)}{2(2\pi)^{1-s}}\left(\zeta\left(1-s,\frac{a}{q}\right) + \zeta\left(1-s,1-\frac{a}{q}\right)\right).$$

We show that Theorem 14.1 is equivalent to Theorem 14.3, which is a special case of the following theorem.

Theorem 14.2. *If χ is a non-principal even primitive character modulo q, $x > 0$, $|\sigma| < 1/2$, and k is a non-negative integer, then*

$$\frac{1}{\Gamma(k+1)}\sideset{}{'}\sum_{n\le x}\sigma_{-s}(\chi,n)(x-n)^k$$
$$= \frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)} - \frac{x^k L(s,\chi)}{2\Gamma(k+1)} + 2\sum_{n=1}^{\lfloor\frac{k+1}{2}\rfloor}\frac{(-1)^{n-1}x^{k-2n+1}}{\Gamma(k-2n+2)}\frac{\zeta(2n)}{(2\pi)^{2n}}L(1-2n+s,\chi)$$
$$+ \frac{1}{\tau(\bar\chi)(2\pi)^k}\sum_{n=1}^{\infty}\sigma_{-s}(\bar\chi,n)\left(\frac{qx}{n}\right)^{(1-s+k)/2} G_{1-s+k}\left(4\pi\sqrt{\frac{nx}{q}}\right),$$

where $G_{\lambda-s}(z)$ is defined in (8.1). The series on the right-hand side converges uniformly on any interval for $x > 0$ where the left-hand side is continuous. The convergence is bounded on any interval $0 < x_1 \le x \le x_2 < \infty$ when $k = 0$.

Proof. From (12.1) and Lemma 12.4, for fixed $x > 0$, we see that

$$\frac{1}{\Gamma(1+k)}\sideset{}{'}\sum_{n\le x}\sigma_{-s}(\chi,n)(x-n)^k = \frac{1}{2\pi i}\int_{(c)}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}dw,$$

where $\max\{1, 1-\sigma, \sigma\} < c$ and $k \geq 0$. Proceeding as we did in the proof of Theorem 13.2, we find that

$$\frac{1}{\Gamma(k+1)}\sideset{}{'}\sum_{n\leq x}\sigma_{-s}(\chi,n)(x-n)^k = \frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)} - \frac{L(s,\chi)x^k}{2\Gamma(k+1)} + \frac{1}{2\pi i}\int_{(1-c)}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}dw, \tag{14.2}$$

provided that $k \geq 0$ and $k > 2c - \sigma - 2$. The functional equation for $L(2s,\chi)$ for an even primitive Dirichlet character χ is given by [28, p. 69]

$$\left(\frac{\pi}{q}\right)^{-s}\Gamma(s)L(2s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\left(\frac{\pi}{q}\right)^{-(\frac{1}{2}-s)}\Gamma\left(\frac{1}{2}-s\right)L(1-2s,\overline{\chi}), \tag{14.3}$$

where $\tau(\chi)$ is the Gauss sum defined in (12.6). Combining the functional equation (2.6) of $\zeta(2w)$ and the functional equation (14.3) of $L(2w+s,\chi)$ for even primitive χ, we deduce the functional equation

$$\zeta(w)L(w+s,\chi) = \frac{\pi^{2w+s-1}}{\tau(\bar{\chi})q^{w+s-1}}\eta(w,s)\zeta(1-w)L(1-w-s,\bar{\chi}), \tag{14.4}$$

where

$$\eta(w,s) = \frac{\Gamma\left(\frac{1}{2}(1-w)\right)\Gamma\left(\frac{1}{2}(1-w-s)\right)}{\Gamma\left(\frac{1}{2}w\right)\Gamma\left(\frac{1}{2}(w+s)\right)}.$$

Define

$$I(y) := \frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}y^w\,dw. \tag{14.5}$$

Using the functional equation (14.4) in the integrand on the right-hand side of (14.2) and inverting the order of summation and integration, we find that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(1-c)}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\,dw\\
&= \frac{x^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\zeta(1-w)L(1-w-s,\bar{\chi})\left(\frac{\pi^2x}{q}\right)^w\,dw\\
&= \frac{x^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\left(\frac{\pi^2x}{q}\right)^w\sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi},n)}{n^{1-w}}\,dw\\
&= \frac{x^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi},n)}{n^{1+k}}\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\left(\frac{\pi^2nx}{q}\right)^w\,dw\\
&= \frac{x^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi},n)}{n}I\left(\frac{\pi^2nx}{q}\right),
\end{aligned}\tag{14.6}$$

provided that $k > 2c - \sigma - 1$. We compute the integral $I(y)$ by using the residue calculus, shifting the line of integration to the right, and letting $c \to -\infty$.

Let k be a positive integer and $\sigma \neq 0$. By (14.5), we can write

$$I(y) := \frac{1}{2\pi i}\int_{(1-c)}F(w)\,dw,$$

where

$$F(w) := \frac{\Gamma(w)\Gamma\left(\frac{1}{2}(1-w)\right)\Gamma\left(\frac{1}{2}(1-w-s)\right)y^w}{\Gamma(1+k+w)\Gamma\left(\frac{1}{2}w\right)\Gamma\left(\frac{1}{2}(w+s)\right)}.$$

Note that the poles of the function $F(w)$ on the right side of the line $1-c+it, -\infty < t < \infty$, are at $w = 2m+1$ and $w = 2m+1-s$, $m = 0, 1, 2, \dots$. Calculating the residues, we find that

$$R_{2m+1}(F(w)) = (-1)^{m+1}\frac{2\Gamma(2m+1)\Gamma\left(-m-\frac{1}{2}s\right)y^{2m+1}}{m!\Gamma(2+k+2m)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}(s+1)\right)}$$

and

$$R_{2m+1-s}(F(w)) = (-1)^{m+1}\frac{2\Gamma(2m+1-s)\Gamma\left(-m+\frac{1}{2}s\right)y^{2m+1-s}}{m!\Gamma(2+k+2m-s)\Gamma\left(m+\frac{1}{2}(1-s)\right)\Gamma\left(m+\frac{1}{2}\right)}.$$

With the aid of the duplication formula (2.4) and the reflection formula (2.2), we find that

$$R_{2m+1}(F(w)) = \frac{2^{s-1}}{\sin(\pi s/2)}\frac{(2\sqrt{y})^{4m+2}}{(2m+k+1)!\Gamma(2m+1+s)} \tag{14.7}$$

and

$$R_{2m+1-s}(F(w)) = -\frac{(2y)^{1-s}}{\sin(\pi s/2)}\frac{(2\sqrt{y})^{4m}}{(2m)!\Gamma(2m+k+2-s)}. \tag{14.8}$$

Consequently, from (2.14), (13.18), and (14.7), for k even,

$$\begin{aligned}\sum_{m=0}^{\infty}R_{2m+1}(F(w)) &= \frac{2^{s-1-2k}y^{-k}}{\sin(\pi s/2)}\sum_{m=0}^{\infty}\frac{(2\sqrt{y})^{4m+2k+2}}{(2m+k+1)!\Gamma(2m+1+s)}\\ &= \frac{2^{s-1-2k}y^{-k}}{\sin(\pi s/2)}\left\{\sum_{m=0}^{\infty}\frac{(2\sqrt{y})^{4m+2}}{(2m+1)!\Gamma(2m+1+s-k)}\right.\\ &\quad\left.-\sum_{m=1}^{k/2}\frac{(2\sqrt{y})^{4m-2}}{(2m-1)!\Gamma(2m-1+s-k)}\right\}\\ &= \frac{2^{-1-k}y^{(1-s-k)/2}}{\sin(\pi s/2)}(I_{-1+s-k}(4\sqrt{y}) - J_{-1+s-k}(4\sqrt{y}))\\ &\quad-\frac{2^{s+1}}{\sin(\pi s/2)}\sum_{m=1}^{k/2}\frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.\end{aligned} \tag{14.9}$$

For each odd integer k,

$$\begin{aligned}\sum_{m=0}^{\infty}R_{2m+1}(F(w)) &= \frac{2^{-1-k}y^{(1-s-k)/2}}{\sin(\pi s/2)}(I_{-1+s-k}(4\sqrt{y}) + J_{-1+s-k}(4\sqrt{y}))\\ &\quad-\frac{2^{s+1}}{\sin(\pi s/2)}\sum_{m=1}^{(k+1)/2}\frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.\end{aligned} \tag{14.10}$$

Similarly, from (14.8), we find that

$$\sum_{m=0}^{\infty}R_{2m+1-s}(F(w)) = -\frac{2^{-1-k}y^{(1-s-k)/2}}{\sin(\pi s/2)}(J_{1-s+k}(4\sqrt{y}) + I_{1-s+k}(4\sqrt{y})). \tag{14.11}$$

Utilizing (13.19) in the sum of (14.9), (14.10), and (14.11), we deduce that

$$\sum_{m=0}^{\infty}(R_{2m+1}(F(w))+R_{2m+1-s}(F(w)))=-\frac{\cos(\pi s/2)}{2^ky^{(-1+s+k)/2}}$$
$$\times\left(\frac{J_{1-s+k}(4\sqrt{y})-(-1)^{k+1}J_{-1+s-k}(4\sqrt{y})}{\sin\pi s}-(-1)^{k+1}\frac{2}{\pi}K_{1-s+k}(4\sqrt{y})\right)$$
$$-\frac{2^{s+1}}{\sin(\pi s/2)}\sum_{m=1}^{\lfloor\frac{k+1}{2}\rfloor}\frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}. \tag{14.12}$$

Using (2.15), we can show that

$$\cos(\pi s/2)\left(\frac{J_{1-s+k}(4\sqrt{y})-(-1)^{k+1}J_{-1+s-k}(4\sqrt{y})}{\sin\pi s}-(-1)^{k+1}\frac{2}{\pi}K_{1-s+k}(4\sqrt{y})\right)$$
$$=G_{1+k-s}(4\sqrt{y}). \tag{14.13}$$

Consider the positively oriented contour $\mathcal{R}_N$ formed by the points $\{1-c-iT, 2N+\frac{3}{2}-iT, 2N+\frac{3}{2}+iT, 1-c+iT\}$, where $T>0$ and N is a positive integer. By the residue theorem,

$$\frac{1}{2\pi i}\int_{\mathcal{R}_N}F(w)\,dw=\sum_{0\le k\le N}R_{2k+1}(F(w))+\sum_{0\le k\le N}R_{2k+1-s}(F(w)).$$

By (13.24), for the integral over the right side of the rectangular contour $\mathcal{R}_N$,

$$\int_{2N+3/2-iT}^{2N+3/2+iT}F(w)\,dw\ll_{T,s}y^{2N+3/2}e^{4N-(4N+2+k+\sigma)\log N}=o(1),$$

as $N\to\infty$. Using Stirling's formula (2.5) to estimate the integrals over the horizontal sides of $\mathcal{R}_N$, we find that

$$\int_{1-c\pm iT}^{\infty\pm iT}F(w)\,dw\ll_s\int_{1-c}^{\infty}y^{\sigma}T^{-2\beta-\sigma-k}\,d\sigma\ll_{s,y}\frac{y^{1-c}}{T^{2c-\sigma-k-2}\log T}=o(1),$$

provided that $k>2c-\sigma-2$. Combining (14.2), (14.6), (14.12), and (14.13), we conclude that

$$\frac{1}{\Gamma(k+1)}\sideset{}{'}\sum_{n\le x}\sigma_{-s}(\chi,n)(x-n)^k$$
$$=\frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)}-\frac{x^kL(s,\chi)}{2\Gamma(k+1)}+2\sum_{n=1}^{\lfloor\frac{k+1}{2}\rfloor}\frac{(-1)^{n-1}x^{k-2n+1}}{\Gamma(k-2n+2)}\frac{\zeta(2n)}{(2\pi)^{2n}}L(1-2n+s,\chi)$$
$$+\frac{1}{\tau(\bar\chi)(2\pi)^k}\sum_{n=1}^{\infty}\sigma_{-s}(\bar\chi,n)\left(\frac{qx}{n}\right)^{(1-s+k)/2}G_{1-s+k}\left(4\pi\sqrt{\frac{nx}{q}}\right), \tag{14.14}$$

provided that $k\ge0$, $\sigma\ne0$, and $k>2c-\sigma-1$. By the asymptotic expansions for Bessel functions (2.16), (2.17), and (2.18), Lemma 12.10, (8.6), and an argument like that in the proof in Theorem 13.2, we deduce the identity (14.14) for $k>|\sigma|-\frac{1}{2}$, with $|\sigma|<\frac{1}{2}$. Thus, we complete the proof of Theorem 11.4. □

From the definition (11.12) and (13.29), we find that $\cos(\pi s/2)M_{1-s}(z)=G_{1-s}(z)$. The case $k=0$ of Theorem 14.2 provides the following corollary.

Corollary 14.3. *If χ is a non-principal even primitive character modulo q, $x>0$, and $|\sigma|<1/2$, then*

$$\sideset{}{'}\sum_{n\leq x}\sigma_{-s}(\chi,n)=xL(1+s,\chi)-\tfrac{1}{2}L(s,\chi)$$
$$+\frac{\cos{(\pi s)/2}}{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\sigma_{-s}(\bar{\chi},n)\left(\frac{qx}{n}\right)^{\frac{1-s}{2}}H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right),$$

where $H_{1-s}(z)$ is defined in (11.12).

Next we show that Theorem 14.3 implies Theorem 14.1.

Proof. First we write (14.1) as a sum over Dirichlet characters. To that end, for any prime q and $0<a<q$,

$$\begin{aligned}
G_s(a,q,x)&=\frac{x}{2}\cos\left(\frac{\pi s}{2}\right)\\
&\times\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{H_{1-s}\left(4\pi\sqrt{mx\left(n+\frac{a}{q}\right)}\right)}{(mx)^{(1+s)/2}(n+a/q)^{(1-s)/2}}+\frac{H_{1-s}\left(4\pi\sqrt{mx\left(n+1-\frac{a}{q}\right)}\right)}{(mx)^{(1+s)/2}(n+1-a/q)^{(1-s)/2}}\right\}\\
&=\frac{x}{2}\cos\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\sum_{\substack{n=1\\ n\equiv\pm a \bmod q}}^{\infty}\frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{(1+s)/2}(n/q)^{(1-s)/2}}\\
&=\frac{(qx)^{(1-s)/2}}{2\phi(q)}\cos\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{m^{(1+s)/2}n^{(1-s)/2}}\sum_{\chi \bmod q}\bar{\chi}(n)(\chi(a)+\chi(-a))\\
&=\frac{(qx)^{(1-s)/2}}{\phi(q)}\cos\left(\frac{\pi s}{2}\right)\sum_{\substack{\chi \bmod q\\ \chi\text{ even}}}\chi(a)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{\chi}(n)n^s\frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mn)^{(1+s)/2}}\\
&=\frac{(qx)^{(1-s)/2}}{\phi(q)}\cos\left(\frac{\pi s}{2}\right)\sum_{\substack{\chi \bmod q\\ \chi\text{ even}}}\chi(a)\sum_{n=1}^{\infty}\sum_{d|n}\bar{\chi}(d)d^s\frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{(1+s)/2}}\\
&=\frac{(qx)^{(1-s)/2}}{\phi(q)}\cos\left(\frac{\pi s}{2}\right)\sum_{\substack{\chi \bmod q\\ \chi\text{ even}}}\chi(a)\sum_{n=1}^{\infty}\sigma_s(\bar{\chi},n)\frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{(1+s)/2}},
\end{aligned}\tag{14.15}$$

where in the penultimate step we recall our assumption that the double series converges in the sense that the product of the indices mn tends to infinity. For the principal character χ_0,

$$\begin{aligned}
\sum_{n=1}^{\infty}\sigma_s(\chi_0,n)\frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{(1+s)/2}}&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\chi_0(n)n^s\frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mn)^{(1+s)/2}}\\
&=\sum_{m=1}^{\infty}\sum_{\substack{n=1\\ q\nmid n}}^{\infty}n^s\frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mn)^{(1+s)/2}}
\end{aligned}\tag{14.16}$$

$$
\begin{aligned}
&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}n^{s}\frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mn)^{(1+s)/2}}-q^{(s-1)/2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}n^{s}\frac{H_{1-s}\left(4\pi\sqrt{mnx}\right)}{(mn)^{(1+s)/2}}\\
&=\sum_{n=1}^{\infty}\sigma_s(n)\frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{(1+s)/2}}-q^{(s-1)/2}\sum_{n=1}^{\infty}\sigma_s(n)\frac{H_{1-s}\left(4\pi\sqrt{nx}\right)}{n^{(1+s)/2}}.
\end{aligned}
$$

Combining (14.15) and (14.16) and applying Lemma 12.1, we find that

$$
\begin{aligned}
G_s(a,q,x)=&\frac{(qx)^{(1-s)/2}}{\phi(q)}\cos\left(\frac{\pi s}{2}\right)\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\text{ even}}}\chi(a)\sum_{n=1}^{\infty}\sigma_s(\bar{\chi},n)\frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{(1+s)/2}}\\
&+\frac{1}{\phi(q)}\left(\sideset{}{'}\sum_{n\leq x}\sigma_{-s}(n)-xZ(s,x)+\frac{1}{2}\zeta(s)\right)\\
&-\frac{q^{1-s}}{\phi(q)}\left(\sideset{}{'}\sum_{n\leq x/q}\sigma_{-s}(n)-\frac{x}{q}Z(s,x/q)+\frac{1}{2}\zeta(s)\right)\\
=&\frac{(qx)^{\frac{1-s}{2}}}{\phi(q)}\cos\left(\frac{\pi s}{2}\right)\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\text{ even}}}\chi(a)\sum_{n=1}^{\infty}\sigma_s(\bar{\chi},n)\frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{(1+s)/2}}\\
&+\frac{1}{\phi(q)}\sideset{}{'}\sum_{n\leq x}\sigma_{-s}(n)-\frac{q^{1-s}}{\phi(q)}\sideset{}{'}\sum_{n\leq x/q}\sigma_{-s}(n)\\
&+\frac{x}{\phi(q)q^{s}}\zeta(1+s)\left(1-\frac{1}{q^{-s}}\right)-\frac{\zeta(s)}{2\phi(q)q^{s-1}}\left(1-\frac{1}{q^{1-s}}\right).
\end{aligned}\tag{14.17}
$$

For each prime q, by Lemma 12.3,

$$
\begin{aligned}
\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\cos\left(2\pi na/q\right)}{n^{s}}&=q^{-s}\sideset{}{'}\sum_{1\leq n\leq x/q}\sigma_{-s}(n)+\frac{1}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\text{ even}}}\chi(a)\tau(\bar{\chi})\sideset{}{'}\sum_{1\leq n\leq x}\sigma_{-s}(\chi,n)\\
&=q^{-s}\sideset{}{'}\sum_{1\leq n\leq x/q}\sigma_{-s}(n)-\frac{1}{\phi(q)}\sideset{}{'}\sum_{1\leq n\leq x}\sigma_{-s}(\chi_0,n)\\
&\quad+\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\text{ even}}}\chi(a)\tau(\bar{\chi})\sideset{}{'}\sum_{1\leq n\leq x}\sigma_{-s}(\chi,n).
\end{aligned}\tag{14.18}
$$

Now,

$$
\begin{aligned}
\sideset{}{'}\sum_{1\leq n\leq x}\sigma_{-s}(\chi_0,n)=\sideset{}{'}\sum_{1\leq n\leq x}\sum_{\substack{d|n\\ q\nmid d}}d^{-s}&=\sideset{}{'}\sum_{1\leq n\leq x}\sum_{d|n}d^{-s}-q^{-s}\sideset{}{'}\sum_{1\leq n\leq x/q}\sum_{d|n}d^{-s}\\
&=\sideset{}{'}\sum_{1\leq n\leq x}\sigma_{-s}(n)-q^{-s}\sideset{}{'}\sum_{1\leq n\leq x/q}\sigma_{-s}(n).
\end{aligned}\tag{14.19}
$$

Substituting (14.19) into (14.18), we find that

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\frac{\cos(2\pi na/q)}{n^s} = \frac{q^{1-s}}{\phi(q)}\sideset{}{'}\sum_{1\leq n\leq x/q}\sigma_{-s}(n) - \frac{1}{\phi(q)}\sideset{}{'}\sum_{1\leq n\leq x}\sigma_{-s}(n) + \frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\text{ even}}}\chi(a)\tau(\bar{\chi})\sideset{}{'}\sum_{1\leq n\leq x}\sigma_{-s}(\chi,n). \tag{14.20}$$

Adding (14.17) and (14.20) and using Theorem 14.3, we find that

$$G_s(a,q,x) + \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\frac{\cos(2\pi na/q)}{n^s} = \frac{x}{\phi(q)q^s}\zeta(1+s)\left(1-\frac{1}{q^{-s}}\right) - \frac{\zeta(s)}{2\phi(q)q^{s-1}}\left(1-\frac{1}{q^{1-s}}\right) + \frac{x}{\phi(q)}\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\text{ even}}}\chi(a)\tau(\bar{\chi})L(1+s,\chi) - \frac{1}{2\phi(q)}\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\text{ even}}}\chi(a)\tau(\bar{\chi})L(s,\chi). \tag{14.21}$$

Recall that if χ_0 is the principal character modulo the prime q, then

$$L(s,\chi_0) = \zeta(s)\left(1-\frac{1}{q^s}\right). \tag{14.22}$$

Using the functional equations of $\zeta(s)$ and $L(s,\chi)$ for even primitive Dirichlet characters, (2.6) and (14.3), respectively, and (14.22), we find from (14.21) that

$$\begin{aligned}G_s(a,q,x) + \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\frac{\cos(2\pi na/q)}{n^s} &= \frac{x\pi^{s+1/2}\Gamma\left(-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}(1+s)\right)}\frac{q^{-s}}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\text{ even}}}\chi(a)L(-s,\bar{\chi})\\ &\quad - \frac{\pi^{s-1/2}\Gamma\left(\frac{1}{2}(1-s)\right)}{2\Gamma\left(\frac{1}{2}s\right)}\frac{q^{1-s}}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\text{ even}}}\chi(a)L(1-s,\bar{\chi})\\ &= \frac{x\pi^{s+1/2}\Gamma\left(-\frac{1}{2}s\right)}{2\Gamma\left(\frac{1}{2}(1+s)\right)}\frac{q^{-s}}{\phi(q)}\sum_{\chi \bmod q}(\chi(a)+\chi(q-a))L(-s,\bar{\chi})\\ &\quad - \frac{\pi^{s-1/2}\Gamma\left(\frac{1}{2}(1-s)\right)}{4\Gamma\left(\frac{1}{2}s\right)}\frac{q^{1-s}}{\phi(q)}\sum_{\chi \bmod q}(\chi(a)+\chi(q-a))L(1-s,\bar{\chi}).\end{aligned} \tag{14.23}$$

We complete the proof of Theorem 14.1 by using (13.34) and (13.35) in (14.23). □

Next we prove that Theorem 11.4 implies Theorem 14.3.

Proof. Let χ be an even primitive character modulo q. Set $\theta = h/q$, where $1 \leq h < q$. The Gauss sum $\tau(n,\chi)$ is defined by

$$\tau(n,\chi) = \sum_{m=1}^{q}\chi(m)e^{2\pi imn/q}.$$

Note that $\tau(1,\chi) = \tau(\chi)$, which is defined in (12.6). For any character χ [5, p. 165, Theorem 8.9]

$$\tau(n,\chi) = \bar{\chi}(n)\tau(\chi).$$

Multiplying both sides of (11.11) by $\bar{\chi}(h)/\tau(\bar{\chi})$ and summing over h, $1 \leq h < q$, we find that the left-hand side yields

$$\frac{1}{\tau(\bar{\chi})}\sum_{h=1}^{q-1}\bar{\chi}(h)\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\cos\left(2\pi nh/q\right)}{n^s} = \frac{1}{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{F\left(\frac{x}{n}\right)}{n^s}\sum_{h=1}^{q-1}\bar{\chi}(h)\cos\left(\frac{2\pi nh}{q}\right)$$

$$= \frac{1}{2\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{F\left(\frac{x}{n}\right)}{n^s}\sum_{h=1}^{q-1}\bar{\chi}(h)\left(e^{2\pi inh/q}+e^{-2\pi inh/q}\right)$$

$$= \frac{1}{2\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{F\left(\frac{x}{n}\right)}{n^s}\tau(\bar{\chi})(\chi(n)+\chi(-n))$$

$$= {\sum_{n\leq x}}' \sigma_{-s}(\chi, n). \tag{14.24}$$

On the other hand, summing over h, $1 \leq h \leq q$, on the right-hand side of (11.11) gives

$$\frac{x}{2\tau(\bar{\chi})}\cos\left(\frac{\pi s}{2}\right)\sum_{h=1}^{q-1}\sum_{m=1}^{\infty}\sum_{\substack{n=1\\ n\equiv \pm h \bmod q}}^{\infty}\bar{\chi}(h)\frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{(1+s)/2}(n/q)^{(1-s)/2}}$$

$$= \frac{x}{2\tau(\bar{\chi})}\cos\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{(1+s)/2}(n/q)^{(1-s)/2}}\sum_{\substack{h=1\\ h\equiv \pm n \bmod q}}^{q-1}\bar{\chi}(h)$$

$$= \frac{x}{\tau(\bar{\chi})}\cos\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{\chi}(n)\frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{(1+s)/2}(n/q)^{(1-s)/2}}$$

$$= \frac{(qx)^{(1-s)/2}\cos(\frac{1}{2}\pi s)}{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi}, n)}{n^{(1+s)/2}}H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right). \tag{14.25}$$

Combining (14.24), (14.25), and (13.33) with the functional equation (14.3) of $L(s,\chi)$ for even primitive χ, we obtain the equality in (11.11), which completes the proof of Theorem 14.3. □

15. Koshliakov Transforms and Modular-type Transformations

In Section 6, we studied a generalization of the Voronoï summation formula, namely (6.8), the right-hand side of which consists of summing infinitely many integrals involving the kernel

$$\left(\frac{2}{\pi}K_s(4\pi\sqrt{nt}) - Y_s(4\pi\sqrt{nt})\right)\cos\left(\frac{\pi s}{2}\right) - J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right). \tag{15.1}$$

Koshliakov [57] remarkably found that for $-\frac{1}{2} < \nu < \frac{1}{2}$, the modified Bessel function $K_\nu(x)$ is self-reciprocal with respect to this kernel, i.e.,

$$\int_0^\infty K_\nu(t)\left(\cos(\nu\pi)\tilde{M}_{2\nu}(2\sqrt{xt}) - \sin(\nu\pi)J_{2\nu}(2\sqrt{xt})\right)dt = K_\nu(x). \tag{15.2}$$

He also showed that for the same values of ν, $xK_\nu(x)$ is self-reciprocal with respect to the companion kernel $\sin(\nu\pi)J_{2\nu}(2\sqrt{xt}) - \cos(\nu\pi)L_{2\nu}(2\sqrt{xt})$, i.e.,

$$\int_0^\infty tK_\nu(t)\left(\sin(\nu\pi)J_{2\nu}(2\sqrt{xt}) - \cos(\nu\pi)L_{2\nu}(2\sqrt{xt})\right)dt = xK_\nu(x). \tag{15.3}$$

Here

$$\tilde{M}_\nu(x) := \frac{2}{\pi}K_\nu(x) - Y_\nu(x) \quad \text{and} \quad L_\nu(x) := -\frac{2}{\pi}K_\nu(x) - Y_\nu(x),$$

It is easy to see that these identities actually hold for complex ν with $-\frac{1}{2} < \text{Re}\ \nu < \frac{1}{2}$. It must be mentioned here that the special case $z = 0$ of (15.2) was obtained by Dixon and Ferrar [33, p. 164, equation (4.1)].

Motivated by these results of Koshliakov, we begin with some definitions.

Definition 15.1. *Let $f(t,\nu)$ be a function analytic in the real variable t and in the complex variable ν. Then, we define the first Koshliakov transform of a function $f(t,\nu)$ to be*

$$\int_0^\infty f(t,\nu)\left(\cos(\nu\pi)\tilde{M}_{2\nu}(2\sqrt{xt}) - \sin(\nu\pi)J_{2\nu}(2\sqrt{xt})\right)dt,$$

and the second Koshliakov transform of a function $f(t,\nu)$ to be

$$\int_0^\infty f(t,\nu)\left(\sin(\nu\pi)J_{2\nu}(2\sqrt{xt}) - \cos(\nu\pi)L_{2\nu}(2\sqrt{xt})\right)dt,$$

provided, of course, that the integrals converge.

Remark. We note here that the first Koshliakov transform is the integral transform that arises naturally when one considers a function corresponding to the functional equation of an even Maass form in conjunction with a summation formula of Ferrar; see for example, the work of J. Lewis and D. Zagier [60, p. 216–217][5].

The following two theorems, obtained in [30, Theorems 5.3, 5.5], give the necessary conditions for functions to equal their first or second Koshliakov transforms. The corollaries resulting from them [30, Corollaries 5.4 and 5.6] give associated modular transformations. These results are stated below.

Theorem 15.2. *Assume $\pm\frac{1}{2}\sigma < c = \text{Re}\ z < 1 \pm \frac{1}{2}\sigma$. Define $f(x,s)$ by*

$$f(x,s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z}F(z,s)\zeta(1-z-s/2)\zeta(1-z+s/2)\,dz,$$

where $F(z,s)$ is a function satisfying $F(z,s) = F(1-z,s)$ and is such that the integral above converges. Then f is self-reciprocal (as a function of x) with respect to the kernel

$$2\pi\left(\cos\left(\tfrac{1}{2}\pi s\right)\tilde{M}_s(4\pi\sqrt{xy}) - \sin\left(\tfrac{1}{2}\pi s\right)J_s(4\pi\sqrt{xy})\right),$$

that is,

$$f(y,s) = 2\pi\int_0^\infty f(x,s)\left[\cos\left(\tfrac{1}{2}\pi s\right)\tilde{M}_s(4\pi\sqrt{xy}) - \sin\left(\tfrac{1}{2}\pi s\right)J_s(4\pi\sqrt{xy})\right]dx.$$

Corollary 15.3. *Let $f(x,s)$ be as in the previous theorem. Then, if α, $\beta > 0$ and $\alpha\beta = 1$, and if $-1 < \sigma < 1$,*

$$\sqrt{\alpha}\int_0^\infty K_{s/2}(2\pi\alpha x)f(x,s)\,dx = \sqrt{\beta}\int_0^\infty K_{s/2}(2\pi\beta x)f(x,s)\,dx. \tag{15.4}$$

[5]The kernel $F_s(\xi)$ in [60, p. 217] is equal to 2π times the kernel in (15.1) with s replaced by $2s-1$. This is immediately seen by an application of (2.15).

Theorem 15.4. *Assume* $\pm\frac{1}{2}\sigma < c = \mathrm{Re}\, z < 1 \pm \frac{1}{2}\sigma$. *Define* $f(x,s)$ *by*

$$f(x,s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{F(z,s)}{(2\pi)^{2z}}\Gamma\left(z-\frac{1}{2}s\right)\Gamma\left(z+\frac{1}{2}s\right)\zeta\left(z-\frac{s}{2}\right)\zeta\left(z+\frac{s}{2}\right)x^{-z}\,dz, \quad (15.5)$$

where $F(z,s)$ *is a function satisfying* $F(z,s) = F(1-z,s)$ *and is such that the integral above converges. Then* f *is self-reciprocal (as a function of* x*) with respect to the kernel*

$$2\pi\left(\sin\left(\tfrac{1}{2}\pi s\right)J_s(4\pi\sqrt{xy}) - \cos\left(\tfrac{1}{2}\pi s\right)L_s(4\pi\sqrt{xy})\right),$$

that is,

$$f(y,s) = 2\pi\int_0^\infty f(x,s)\left[\sin\left(\tfrac{1}{2}\pi s\right)J_s(4\pi\sqrt{xy}) - \cos\left(\tfrac{1}{2}\pi s\right)L_s(4\pi\sqrt{xy})\right]dx.$$

Corollary 15.5. *Let* $f(x,s)$ *be as in the previous theorem. Then, if* α, $\beta > 0$ *and* $\alpha\beta = 1$, *and* $-1 < \sigma < 1$,

$$\alpha^{3/2}\int_0^\infty xK_{s/2}(2\pi\alpha x)f(x,s)\,dx = \beta^{3/2}\int_0^\infty xK_{s/2}(2\pi\beta x)f(x,s)\,dx. \quad (15.6)$$

The identity in (15.2) can be proved using Theorem 15.2 by taking

$$F(z,s) = \frac{\pi^{-z}}{4}\frac{\Gamma\left(\frac{1}{2}z-\frac{1}{4}s\right)\Gamma\left(\frac{1}{2}z+\frac{1}{4}s\right)}{\zeta\left(1-z-\frac{1}{2}s\right)\zeta\left(1-z+\frac{1}{2}s\right)}$$

and then using the fact [64, p. 115, formula 11.1] that for $c = \mathrm{Re}\, z > \pm\, \mathrm{Re}\, \nu$ and $a > 0$,

$$\frac{1}{2\pi i}\int_{(c)} 2^{z-2}a^{-z}\Gamma\left(\frac{z}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{z}{2}+\frac{\nu}{2}\right)x^{-z}\,dz = K_\nu(ax). \quad (15.7)$$

Note that (2.2), (2.4), and (2.6) imply $F(z,s) = F(1-z,s)$. In the last step, replace s by 2ν, x by $x/(2\pi)$, and y by $y/(2\pi)$ to obtain (15.2). Similarly, (15.3) can be proved using Theorem 15.4 by taking

$$F(z,s) = \frac{\pi^{z+1}}{\Gamma\left(\frac{1}{2}z-\frac{1}{4}s\right)\Gamma\left(\frac{1}{2}z+\frac{1}{4}s\right)\zeta\left(z-\frac{1}{2}s\right)\zeta\left(z+\frac{1}{2}s\right)},$$

and then using (15.7) with z replaced by $z+1$. The equality $F(z,s) = F(1-z,s)$ can be proved using (2.3), (2.4), and (2.6). As before, in the final step, we replace s by 2ν, x by $x/(2\pi)$, and y by $y/(2\pi)$ to obtain (15.3).

If we let $f(x,s) = K_{s/2}(2\pi x)$ in (15.4), we obtain

$$\sqrt{\alpha}\int_0^\infty K_{s/2}(2\pi\alpha x)K_{s/2}(2\pi x)\,dx = \sqrt{\beta}\int_0^\infty K_{s/2}(2\pi\beta x)K_{s/2}(2\pi x)\,dx, \quad (15.8)$$

which is really a special case of Pfaff's transformation [77, p. 110]

$${}_2F_1(a,b;c;w) = (1-w)^{-a}{}_2F_1\left(a, c-b; c; \frac{w}{w-1}\right), \quad (15.9)$$

as can be checked using the evaluation [69, p. 384, Formula **2.16.33.1**]

$$\int_0^\infty x^{a-1}K_\mu(bx)K_\nu(cx)\,dx$$
$$= 2^{a-3}c^{-a-\mu}\frac{b^\mu}{\Gamma(a)}\Gamma\left(\frac{a+\mu+\nu}{2}\right)\Gamma\left(\frac{a+\mu-\nu}{2}\right)\Gamma\left(\frac{a-\mu+\nu}{2}\right)\Gamma\left(\frac{a-\mu-\nu}{2}\right)$$
$$\times\, {}_2F_1\left(\frac{a+\mu+\nu}{2}, \frac{a-\mu+\nu}{2}; a; 1-\frac{b^2}{c^2}\right),$$

valid for Re $(b+c)>0$ and Re $a>|\text{Re }\mu|+|\text{Re }\nu|$. Similarly, letting $f(x,s)=xK_{s/2}(2\pi x)$ in (15.6) yields

$$\alpha^{3/2}\int_0^\infty x^2K_{s/2}(2\pi\alpha x)K_{s/2}(2\pi x)\,dx=\beta^{3/2}\int_0^\infty x^2K_{s/2}(2\pi\beta x)K_{s/2}(2\pi x)\,dx,$$

which is again a special case of Pfaff's transformation (15.9).

15.1. **A New Modular Transformation.** In [30, Theorems 4.5, 4.9], transformations of the type given in Corollary 15.3 resulting from the choices $F(z,s)=\Gamma\left(z+\frac{1}{2}s\right)\Gamma\left(1-z+\frac{1}{2}s\right)$ and $F(z,s)=\Gamma\left(\frac{1}{2}z+\frac{1}{4}s\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}z+\frac{1}{4}s\right)$ were obtained. In the following theorem, we give a new example of a function $f(x,s)$, equal to its first Koshliakov transform, constructed by choosing

$$F(z,s)=\frac{1}{\sin(\pi z)-\sin\left(\frac{1}{2}\pi s\right)}$$

which, with the help of Corollary 15.3, gives a new modular transformation. An integral involving a product of Riemann Ξ-functions (defined in (2.10)) at two different arguments linked with this transformation is also obtained.

Theorem 15.6. *Let $-1<\sigma<1$. Let*

$$\begin{aligned}f(x,s):=&\frac{x^{s/2}\zeta(1+s)}{2\sin\left(\frac{1}{2}\pi s\right)}+\frac{x^{2-s/2}}{\pi\cos\left(\frac{1}{2}\pi s\right)}\sum_{n=1}^\infty\sigma_{-s}(n)\left(\frac{n^{s-1}-x^{s-1}}{n^2-x^2}\right)\\&-2^{-s}\pi^{-1-s}x^{-s/2}\left\{\Gamma(s)\zeta(s)\left(\pi\tan\left(\frac{\pi s}{2}\right)-2(\gamma+\log x)\right)-\frac{(2\pi)^s\zeta'(1-s)}{\cos\left(\frac{1}{2}\pi s\right)}\right\}.\end{aligned}\tag{15.10}$$

Then, for $\alpha,\beta>0$ and $\alpha\beta=1$,

$$\begin{aligned}&\sqrt{\alpha}\int_0^\infty K_{s/2}(2\pi\alpha x)f(x,s)\,dx=\sqrt{\beta}\int_0^\infty K_{s/2}(2\pi\beta x)f(x,s)\,dx\\&=\frac{1}{4\pi^3}\int_0^\infty\Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Gamma\left(\frac{-s+1+it}{4}\right)\Gamma\left(\frac{-s+1-it}{4}\right)\\&\qquad\times\Xi\left(\frac{t-is}{2}\right)\Xi\left(\frac{t+is}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{t^2+(s+1)^2}\,dt.\end{aligned}\tag{15.11}$$

Remark: When x is an integer m, we adhere to the interpretation

$$\lim_{x\to m}\frac{m^{s-1}-x^{s-1}}{m^2-x^2}=\frac{(s-1)}{2}m^{s-3}.$$

Proof. Let

$$F(z,s)=\frac{1}{\sin(\pi z)-\sin\left(\frac{1}{2}\pi s\right)}.\tag{15.12}$$

We first show that for $\pm\frac{1}{2}\sigma<c=\text{Re }z<1\pm\frac{1}{2}\sigma$,

$$\mathfrak{I}(x,s):=\frac{1}{2\pi i}\int_{(c)}F(z,s)\zeta(1-z-s/2)\zeta(1-z+s/2)x^{-z}\,dz=f(x,s),\tag{15.13}$$

where $f(x,s)$ is defined in (15.10).

To prove (15.13), first assume that Re $z > 1 \pm \frac{1}{2}\sigma$, let $z = 1 - w$, and consider, for $\lambda =$ Re $w < \pm\, \frac{\sigma}{2}$, the integral

$$\mathfrak{H}(x,s) := \frac{1}{2\pi i}\int_{(\lambda)} \frac{\zeta\left(w-\frac{s}{2}\right)\zeta\left(w+\frac{s}{2}\right)}{\sin(\pi w)-\sin\left(\frac{1}{2}\pi s\right)} x^{w-1}\, dw.$$

In order to use the formula

$$\zeta\left(w-\frac{s}{2}\right)\zeta\left(w+\frac{s}{2}\right) = \sum_{n=1}^{\infty}\frac{\sigma_{-s}(n)}{n^{w-\frac{s}{2}}}, \tag{15.14}$$

which is valid for Re $w > 1 \pm \frac{1}{2}\sigma$, we need to shift the line of integration to $\lambda' =$ Re $w > 1 \pm$ Re $\frac{1}{2}\sigma$. Considering a positively oriented rectangular contour with vertices $[\lambda - iT, \lambda' - iT, \lambda' + iT, \lambda + iT, \lambda - iT]$ for $T > 0$, shifting the line of integration, considering the contributions of the simple poles at $\frac{1}{2}s$ and $1 + \frac{1}{2}s$ and of the double pole at $1 - \frac{1}{2}s$, and noting that the integrals along the horizontal segments tend to zero as $T \to \infty$, by Cauchy's residue theorem, we find that

$$\begin{aligned}\mathfrak{H}(x,s) = \frac{1}{2\pi i}\Bigg(&\frac{1}{x}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}\int_{(\lambda')}\frac{(n/x)^{-w}}{\sin(\pi w)-\sin\left(\frac{\pi s}{2}\right)}\,dw\\ &- 2\pi i\left(R_{s/2}+R_{1+s/2}+R_{1-s/2}\right)\Bigg),\end{aligned} \tag{15.15}$$

where the interchange of the order of summation and integration can be easily justified. The residues $R_{s/2}$, $R_{1+s/2}$, and $R_{1-s/2}$ are computed as

$$\begin{aligned} R_{s/2} &= \lim_{w\to\frac{1}{2}s}\frac{(w-\frac{1}{2}s)}{\sin(\pi w)-\sin\left(\frac{1}{2}\pi s\right)}\zeta\left(w-\frac{s}{2}\right)\zeta\left(w+\frac{s}{2}\right)x^{w-1}\\ &= -\frac{\zeta(s)x^{s/2-1}}{2\pi\cos\left(\frac{1}{2}\pi s\right)}, \end{aligned} \tag{15.16}$$

$$\begin{aligned} R_{1+s/2} &= \lim_{w\to 1+\frac{1}{2}s}\frac{(w-1-\frac{1}{2}s)}{\sin(\pi w)-\sin\left(\frac{1}{2}\pi s\right)}\zeta\left(w-\frac{s}{2}\right)\zeta\left(w+\frac{s}{2}\right)x^{w-1}\\ &= -\frac{\zeta(1+s)x^{s/2}}{2\sin\left(\frac{1}{2}\pi s\right)}, \end{aligned} \tag{15.17}$$

$$\begin{aligned} R_{1-s/2} &= \lim_{w\to 1-\frac{1}{2}s}\frac{d}{dw}\left\{\frac{(w-1+\frac{1}{2}s)^2}{\sin(\pi w)-\sin\left(\frac{1}{2}\pi s\right)}\zeta\left(w-\frac{s}{2}\right)\zeta\left(w+\frac{s}{2}\right)x^{w-1}\right\}\\ &= 2^{-s}\pi^{-1-s}x^{-s/2}\left\{\Gamma(s)\zeta(s)\left(\pi\tan\left(\frac{\pi s}{2}\right)-2(\gamma+\log x)\right)-\frac{(2\pi)^s\zeta'(1-s)}{\cos\left(\frac{1}{2}\pi s\right)}\right\}. \end{aligned} \tag{15.18}$$

Next,

$$\begin{aligned} &\int_{(\lambda')}\frac{(n/x)^{-w}}{\sin(\pi w)-\sin\left(\frac{1}{2}\pi s\right)}\,dw\\ &\qquad = \frac{1}{2\pi^2}\int_{(\lambda')}\frac{\pi}{\sin\left(\frac{1}{2}\pi\left(w-\frac{1}{2}s\right)\right)}\frac{\pi}{\cos\left(\frac{1}{2}\pi\left(w+\frac{1}{2}s\right)\right)}\left(\frac{n}{x}\right)^{-w}dw. \end{aligned} \tag{15.19}$$

For $0<d=$ Re $z<2$ [38, p. 345, formula **(12)**],

$$\frac{1}{2\pi i}\int_{(d)}\frac{\pi}{\sin\left(\frac{1}{2}\pi z\right)}x^{-z}\,dz=\frac{2}{1+x^2}.$$

Replace z by $w-\frac{1}{2}s$ to obtain, for $\frac{1}{2}\sigma<d'=\text{Re }w<2+\frac{1}{2}\sigma$,

$$\frac{1}{2\pi i}\int_{(d')}\frac{\pi}{\sin\left(\frac{1}{2}\pi\left(w-\frac{1}{2}s\right)\right)}x^{-w}\,dw=\frac{2x^{-s/2}}{1+x^2}. \tag{15.20}$$

Also, replace w by $w+1+s$ in (15.20), so that for $-1-\frac{1}{2}\sigma<d''=\text{Re }w<1-\frac{1}{2}\sigma$,

$$\frac{1}{2\pi i}\int_{(d'')}\frac{\pi}{\cos\left(\frac{1}{2}\pi\left(w+\frac{1}{2}s\right)\right)}x^{-w}\,dw=\frac{2x^{1+s/2}}{1+x^2}. \tag{15.21}$$

Employing (15.20) and (15.21) in (15.19) and using (2.13), we deduce that, for $\frac{1}{2}\sigma<c'=$ Re $w<1-\frac{1}{2}\sigma$,

$$\begin{aligned}\int_{(c')}\frac{(n/x)^{-w}}{\sin(\pi w)-\sin\left(\frac{1}{2}\pi s\right)}\,dw&=\frac{4i}{\pi}\int_0^\infty\frac{t^{-s/2}}{(1+t^2)}\frac{(n/(xt))^{1+s/2}}{(1+n^2/(x^2t^2))}\frac{dt}{t}\\&=\frac{4i}{\pi}\left(\frac{n}{x}\right)^{1+s/2}\int_0^\infty\frac{t^{-s}}{(1+t^2)\,(n^2/x^2+t^2)}\,dt.\end{aligned} \tag{15.22}$$

From [41, p. 330, formula **3.264.2**], for $0<\text{Re }\mu<4$, $|\arg b|<\pi$, and $|\arg h|<\pi$,

$$\int_0^\infty\frac{t^{\mu-1}}{(b+t^2)(h+t^2)}\,dt=\frac{\pi}{2\sin\left(\frac{1}{2}\mu\pi\right)}\frac{h^{\mu/2-1}-b^{\mu/2-1}}{b-h}.$$

Letting $\mu=-s+1$, $b=1$, and $h=n^2/x^2$ above, employing the resulting identity in (15.22), and simplifying, we see that, for $\frac{1}{2}\sigma<c'=\text{Re }w<1-\frac{1}{2}\sigma$,

$$\int_{(c')}\frac{(n/x)^{-w}}{\sin(\pi w)-\sin\left(\frac{1}{2}\pi s\right)}\,dw=\frac{2i}{\cos\left(\frac{1}{2}\pi s\right)}\left(\frac{(n/x)^{-s/2}-(n/x)^{1+s/2}}{1-n^2/x^2}\right).$$

Employing the residue theorem again, we find that for $\lambda'=\text{Re }w>1\pm\frac{1}{2}\sigma$,

$$\begin{aligned}&\int_{(\lambda')}\frac{(n/x)^{-w}}{\sin(\pi w)-\sin\left(\frac{1}{2}\pi s\right)}\,dw\\&\quad=\int_{(c')}\frac{(n/x)^{-w}}{\sin(\pi w)-\sin\left(\frac{1}{2}\pi s\right)}\,dw+2\pi i\lim_{w\to1-s/2}\frac{(w-1+s/2)\,(n/x)^{-w}}{\sin(\pi w)-\sin\left(\frac{1}{2}\pi s\right)}\\&\quad=\frac{2i}{\cos\left(\frac{1}{2}\pi s\right)}\left(\frac{(n/x)^{-s/2}-(n/x)^{1+s/2}}{1-n^2/x^2}\right)-\frac{2i}{\cos\left(\frac{1}{2}\pi s\right)}\,(n/x)^{s/2-1}\\&\quad=\frac{2in^{-s/2}x^{3-s/2}}{\cos\left(\frac{1}{2}\pi s\right)}\left(\frac{n^{s-1}-x^{s-1}}{n^2-x^2}\right).\end{aligned} \tag{15.23}$$

Now substitute (15.16), (15.17), (15.18), and (15.23) in (15.15) to see that, for $c''=\text{Re }z>1\pm\frac{1}{2}\sigma$,

$$\mathfrak{H}(x,z)=\frac{1}{2\pi i}\int_{(c'')}\frac{\zeta(1-z-s/2)\zeta(1-z+s/2)}{\sin(\pi z)-\sin\left(\frac{1}{2}\pi s\right)}x^{-z}\,dz$$

$$
\begin{aligned}
&= \frac{x^{s/2}\zeta(1+s)}{2\sin\left(\frac{1}{2}\pi s\right)} - \frac{\zeta(s)x^{s/2-1}}{2\pi\cos\left(\frac{1}{2}\pi s\right)} + \frac{x^{2-s/2}}{\pi\cos\left(\frac{1}{2}\pi s\right)}\sum_{n=1}^{\infty}\sigma_{-s}(n)\left(\frac{n^{s-1}-x^{s-1}}{n^2-x^2}\right)\\
&\quad - 2^{-s}\pi^{-1-s}x^{-s/2}\left\{\Gamma(s)\zeta(s)\left(\pi\tan\left(\frac{\pi s}{2}\right)-2(\gamma+\log x)\right) - \frac{(2\pi)^s\zeta'(1-s)}{\cos\left(\frac{1}{2}\pi s\right)}\right\}.
\end{aligned}
$$

Using the residue theorem again, we see that for $\pm\frac{1}{2}\sigma < c = \text{Re } z < 1 \pm \frac{1}{2}\sigma$,

$$
\begin{aligned}
\mathfrak{I}(x,s) &= \mathfrak{H}(x,s) - \lim_{z\to 1-s/2}\frac{(z-1+s/2)}{\sin(\pi z)-\sin\left(\frac{1}{2}\pi s\right)}\zeta(1-z-s/2)\zeta(1-z+s/2)x^{-z}\\
&= \frac{x^{s/2}\zeta(1+s)}{2\sin\left(\frac{1}{2}\pi s\right)} + \frac{x^{2-s/2}}{\pi\cos\left(\frac{1}{2}\pi s\right)}\sum_{n=1}^{\infty}\sigma_{-s}(n)\left(\frac{n^{s-1}-x^{s-1}}{n^2-x^2}\right)\\
&\quad - 2^{-s}\pi^{-1-s}x^{-s/2}\left\{\Gamma(s)\zeta(s)\left(\pi\tan\left(\frac{\pi s}{2}\right)-2(\gamma+\log x)\right) - \frac{(2\pi)^s\zeta'(1-s)}{\cos\left(\frac{1}{2}\pi s\right)}\right\}\\
&= f(x,s),
\end{aligned}
$$

where $f(x,s)$ is defined in (15.10). The proof of the first equality in (15.11) now follows from (15.13), Theorem 15.2, and Corollary 15.3.

In order to establish the equality between the extreme sides of (15.11), note that by (15.7), for $c = \text{Re } z > \pm\frac{1}{2}\sigma$,

$$
\frac{1}{2\pi i}\int_{(c)} 2^{z-2}(2\pi\alpha)^{-z}\Gamma\left(\frac{z}{2}-\frac{s}{4}\right)\Gamma\left(\frac{z}{2}+\frac{s}{4}\right)x^{-z}\,dz = K_{s/2}(2\pi\alpha x). \tag{15.24}
$$

From (15.24) and (2.12), for $\pm\frac{1}{2}\sigma < c = \text{Re } z < 1 \pm \frac{1}{2}\sigma$,

$$
\begin{aligned}
\sqrt{\alpha}\int_0^{\infty}K_{s/2}(2\pi\alpha x)f(x,s)\,dx &= \frac{\sqrt{\alpha}}{2\pi i}\int_{(c)}2^{-z-1}(2\pi\alpha)^{z-1}\\
&\times\frac{\Gamma\left(\frac{1}{2}-\frac{1}{2}z-\frac{1}{4}s\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}z+\frac{1}{4}s\right)}{\sin(\pi z)-\sin\left(\frac{1}{2}\pi s\right)}\zeta(1-z-\tfrac{1}{2}s)\zeta(1-z+\tfrac{1}{2}s)\,dz.
\end{aligned}
$$

Now use the functional equation (2.7) for $\zeta(1-z-s/2)$ and for $\zeta(1-z+s/2)$, (2.1), (2.2), and (2.9) to simplify the integrand, thereby obtaining

$$
\begin{aligned}
&\sqrt{\alpha}\int_0^{\infty}K_{s/2}(2\pi\alpha x)f(x,s)\,dx\\
&= \frac{1}{64\pi^3 i\sqrt{\alpha}}\int_{(c)}\frac{\Gamma\left(\frac{1}{2}z+\frac{1}{4}s-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}z+\frac{1}{4}s\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}z-\frac{1}{4}s\right)\Gamma\left(\frac{1}{2}z-\frac{1}{4}s\right)}{\left(\frac{1}{2}z-\frac{1}{4}s-\frac{1}{2}\right)\left(-\frac{1}{2}z-\frac{1}{4}s\right)}\\
&\quad\times\xi\left(z-\frac{s}{2}\right)\xi\left(z+\frac{s}{2}\right)\alpha^z\,dz.
\end{aligned} \tag{15.25}
$$

From [29, equation (2.8)], if $f(s,t) = \phi(s,it)\phi(s,-it)$, where ϕ is analytic as a function of a real variable t and of a complex variable s, then

$$
\begin{aligned}
&\int_0^{\infty} f\left(s,\frac{t}{2}\right)\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\cos\left(\tfrac{1}{2}t\log\alpha\right)\,dt\\
&= \frac{1}{i\sqrt{\alpha}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\left(s,z-\frac{1}{2}\right)\phi\left(s,\frac{1}{2}-z\right)\xi\left(z-\frac{s}{2}\right)\xi\left(z+\frac{s}{2}\right)\alpha^z\,dz.
\end{aligned} \tag{15.26}
$$

It is easy to see that with

$$\phi(z,s)=\frac{\Gamma\left(\frac{1}{2}z+\frac{1}{4}s-\frac{1}{4}\right)\Gamma\left(\frac{1}{2}z-\frac{1}{4}s+\frac{1}{4}\right)}{\left(\frac{1}{2}z-\frac{1}{4}s-\frac{1}{4}\right)},$$

and the fact that shifting the line of integration from Re $z=c$ to Re $z=\frac{1}{2}$ and using the residue theorem leaves the integral in (15.25) unchanged, this integral can be written in the form given on the right-hand side of (15.26), whence (15.26) proves the equality between the extreme sides of (15.11). □

Theorem 15.6 gives a new generalization of the following formula due to Koshliakov [56, equations (36), (40)], different from the one given in [30, Theorem 4.5].

Theorem 15.7. *Define*

$$\Lambda(x)=\frac{\pi^2}{6}+\gamma^2-2\gamma_1+2\gamma\log x+\frac{1}{2}\log^2 x+\sum_{n=1}^{\infty}d(n)\left(\frac{1}{x+n}-\frac{1}{n}\right),$$

where γ_1 is a Stieltjes constant defined in (12.5). *Then, for $\alpha,\beta>0$ and $\alpha\beta=1$,*

$$\begin{aligned}\sqrt{\alpha}\int_0^{\infty}K_0(2\pi\alpha x)\Lambda(x)\,dx&=\sqrt{\beta}\int_0^{\infty}K_0(2\pi\beta x)\Lambda(x)\,dx\\&=8\int_0^{\infty}\frac{\left(\Xi\left(\frac{1}{2}t\right)\right)^2}{(1+t^2)^2}\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{\cosh\left(\frac{1}{2}\pi t\right)}\,dt.\end{aligned}$$

Koshliakov's result can be obtained by letting $s\to 0$ in Theorem 15.6.

15.2. Transformation Involving Modified Lommel Functions: A Series Analogue of Theorem 15.6.

In [42, Theorem 6], Guinand proved the following theorem.

Theorem 15.8. *If $f(x)$ and $f'(x)$ are integrals, $f(x), xf'(x)$, and $x^2f''(x)$ belong to $L^2(0,\infty)$, and $0<|r|<1$, then*

$$\begin{aligned}&\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\left(1-\frac{n}{N}\right)^2\sigma_r(n)n^{-r/2}f(n)\right.\\&\qquad\left.-\int_0^N\left(1-\frac{x}{N}\right)^2 f(x)\left(x^{-r/2}\zeta(1-r)+x^{r/2}\zeta(1+r)\right)\,dx\right\}\\&=\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\left(1-\frac{n}{N}\right)^2\sigma_r(n)n^{-r/2}g(n)\right.\\&\qquad\left.-\int_0^N\left(1-\frac{x}{N}\right)^2 g(x)\left(x^{-r/2}\zeta(1-r)+x^{r/2}\zeta(1+r)\right)\,dx\right\},\end{aligned}$$

where

$$\int_0^x g(y)y^{r/2}\,dy=x^{(r+1)/2}\int_0^{\infty}y^{-\frac{1}{2}}f(y)\phi_{r+1}(4\pi\sqrt{xy})\,dy,$$

$$\phi_\nu(z)=\cos\left(\frac{1}{2}\pi\nu\right)J_\nu(z)-\sin\left(\frac{1}{2}\pi\nu\right)\left(Y_\nu(z)+\frac{2}{\pi}K_\nu(z)\right),$$

and $g(x)$ is chosen so that it is the integral of its derivative.

As discussed by Guinand [43, equation (1)], this gives

$$\begin{aligned}&\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}G(n,s)-\zeta(1+s)\int_0^{\infty}t^{s/2}G(t,s)\,dt-\zeta(1-s)\int_0^{\infty}t^{-s/2}G(t,s)\,dt\\&=\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}H(n,s)-\zeta(1+s)\int_0^{\infty}t^{s/2}H(t,s)\,dt-\zeta(1-s)\int_0^{\infty}t^{-s/2}H(t,s)\,dt,\end{aligned}\tag{15.27}$$

where $G(x,s)$ satisfies the same conditions as those of f in Theorem 15.8, and where

$$\begin{aligned}H(x,s)=\int_0^{\infty}G(t,s)\Big(&-2\pi\sin\left(\tfrac{1}{2}\pi s\right)J_s(4\pi\sqrt{xt})\\&-\cos\left(\tfrac{1}{2}\pi s\right)\Big(2\pi Y_s(4\pi\sqrt{xt})-4K_s(4\pi\sqrt{xt})\Big)\Big)\,dt,\end{aligned}$$

that is, $H(x,s)$ is essentially the first Koshliakov transform of $G(x,s)$.

Even though (15.4) and (15.27) are both modular transformations and both involve the first Koshliakov transform of a function, they are very different in nature as can be seen from the fact that if $f(x,s)=K_{s/2}(2\pi x)$ in (15.4), we obtain Pfaff's transformation, as discussed in (15.8) and (15.9), whereas, letting $G(x,s)=K_{s/2}(2\pi\alpha x)$ in (15.27) yields the Ramanujan–Guinand formula, as discussed in [30, Section 7].

The function $f(x,s)$ of Theorem 15.6 is equal to its first Koshliakov transform, as can be seen from (15.12), (15.13), and Theorem 15.2. Thus there are two series transformations associated with this $f(x,s)$ for a fixed s such that $-1<\sigma<1$ – one resulting from letting $G(x,s)=H(x,s)=f(x,s)$ in (15.27), and the other being the series analogue of Theorem 15.6 obtainable by interchanging the order of summation and integration in both expressions in the first equality in (15.11). The former seems more formidable than the latter. Thus we attempt the latter, which gives a beautiful transformation involving the modified Lommel functions.

There are several functions in the literature called Lommel functions. However, the ones that are important for us are those defined by [83, p. 346, equation (10)]

$$s_{\mu,\nu}(w):=\frac{w^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}{}_1F_2\left(1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2};-\frac{1}{4}w^2\right)\tag{15.28}$$

and [83, p. 347, equation (2)]

$$\begin{aligned}S_{\mu,\nu}(w)=s_{\mu,\nu}(w)&+\frac{2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\sin(\nu\pi)}\\&\times\left\{\cos\left(\frac{1}{2}(\mu-\nu)\pi\right)J_{-\nu}(w)-\cos\left(\frac{1}{2}(\mu+\nu)\pi\right)J_{\nu}(w)\right\}\end{aligned}\tag{15.29}$$

for $\nu\notin\mathbb{Z}$, and

$$\begin{aligned}S_{\mu,\nu}(w)=s_{\mu,\nu}(w)&+2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)\\&\times\left\{\sin\left(\frac{1}{2}(\mu-\nu)\pi\right)J_{\nu}(w)-\cos\left(\frac{1}{2}(\mu-\nu)\pi\right)Y_{\nu}(w)\right\}\end{aligned}\tag{15.30}$$

for $\nu\in\mathbb{Z}$. These functions are the solutions of an inhomogeneous form of the Bessel differential equation [83, p. 345], namely,

$$w^2\frac{d^2y}{dw^2}+w\frac{dy}{dw}+(w^2-\nu^2)y=w^{\mu+1}.$$

Even though $s_{\mu,\nu}(w)$ is undefined when $\mu \pm \nu$ is an odd negative integer, $S_{\mu,\nu}(w)$ has a limit at those values [83, p. 347]. These are the exceptional cases of the Lommel function $S_{\mu,\nu}(w)$. For more details on the exceptional cases, the reader is referred to [83, pp. 348–349, Section 10.73] and to a more recent article [39].

The modified Lommel functions or the Lommel functions of imaginary argument [87] are defined by

$$T_{\mu,\nu}(y) := -i^{1-\mu} S_{\mu,\nu}(iy), \tag{15.31}$$

where y is real. For further information on modified Lommel functions, the reader is referred to [72] and [74].

Lommel functions, as well as modified Lommel functions, are very useful in physics and mathematical physics. For example, see [2, 40, 75, 78, 80].

As a series analogue of Theorem 15.6, we will now obtain the following modular transformation consisting of infinite series of modified Lommel functions, of which one is an exceptional case of Lommel functions.

Theorem 15.9. *For $\alpha > 0$ and $-1 < \sigma < 1$, let*

$$\begin{aligned}\mathfrak{L}(s,\alpha) := & \frac{\pi^{s/2}\Gamma\left(-\frac{1}{2}s\right)\zeta(-s)}{8\sin\left(\frac{1}{2}\pi s\right)}\alpha^{-(s+1)/2} - 2^{-2-s}\pi^{-(s+3)/2}\alpha^{-1+s/2}\Gamma\left(\frac{1-s}{2}\right)\\ & \times\left\{\Gamma(s)\zeta(s)\left(\pi\tan\left(\frac{\pi s}{2}\right) - \gamma + 2\log(2\pi\alpha) - \psi\left(\frac{1-s}{2}\right)\right) - \frac{(2\pi)^s\zeta'(1-s)}{\cos\left(\frac{1}{2}\pi s\right)}\right\}\\ & + \frac{\sqrt{\alpha}}{\pi\cos\left(\frac{1}{2}\pi s\right)}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}\left\{2^{s/2}\Gamma\left(1+\frac{s}{2}\right)T_{-1-s/2,-s/2}(2\pi n\alpha)\right.\\ & \left. \qquad - \sqrt{\pi}2^{-s/2}\Gamma\left(\frac{3-s}{2}\right)T_{-2+s/2,s/2}(2\pi n\alpha)\right\}.\end{aligned}$$

Then, for $\alpha\beta = 1$,

$$\begin{aligned}\mathfrak{L}(s,\alpha) &= \mathfrak{L}(s,\beta)\\ &= \frac{1}{4\pi^3}\int_0^{\infty}\Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Gamma\left(\frac{-s+1+it}{4}\right)\Gamma\left(\frac{-s+1-it}{4}\right)\\ &\qquad \times \Xi\left(\frac{t-is}{2}\right)\Xi\left(\frac{t+is}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{t^2+(s+1)^2}\,dt.\end{aligned} \tag{15.32}$$

Remark: We emphasize that there are very few results in the literature involving infinite series of Lommel functions. Papers by R. G. Cooke [25] and Lewis and Zagier [60, p. 213–218] are two examples. The special case of the Lommel function $S_{\mu,\nu}(z)$ that Lewis and Zagier consider in [60, p. 214, Equation (2.15)] is

$$\mathcal{C}_s(z) = \sqrt{z}\Gamma(2s+1)S_{-2s-\frac{1}{2},\frac{1}{2}}(z).$$

To the best of our knowledge, none of these papers involve infinite series of exceptional cases of Lommel functions.

Proof. Consider the extreme left side of (15.11). Using (15.7) and (2.7), we find that

$$\int_0^{\infty}\frac{x^{s/2}\zeta(1+s)}{2\sin\left(\frac{1}{2}\pi s\right)}K_{s/2}(2\pi\alpha x)\,dx = \frac{\pi^{s/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)}{8\alpha^{1+s/2}\sin\left(\frac{1}{2}\pi s\right)}. \tag{15.33}$$

Also note that formula **2.16.20.1** of [69, p. 365] asserts that, for $|\mathrm{Re}\, w| > \mathrm{Re}\ \nu$ and real $m > 0$,

$$\int_0^\infty x^{w-1} K_\nu(mx) \log x \, dx = \frac{2^{w-3}}{m^w} \Gamma\left(\frac{w+\nu}{2}\right) \Gamma\left(\frac{w-\nu}{2}\right) \times \left\{ \psi\left(\frac{w+\nu}{2}\right) + \psi\left(\frac{w-\nu}{2}\right) - 2\log\left(\frac{m}{2}\right) \right\}. \tag{15.34}$$

Now use (15.7) and (15.34) to find, upon simplification, that

$$\begin{aligned}
&-2^{-s}\pi^{-1-s} \int_0^\infty x^{-s/2} K_{s/2}(2\pi\alpha x) \\
&\qquad \times \left\{ \Gamma(s)\zeta(s)\left(\pi\tan\left(\frac{\pi s}{2}\right) - 2(\gamma + \log x)\right) - \frac{(2\pi)^s \zeta'(1-s)}{\cos\left(\frac{\pi s}{2}\right)} \right\} dx \\
&= -2^{-2-s}\pi^{-(s+3)/2}\alpha^{-1+s/2}\Gamma\left(\frac{1-s}{2}\right) \qquad (15.35)\\
&\quad \times \left\{ \Gamma(s)\zeta(s)\left(\pi\tan\left(\frac{\pi s}{2}\right) - \gamma + 2\log(2\pi\alpha) - \psi\left(\frac{1-s}{2}\right)\right) - \frac{(2\pi)^s\zeta'(1-s)}{\cos\left(\frac{\pi s}{2}\right)} \right\}.
\end{aligned}$$

In [69, p. 347, formula **2.16.3.18**], we find that, for $y > 0$, Re $c > 0$, and Re $a > |\mathrm{Re}\ \nu|$, the integral evaluation (in corrected form)

$$\begin{aligned}
PV \int_0^\infty \frac{x^{a-1}}{x^2 - y^2} K_\nu(cx)\, dx &= \frac{\pi^2 y^{a-2}}{4\sin(\nu\pi)} \left(\cot\left(\frac{\pi(a+\nu)}{2}\right) I_\nu(cy) - \cot\left(\frac{\pi(a-\nu)}{2}\right) I_{-\nu}(cy) \right) \\
&\quad + 2^{a-4}c^{2-a}\Gamma\left(\frac{a+\nu}{2} - 1\right)\Gamma\left(\frac{a-\nu}{2} - 1\right) \\
&\quad \times {}_1F_2\left(1; 2 - \frac{\nu+a}{2}, 2 + \frac{\nu-a}{2}; \frac{c^2y^2}{4}\right).
\end{aligned} \tag{15.36}$$

(In [69], the principal value designation PV is missing.) Next we show that

$$\begin{aligned}
&\int_0^\infty x^{2-s/2} K_{s/2}(2\pi\alpha x) \sum_{n=1}^\infty \sigma_{-s}(n) \left(\frac{n^{s-1} - x^{s-1}}{n^2 - x^2}\right) dx \qquad (15.37)\\
&= \sum_{n=1}^\infty \sigma_{-s}(n) \left\{ n^{s-1} PV \int_0^\infty \frac{x^{2-s/2}K_{s/2}(2\pi\alpha x)}{n^2 - x^2}\, dx \right. \\
&\qquad \left. - PV \int_0^\infty \frac{x^{1+s/2}K_{s/2}(2\pi\alpha x)}{n^2 - x^2}\, dx \right\}.
\end{aligned}$$

Let $w(t) \in C_0^\infty$ be a smooth function so that $0 \le w(t) \le 1$ for all $t \in \mathbb{R}$, $w(t)$ has compact support in $(-\frac{1}{3}, \frac{1}{3})$, and $w(t) = 1$ in $(-\frac{1}{4}, \frac{1}{4})$. Then the right-hand side of (15.37) is equal to

$$\begin{aligned}
&\sum_{n=1}^\infty \sigma_{-s}(n) PV \int_0^\infty x^{2-s/2} K_{s/2}(2\pi\alpha x) \left(\frac{n^{s-1} - x^{s-1}}{n^2 - x^2}\right) dx \qquad (15.38)\\
&= \sum_{n=1}^\infty \sigma_{-s}(n) \int_0^\infty x^{2-s/2} K_{s/2}(2\pi\alpha x) \left(\frac{n^{s-1} - x^{s-1}}{n^2 - x^2}\right) (1 - w(x-n))\, dx
\end{aligned}$$

$$+\sum_{n=1}^{\infty}\sigma_{-s}(n)PV\int_0^{\infty}x^{2-s/2}K_{s/2}(2\pi\alpha x)\left(\frac{n^{s-1}-x^{s-1}}{n^2-x^2}\right)w(x-n)\,dx.$$

Since the series in the integrand on the left-hand side of (15.37) is absolutely convergent, we can interchange the summation and integration of the first expression on the right-hand side of (15.38). Note that if m is a positive integer and $m-\frac{1}{2}\le x\le m+\frac{1}{2}$,

$$\sum_{n=1}^{\infty}\sigma_{-s}(n)\left(\frac{n^{s-1}-x^{s-1}}{n^2-x^2}\right)w(x-n)=\sigma_{-s}(m)\left(\frac{m^{s-1}-x^{s-1}}{m^2-x^2}\right)w(x-m).$$

Therefore,

$$\begin{aligned}
&PV\int_0^{\infty}x^{2-s/2}K_{s/2}(2\pi\alpha x)\sum_{n=1}^{\infty}\sigma_{-s}(n)\left(\frac{n^{s-1}-x^{s-1}}{n^2-x^2}\right)w(x-n)\,dx\\
&=\sum_{m=1}^{\infty}PV\int_{m-1/2}^{m+1/2}x^{2-s/2}K_{s/2}(2\pi\alpha x)\sum_{n=1}^{\infty}\sigma_{-s}(n)\left(\frac{n^{s-1}-x^{s-1}}{n^2-x^2}\right)w(x-n)\,dx\\
&=\sum_{m=1}^{\infty}\sigma_{-s}(m)PV\int_{m-1/2}^{m+1/2}x^{2-s/2}K_{s/2}(2\pi\alpha x)\left(\frac{m^{s-1}-x^{s-1}}{m^2-x^2}\right)w(x-m)\,dx\\
&=\sum_{m=1}^{\infty}\sigma_{-s}(m)PV\int_0^{\infty}x^{2-s/2}K_{s/2}(2\pi\alpha x)\left(\frac{m^{s-1}-x^{s-1}}{m^2-x^2}\right)w(x-m)\,dx.
\end{aligned}$$

This justifies the interchange of summation and integration in (15.37). Using (15.36), we have

$$\begin{aligned}
&n^{s-1}\cdot PV\int_0^{\infty}\frac{x^{2-s/2}K_{s/2}(2\pi\alpha x)}{n^2-x^2}\,dx \qquad (15.39)\\
&=-\frac{\pi^2 n^{s/2}}{4\sin\left(\frac{1}{2}\pi s\right)}\left\{\cot\left(\tfrac{3\pi}{2}\right)I_{s/2}(2\pi n\alpha)-\cot\left(\tfrac{\pi}{2}(3-s)\right)I_{-s/2}(2\pi n\alpha)\right\}\\
&\quad-n^{s-1}2^{-1-s/2}(2\pi\alpha)^{-1+s/2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1-s}{2}\right){}_1F_2\left(1;\frac{1}{2},\frac{1+s}{2};\pi^2n^2\alpha^2\right)\\
&=\frac{\pi^2 n^{s/2}}{4\cos\left(\frac{1}{2}\pi s\right)}I_{-s/2}(2\pi n\alpha)-\frac{\pi^{(s-1)/2}\alpha^{-1+s/2}n^{s-1}}{4}\Gamma\left(\frac{1-s}{2}\right){}_1F_2\left(1;\frac{1}{2},\frac{1+s}{2};\pi^2n^2\alpha^2\right).
\end{aligned}$$

Another application of (15.36) yields

$$\begin{aligned}
&PV\int_0^{\infty}\frac{x^{1+s/2}K_{s/2}(2\pi\alpha x)}{x^2-n^2}\,dx=\frac{\pi^2 n^{s/2}}{4\sin\left(\frac{1}{2}\pi s\right)}\cot\left(\frac{\pi}{2}(2+s)\right)I_{s/2}(2\pi n\alpha)\\
&\quad+\lim_{\nu\to s/2}\Bigg\{2^{s/2-2}(2\pi\alpha)^{-s/2}\Gamma\left(\frac{s}{4}+\frac{\nu}{2}\right)\Gamma\left(\frac{s}{4}-\frac{\nu}{2}\right)\\
&\quad\times{}_1F_2\left(1;1-\frac{s}{4}-\frac{\nu}{2},1-\frac{s}{4}+\frac{\nu}{2};\pi^2n^2\alpha^2\right)\\
&\quad-\frac{\pi^2 n^{s/2}}{4\sin(\nu\pi)}\cot\left(\frac{\pi}{2}\left(2+\frac{s}{2}-\nu\right)\right)I_{-\nu}(2\pi n\alpha)\Bigg\}\\
&=:\frac{\pi^2 n^{s/2}\cot\left(\frac{\pi s}{2}\right)}{4\sin\left(\frac{\pi s}{2}\right)}I_{s/2}(2\pi n\alpha)+L, \qquad (15.40)
\end{aligned}$$

where we have denoted the limit by L. Note that

$$\begin{aligned}L&=\frac{1}{4}\lim_{\nu\to s/2}\Gamma\left(\frac{s}{4}-\frac{\nu}{2}\right)\Bigg\{(\pi\alpha)^{-s/2}\Gamma\left(\frac{s}{4}+\frac{\nu}{2}\right){}_1F_2\left(1;1-\frac{s}{4}-\frac{\nu}{2},1-\frac{s}{4}+\frac{\nu}{2};\pi^2n^2\alpha^2\right)\\&\quad-\frac{\pi n^{s/2}}{\sin(\nu\pi)}\cos\left(\pi\left(\frac{s}{4}-\frac{\nu}{2}\right)\right)\Gamma\left(1-\frac{s}{4}+\frac{\nu}{2}\right)I_{-\nu}(2\pi n\alpha)\Bigg\}\\&=\frac{1}{4}\lim_{\nu\to s/2}\frac{1}{\left(\frac{s}{4}-\frac{\nu}{2}\right)}\Bigg\{(\pi\alpha)^{-s/2}\Gamma\left(\frac{s}{4}+\frac{\nu}{2}\right){}_1F_2\left(1;1-\frac{s}{4}-\frac{\nu}{2},1-\frac{s}{4}+\frac{\nu}{2};\pi^2n^2\alpha^2\right)\\&\quad-\frac{\pi n^{s/2}}{\sin(\nu\pi)}\cos\left(\pi\left(\frac{s}{4}-\frac{\nu}{2}\right)\right)\Gamma\left(1-\frac{s}{4}+\frac{\nu}{2}\right)I_{-\nu}(2\pi n\alpha)\Bigg\},\end{aligned}$$

where we multiplied the expression on the right side of the first equality above by $\left(\frac{s}{4}-\frac{\nu}{2}\right)$ and used the functional equation of the Gamma function.

The last expression inside the limit symbol is of the form $\frac{0}{0}$, since

$$\begin{aligned}&\lim_{\nu\to s/2}\Bigg\{(\pi\alpha)^{-s/2}\Gamma\left(\frac{s}{4}+\frac{\nu}{2}\right){}_1F_2\left(1;1-\frac{s}{4}-\frac{\nu}{2},1-\frac{s}{4}+\frac{\nu}{2};\pi^2n^2\alpha^2\right)\\&\qquad\qquad-\frac{\pi n^{s/2}}{\sin(\nu\pi)}\cos\left(\pi\left(\frac{s}{4}-\frac{\nu}{2}\right)\right)\Gamma\left(1-\frac{s}{4}+\frac{\nu}{2}\right)I_{-\nu}(2\pi n\alpha)\Bigg\}\\&=(\pi\alpha)^{-s/2}\Gamma\left(\frac{s}{2}\right){}_0F_1\left(-;1-\frac{s}{2};\pi^2n^2\alpha^2\right)-\frac{\pi n^{s/2}}{\sin\left(\frac{1}{2}\pi s\right)}I_{-s/2}(2\pi n\alpha)\\&=0,\end{aligned}$$

as can be seen by applying the reflection formula (2.2) and the definition [41, p. 911, formula **8.406, nos. 1-2**]

$$I_\nu(w):=\begin{cases}e^{-\pi\nu i/2}J_\nu(e^{\pi i/2}w), & \text{if } -\pi<\arg w\le\frac{1}{2}\pi,\\ e^{3\pi\nu i/2}J_\nu(e^{-3\pi i/2}w), & \text{if } \frac{1}{2}\pi<\arg w\le\pi,\end{cases}\tag{15.41}$$

where $J_\nu(w)$ is defined in (2.14). Thus L'Hopital's rule yields

$$\begin{aligned}L&=-\frac{1}{2}\lim_{\nu\to s/2}\frac{d}{d\nu}\Bigg\{(\pi\alpha)^{-s/2}\Gamma\left(\frac{s}{4}+\frac{\nu}{2}\right){}_1F_2\left(1;1-\frac{s}{4}-\frac{\nu}{2},1-\frac{s}{4}+\frac{\nu}{2};\pi^2n^2\alpha^2\right)\\&\qquad\qquad-\frac{\pi n^{s/2}}{\sin(\nu\pi)}\cos\left(\pi\left(\frac{s}{4}-\frac{\nu}{2}\right)\right)\Gamma\left(1-\frac{s}{4}+\frac{\nu}{2}\right)I_{-\nu}(2\pi n\alpha)\Bigg\}\\&=-\frac{1}{2}\lim_{\nu\to s/2}\Bigg[(\pi\alpha)^{-s/2}\Bigg\{\Gamma\left(\frac{s}{4}+\frac{\nu}{2}\right)\frac{d}{d\nu}{}_1F_2\left(1;1-\frac{s}{4}-\frac{\nu}{2},1-\frac{s}{4}+\frac{\nu}{2};\pi^2n^2\alpha^2\right)\\&\qquad\qquad+\frac{1}{2}\Gamma'\left(\frac{s}{4}+\frac{\nu}{2}\right){}_1F_2\left(1;1-\frac{s}{4}-\frac{\nu}{2},1-\frac{s}{4}+\frac{\nu}{2};\pi^2n^2\alpha^2\right)\Bigg\}\\&\qquad-\pi n^{s/2}\frac{d}{d\nu}\Bigg\{\frac{\cos\left(\pi\left(\frac{s}{4}-\frac{\nu}{2}\right)\right)}{\sin(\nu\pi)}\Gamma\left(1-\frac{s}{4}+\frac{\nu}{2}\right)I_{-\nu}(2\pi n\alpha)\Bigg\}\Bigg].\end{aligned}\tag{15.42}$$

Using the series definition of ${}_1F_2$, we see that

$$\frac{d}{d\nu}{}_1F_2\left(1;1-\frac{s}{4}-\frac{\nu}{2},1-\frac{s}{4}+\frac{\nu}{2};\pi^2n^2\alpha^2\right)\Bigg|_{\nu=s/2}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty}(\pi n\alpha)^{2m}\frac{d}{d\nu}\left(\frac{1}{\left(1-\frac{s}{4}-\frac{\nu}{2}\right)_m\left(1-\frac{s}{4}+\frac{\nu}{2}\right)_m}\right)\Bigg|_{\nu=s/2}\\
&= \frac{1}{2}\sum_{m=0}^{\infty}\frac{\begin{pmatrix}-\psi\left(1-\frac{s}{4}-\frac{\nu}{2}\right)+\psi\left(1+m-\frac{s}{4}-\frac{\nu}{2}\right)\\+\psi\left(1-\frac{s}{4}+\frac{\nu}{2}\right)-\psi\left(1+m-\frac{s}{4}+\frac{\nu}{2}\right)\end{pmatrix}}{\left(1-\frac{s}{4}-\frac{\nu}{2}\right)_m\left(1-\frac{s}{4}+\frac{\nu}{2}\right)_m}(\pi n\alpha)^{2m}\Bigg|_{\nu=s/2}\\
&= \frac{1}{2}\Gamma\left(1-\frac{s}{2}\right)\sum_{m=0}^{\infty}\frac{\left(-\psi\left(1-\frac{s}{2}\right)+\psi\left(1+m-\frac{s}{2}\right)-\gamma-\psi\left(1+m\right)\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m}\\
&= -\frac{(\pi n\alpha)^{s/2}}{2}\Gamma\left(1-\frac{s}{2}\right)\left(\psi\left(1-\frac{s}{2}\right)+\gamma\right)I_{-s/2}(2\pi n\alpha)\\
&\quad+\frac{1}{2}\Gamma\left(1-\frac{s}{2}\right)\sum_{m=0}^{\infty}\frac{\left(\psi\left(1+m-\frac{s}{2}\right)-\psi\left(1+m\right)\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m},
\end{aligned}
\tag{15.43}
$$

where in the last step we used the fact $\psi(1)=-\gamma$, and also (15.41) and (2.14). Next, as $\nu\to s/2$, the last expression in (15.42) simplifies to

$$
\begin{aligned}
&\frac{d}{d\nu}\left\{\frac{\cos\left(\pi\left(\frac{s}{4}-\frac{\nu}{2}\right)\right)}{\sin(\nu\pi)}\Gamma\left(1-\frac{s}{4}+\frac{\nu}{2}\right)I_{-\nu}(2\pi n\alpha)\right\}\Bigg|_{\nu=s/2}\\
&= \lim_{\nu\to s/2}\left[\frac{\Gamma\left(1-\frac{s}{4}+\frac{\nu}{2}\right)}{2\sin(\nu\pi)}I_{-\nu}(2\pi n\alpha)\left\{\pi\left(\sin\left(\frac{\pi}{4}(s-2\nu)\right)\ -2\cos\left(\frac{\pi}{4}(s-2\nu)\right)\cot(\nu\pi)\right)\right.\right.\\
&\quad\left.\left.+\cos\left(\frac{\pi}{4}(s-2\nu)\right)\psi\left(1-\frac{s}{4}+\frac{\nu}{2}\right)\right\}+\frac{\cos\left(\frac{\pi}{4}(s-2\nu)\right)\Gamma\left(1-\frac{s}{4}+\frac{\nu}{2}\right)}{\sin(\nu\pi)}\frac{d}{d\nu}I_{-\nu}(2\pi n\alpha)\right]\\
&= -\frac{I_{-s/2}(2\pi n\alpha)}{2\sin\left(\frac{1}{2}\pi s\right)}\left(\gamma+2\pi\cot\left(\frac{\pi s}{2}\right)\right)+\frac{1}{\sin\left(\frac{1}{2}\pi s\right)}\ \frac{d}{d\nu}I_{-\nu}(2\pi n\alpha)\Bigg|_{\nu=s/2}.
\end{aligned}
\tag{15.44}
$$

Substituting (15.43) and (15.44) in (15.42), and using (2.2) in the second step below, we find that

$$
\begin{aligned}
L &= -\frac{1}{2}\Bigg[(\pi\alpha)^{-s/2}\Bigg\{\Gamma\left(\frac{s}{2}\right)\Bigg(-\frac{(\pi n\alpha)^{s/2}}{2}\Gamma\left(1-\frac{s}{2}\right)\left(\psi\left(1-\frac{s}{2}\right)+\gamma\right)I_{-s/2}(2\pi n\alpha)\\
&\quad+\frac{1}{2}\Gamma\left(1-\frac{s}{2}\right)\sum_{m=0}^{\infty}\frac{\left(\psi\left(1+m-\frac{s}{2}\right)-\psi\left(1+m\right)\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m}\Bigg)\\
&\quad+\frac{1}{2}\Gamma'\left(\frac{s}{2}\right){}_0F_1\left(-;1-\frac{s}{2};\pi^2n^2\alpha^2\right)\Bigg\}\\
&\quad-\pi n^{s/2}\Bigg\{-\frac{I_{-s/2}(2\pi n\alpha)}{2\sin\left(\frac{1}{2}\pi s\right)}\left(\gamma+2\pi\cot\left(\frac{\pi s}{2}\right)\right)+\frac{1}{\sin\left(\frac{1}{2}\pi s\right)}\ \frac{d}{d\nu}I_{-\nu}(2\pi n\alpha)\Bigg|_{\nu=s/2}\Bigg\}\Bigg]\\
&= -\frac{\pi}{2\sin\left(\frac{1}{2}\pi s\right)}\Bigg[(\pi\alpha)^{-s/2}\Bigg\{-\frac{(\pi n\alpha)^{s/2}}{2}\left(\psi\left(1-\frac{s}{2}\right)+\gamma\right)I_{-s/2}(2\pi n\alpha)\\
&\quad+\frac{1}{2}\sum_{m=0}^{\infty}\frac{\left(\psi\left(1+m-\frac{s}{2}\right)-\psi\left(1+m\right)\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m}+\frac{(\pi n\alpha)^{s/2}}{2}\psi\left(\frac{s}{2}\right)I_{-s/2}(2\pi n\alpha)\Bigg\}\\
&\quad-n^{s/2}\Bigg\{-\frac{1}{2}I_{-s/2}(2\pi n\alpha)\left(\gamma+2\pi\cot\left(\frac{\pi s}{2}\right)\right)+\frac{d}{d\nu}I_{-\nu}(2\pi n\alpha)\Bigg|_{\nu=s/2}\Bigg\}\Bigg]
\end{aligned}
$$

$$= -\frac{\pi}{2\sin\left(\frac{1}{2}\pi s\right)}\Bigg[\frac{n^{s/2}}{2}I_{-s/2}(2\pi n\alpha)\left(\psi\left(\frac{s}{2}\right)-\psi\left(1-\frac{s}{2}\right)+2\pi\cot\left(\frac{\pi s}{2}\right)\right)$$
$$+\frac{(\pi\alpha)^{-s/2}}{2}\sum_{m=0}^{\infty}\frac{\left(\psi\left(1+m-\frac{s}{2}\right)-\psi\left(1+m\right)\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m}$$
$$-n^{s/2}\,\frac{d}{d\nu}I_{-\nu}(2\pi n\alpha)\bigg|_{\nu=s/2}\Bigg]. \tag{15.45}$$

The reflection formula (2.2) implies that

$$\psi\left(\frac{s}{2}\right)-\psi\left(1-\frac{s}{2}\right)=-\pi\cot\left(\frac{\pi s}{2}\right). \tag{15.46}$$

Also, for $\nu\neq k$ or $k+\frac{1}{2}$, where k is an integer, we have [41, p. 929, formula **8.486.4**]

$$\frac{\partial I_{\nu}(w)}{\partial\nu}=I_{\nu}(w)\log\left(\frac{w}{2}\right)-\sum_{m=0}^{\infty}\frac{\psi(\nu+m+1)}{m!\Gamma\left(\nu+m+1\right)}\left(\frac{w}{2}\right)^{\nu+2m}. \tag{15.47}$$

Hence, employing (15.46) and (15.47) in (15.45), we obtain

$$L=-\frac{\pi}{2\sin\left(\frac{1}{2}\pi s\right)}\Bigg[\frac{\pi n^{s/2}\cot\left(\frac{\pi s}{2}\right)}{2}I_{-s/2}(2\pi n\alpha)$$
$$+\frac{(\pi\alpha)^{-s/2}}{2}\sum_{m=0}^{\infty}\frac{\left(\psi\left(1+m-\frac{s}{2}\right)-\psi\left(1+m\right)\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m}$$
$$-n^{s/2}\bigg\{-I_{-s/2}(2\pi n\alpha)\log(\pi n\alpha)+\sum_{m=0}^{\infty}\frac{\psi\left(1+m-\frac{s}{2}\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m-s/2}\bigg\}\Bigg]$$
$$=-\frac{\pi}{4\sin\left(\frac{1}{2}\pi s\right)}\Bigg[n^{s/2}I_{-s/2}(2\pi n\alpha)\left(\pi\cot\left(\frac{\pi s}{2}\right)+2\log(\pi n\alpha)\right)$$
$$-(\pi\alpha)^{-s/2}\sum_{m=0}^{\infty}\frac{\left(\psi\left(1+m-\frac{s}{2}\right)+\psi\left(1+m\right)\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m}\Bigg]. \tag{15.48}$$

Now substitute (15.48) in (15.40) and use the definition [83, p. 78, equation (6)]

$$K_{\nu}(w)=\frac{\pi}{2}\frac{I_{-\nu}(w)-I_{\nu}(w)}{\sin(\nu\pi)} \tag{15.49}$$

to deduce that

$$PV\int_{0}^{\infty}\frac{x^{1+s/2}K_{s/2}(2\pi\alpha x)}{x^2-n^2}\,dx$$
$$=-\frac{\pi n^{s/2}}{2}K_{s/2}(2\pi n\alpha)\cot\left(\frac{\pi s}{2}\right)-\frac{\pi n^{s/2}}{2\sin\left(\frac{\pi s}{2}\right)}I_{-s/2}(2\pi n\alpha)\log(\pi n\alpha)$$
$$+\frac{\pi(\pi\alpha)^{-s/2}}{4\sin\left(\frac{1}{2}\pi s\right)}\sum_{m=0}^{\infty}\frac{\left(\psi\left(1+m-\frac{s}{2}\right)+\psi\left(1+m\right)\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m}. \tag{15.50}$$

Hence, from (15.39) and (15.50),

$$PV\int_{0}^{\infty}x^{2-s/2}K_{s/2}(2\pi\alpha x)\left(\frac{n^{s-1}-x^{s-1}}{n^2-x^2}\right)dx$$

$$= \frac{\pi^2 n^{s/2}}{4\cos\left(\frac{1}{2}\pi s\right)} I_{-s/2}(2\pi n\alpha) - \frac{\pi^{\frac{s-1}{2}}\alpha^{-1+s/2}n^{s-1}}{4}\Gamma\left(\frac{1-s}{2}\right) {}_1F_2\left(1;\frac{1}{2},\frac{1+s}{2};\pi^2n^2\alpha^2\right)$$
$$- \frac{\pi n^{s/2}}{2}K_{s/2}(2\pi n\alpha)\cot\left(\frac{\pi s}{2}\right) - \frac{\pi n^{s/2}}{2\sin\left(\frac{1}{2}\pi s\right)} I_{-s/2}(2\pi n\alpha)\log(\pi n\alpha)$$
$$+ \frac{\pi(\pi\alpha)^{-s/2}}{4\sin\left(\frac{1}{2}\pi s\right)} \sum_{m=0}^{\infty} \frac{\left(\psi\left(1+m-\frac{s}{2}\right)+\psi\left(1+m\right)\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m}. \tag{15.51}$$

From [83, p. 349, equation (3)],

$$S_{\nu-1,\nu}(w) = -2^{\nu-2}\pi\Gamma(\nu)Y_\nu(w) \tag{15.52}$$
$$+ \frac{w^\nu}{4}\Gamma(\nu)\sum_{m=0}^{\infty}\frac{(-1)^m(w/2)^{2m}}{m!\Gamma(\nu+m+1)}\left\{2\log\left(\frac{1}{2}w\right)-\psi(\nu+m+1)-\psi(m+1)\right\},$$

where $S_{\nu-1,\nu}(w)$ is an exceptional case of the Lommel function $S_{\mu,\nu}(w)$ defined in (15.29) and (15.30). Dixon and Ferrar [31, p. 38, equation (3.11)] denote the infinite series on the right-hand side of (15.52) by $Y_{/\nu}(w)$.

Let $w = 2\pi in\alpha$ and $\nu = -\frac{1}{2}s$ in (15.52), and then use (15.41) and the relation [77, p. 233, equation (9.27)]

$$Y_\nu(iw) = e^{\frac{1}{2}(\nu+1)\pi i}I_\nu(w) - \frac{2}{\pi}e^{-\frac{1}{2}\nu\pi i}K_\nu(w), \tag{15.53}$$

which is valid for $-\pi < \arg w \leq \frac{1}{2}\pi$, to find after simplification that

$$\sum_{m=0}^{\infty} \frac{\left(\psi\left(1+m-\frac{s}{2}\right)+\psi\left(1+m\right)\right)}{m!\Gamma\left(1+m-\frac{s}{2}\right)}(\pi n\alpha)^{2m} = 2(\pi n\alpha)^{s/2}I_{-s/2}(2\pi n\alpha)\log(\pi n\alpha)$$
$$+ 2e^{i\pi s/2}(\pi n\alpha)^{s/2}K_{s/2}(2\pi n\alpha) - \frac{4(2\pi in\alpha)^{s/2}}{\Gamma\left(-\frac{1}{2}s\right)}S_{-1-s/2,-s/2}(2\pi in\alpha). \tag{15.54}$$

Also, let $\mu = \frac{s}{2}-2, \nu = \frac{s}{2}$ and $w = 2\pi in\alpha$ in (15.29) and (15.28), then substitute the latter into the former, use (15.53), and simplify to obtain

$${}_1F_2\left(1;\frac{1}{2},\frac{1+s}{2};\pi^2n^2\alpha^2\right) = (2\pi in\alpha)^{1-s/2}(1-s)S_{s/2-2,s/2}(2\pi in\alpha)$$
$$- \pi^{(3-s)/2}i^{1-s/2}\Gamma\left(\frac{s+1}{2}\right)\left(ie^{i\pi s/4}I_{s/2}(2\pi n\alpha) - \frac{2}{\pi}e^{-i\pi s/4}K_{s/2}(2\pi n\alpha)\right).$$

Hence, using (2.3), we see that

$$- \frac{\pi^{(s-1)/2}\alpha^{-1+s/2}n^{s-1}}{4}\Gamma\left(\frac{1-s}{2}\right) {}_1F_2\left(1;\frac{1}{2},\frac{1+s}{2};\pi^2n^2\alpha^2\right)$$
$$= -\frac{\pi^2n^{s/2}}{4\cos\left(\frac{1}{2}\pi s\right)}I_{s/2}(2\pi n\alpha) - \frac{i\pi n^{s/2}e^{-i\pi s/2}}{2\cos\left(\frac{1}{2}\pi s\right)}K_{s/2}(2\pi n\alpha)$$
$$- i^{1-\frac{s}{2}}\sqrt{\pi}\left(\frac{n}{2}\right)^{s/2}\Gamma\left(\frac{3-s}{2}\right)S_{s/2-2,s/2}(2\pi in\alpha). \tag{15.55}$$

Finally substituting (15.54) and (15.55) in (15.51), using (2.2) and (15.49), and simplifying, we obtain

$$PV\int_0^\infty x^{2-s/2}K_{s/2}(2\pi\alpha x)\left(\frac{n^{s-1}-x^{s-1}}{n^2-x^2}\right)dx = (2in)^{s/2}\Gamma\left(1+\frac{s}{2}\right)S_{-1-s/2,-s/2}(2\pi in\alpha)$$

$$- i^{1-s/2}\sqrt{\pi}\left(\frac{n}{2}\right)^{s/2}\Gamma\left(\frac{3-s}{2}\right)S_{-2+s/2,s/2}(2\pi in\alpha), \tag{15.56}$$

that is, the integral in (15.56) can be written simply as a linear combination of two Lommel functions of imaginary arguments, one of which belongs to the exceptional case. (Note that we can replace the second subscript $-s/2$ of the first Lommel function in (15.56) by $s/2$, since the Lommel function $S_{\mu,\nu}(w)$ is an even function of ν (see [83, p. 348]). From (15.11), (15.33), (15.35), and (15.57), we arrive at (15.32). This completes the proof of Theorem 15.9. □

Since the integral evaluation (15.56) is new and has not been recorded in the tables of integrals such as [41] and [69], we record it below as a theorem and rewrite it in terms of the modified Lommel function.

Theorem 15.10. *Let $-2 < \sigma < 3$ and $y, \alpha > 0$. Let the modified Lommel function $T_{\mu,\nu}(y)$ be defined in* (15.31). *Then*

$$PV\int_0^\infty x^{2-s/2}K_{s/2}(2\pi\alpha x)\left(\frac{y^{s-1}-x^{s-1}}{y^2-x^2}\right)dx \tag{15.57}$$
$$= (2y)^{s/2}\Gamma\left(1+\frac{s}{2}\right)T_{-1-s/2,-s/2}(2\pi\alpha y) - \sqrt{\pi}\left(\frac{y}{2}\right)^{s/2}\Gamma\left(\frac{3-s}{2}\right)T_{-2+s/2,s/2}(2\pi\alpha y).$$

Remark: The Ramanujan–Guinand formula [29, Theorem 1.2, Theorem 1.4], containing infinite series involving $\sigma_s(n)$ and the modified Bessel function $K_\nu(z)$, is similar to the series occuring in the Fourier expansion of non-holomorphic Eisenstein series [24], [60, p. 243]. In view of the fact that the modular relation involving series of Lommel functions appearing in Lewis and Zagier's work [60, p. 217] characterizes the Fourier coefficients of even Maass forms [60, p. 216, Proposition 3], we surmise that the modular transformation consisting of the series involving $\sigma_s(n)$ and the modified Lommel function that we have obtained in Theorem 15.9 may also have some important implications in the theory of Maass forms.

15.3. An Integral Representation for $f(x,s)$ Defined in (15.10) and an Equivalent Formulation of Theorem 15.6.

Some elegant integral transformations involving the function $\varphi(x,s)$ defined in (6.6) or, in particular, the function $\varphi(x,s) - x^{s/2-1}\zeta(s)/(2\pi)$, were established in [30, Theorems 6.3, 6.4]. In this subsection, we derive an equivalent form of Theorem 15.6 which yields yet another integral transformation involving this function. These integrals also involve the Lommel function $S_{\nu,\nu}(w)$.

We first derive an integral representation for $f(x,s)$ defined in (15.10).

Theorem 15.11. *Let $f(x,s)$ and $\varphi(x,s)$ be defined in* (15.10) *and* (6.6)*, respectively. Then, for $-1 < \sigma < 1$,*

$$f(x,s) = \frac{2}{\pi}x^{-s/2}\int_0^\infty \frac{t^{1+s/2}}{x^2+t^2}\left(\varphi(t,s) - \frac{\zeta(s)}{2\pi}t^{s/2-1}\right)dt. \tag{15.58}$$

Proof. Note that from [30, equation (6.9)], for $\pm\frac{1}{2}\sigma < c = \text{Re } z < 1 \pm \frac{1}{2}\sigma$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta\left(1-z+\frac{1}{2}s\right)\zeta\left(1-z-\frac{1}{2}s\right)}{2\cos\left(\frac{1}{2}\pi\left(z+\frac{1}{2}s\right)\right)}x^{-z}\,dz = \varphi(x,s) - \frac{\zeta(s)}{2\pi}x^{s/2-1}. \tag{15.59}$$

Let $s = 1$ in (2.11), replace x by x^2 and w by $\frac{1}{2}z - \frac{1}{4}s$, and use (2.2) to find that for $\frac{1}{2}\sigma < d' = \text{Re } z < 2 + \frac{1}{2}\sigma$,

$$\frac{1}{2\pi i}\int_{d'-i\infty}^{d'+i\infty}\frac{\pi}{\sin\left(\frac{1}{2}\pi\left(z-\frac{1}{2}s\right)\right)}x^{-z}\,dz = \frac{2x^{-s/2}}{1+x^2}. \tag{15.60}$$

Hence, for $\pm\frac{1}{2}\sigma < c = \text{Re } z < 1 \pm \frac{1}{2}\sigma$,

$$\begin{aligned}
f(x,s) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta(1-z-\frac{1}{2}s)\zeta(1-z+\frac{1}{2}s)}{\sin(\pi z)-\sin\left(\frac{1}{2}\pi s\right)} x^{-z}\, dz\\
&= \frac{1}{\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta(1-z-\frac{1}{2}s)\zeta(1-z+\frac{1}{2}s)}{2\cos\left(\frac{1}{2}\pi\left(z+\frac{1}{2}s\right)\right)} \frac{\pi}{\sin\left(\frac{1}{2}\pi\left(z-\frac{1}{2}s\right)\right)}\, dz\\
&= \frac{2}{\pi}x^{-s/2}\int_0^\infty \frac{t^{1+s/2}}{x^2+t^2}\left(\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right) dt, \qquad (15.61)
\end{aligned}$$

where in the last step we used (2.13). □

Now we are in a position to state and prove an equivalent formulation of Theorem 15.6.

Theorem 15.12. *Let $S_{\mu,\nu}(w)$ be the Lommel function defined in* (15.29). *For $\alpha,\beta > 0$, $\alpha\beta = 1$, and $-1 < \sigma < 1$,*

$$\begin{aligned}
&\sqrt{\alpha}\int_0^\infty S_{s/2,s/2}(2\pi\alpha t)\left(\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right) dt\\
&= \sqrt{\beta}\int_0^\infty S_{s/2,s/2}(2\pi\beta t)\left(\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right) dt\\
&= \frac{2^{s/2-2}\pi^{-\frac{5}{2}}}{\Gamma\left(\frac{1-s}{2}\right)}\int_0^\infty \Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Gamma\left(\frac{-s+1+it}{4}\right)\Gamma\left(\frac{-s+1-it}{4}\right)\\
&\qquad\qquad \times \Xi\left(\frac{t-is}{2}\right)\Xi\left(\frac{t+is}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{t^2+(s+1)^2}\, dt. \qquad (15.62)
\end{aligned}$$

Proof. Using (15.58), we write the extreme left side in (15.11) as

$$\begin{aligned}
&\sqrt{\alpha}\int_0^\infty K_{s/2}(2\pi\alpha x) f(x,s)\, dx\\
&= \frac{2\sqrt{\alpha}}{\pi}\int_0^\infty x^{-s/2}K_{s/2}(2\pi\alpha x)\int_0^\infty \frac{t^{1+s/2}}{x^2+t^2}\left(\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right) dt\, dx\\
&= \frac{2\sqrt{\alpha}}{\pi}\int_0^\infty t^{1+s/2}\left(\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right)\int_0^\infty \frac{x^{-s/2}K_{s/2}(2\pi\alpha x)}{x^2+t^2}\, dx\, dt. \qquad (15.63)
\end{aligned}$$

The interchange of the order of integration given above is delicate and hence we justify it below.

By Fubini's theorem [77, p. 30, Theorem 2.2], it suffices to show that each of the two double integrals are absolutely convergent. We begin with the first one. Thus,

$$\begin{aligned}
&\int_0^\infty \left|x^{-s/2}K_{s/2}(2\pi\alpha x)\right|\int_0^\infty \left|\frac{t^{1+s/2}}{x^2+t^2}\left(\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right)\right| dt\, dx\\
&= \int_0^1 x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right|\int_0^\infty \frac{t^{1+\sigma/2}}{x^2+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt\, dx\\
&\quad + \int_1^\infty x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right|\int_0^\infty \frac{t^{1+\sigma/2}}{x^2+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt\, dx\\
&=: I_1(s,\alpha)+I_2(s,\alpha).
\end{aligned}$$

Consider $I_2(s,\alpha)$ first. Note that

$$|I_2(s,\alpha)| = \int_1^\infty x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right| \int_0^\infty \frac{t^{1+\sigma/2}}{x^2+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt\,dx$$
$$\leq \int_1^\infty x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right| \int_0^\infty \frac{t^{1+\sigma/2}}{1+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt\,dx.$$

Now use the definition of $\varphi(t,s)$ in (6.6), the asymptotic expansion of $K_\nu(z)$ from (2.18), and the fact that $\sigma<1$ to see that the inner integral converges as $t\to\infty$. To analyze the behavior of this integral as $t\to 0$, we observe from (7.3) and (7.4) that

$$\begin{aligned}\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1} &= -\frac{\Gamma(s)\zeta(s)t^{-s/2}}{(2\pi)^s}-\frac{t^{s/2}}{2}\zeta(s+1)+\frac{t^{s/2+1}}{\pi}\sum_{n=1}^\infty \frac{\sigma_{-s}(n)}{n^2+t^2}\\ &= -\frac{\Gamma(s)\zeta(s)t^{-s/2}}{(2\pi)^s}-\frac{t^{s/2}}{2}\zeta(s+1)\\ &\quad+\frac{t^{s/2+1}}{\pi}\sum_{n=0}^\infty(-1)^n\zeta(2n+2)\zeta(2n+2+s)t^{2n}, \end{aligned}\tag{15.64}$$

where the last equality holds for $|t|<1$. In the last step, we expanded $1/(1+t^2/n^2)$ in a geometric series, interchanged the order of summation, and then used (3.7). Now $\sigma>-1$ implies the convergence of this integral near 0. Thus,

$$\int_0^\infty \frac{t^{1+\sigma/2}}{1+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt = O_s(1). \tag{15.65}$$

Using (2.18) and (15.65), we conclude that $I_2(s,\alpha)$ converges.

Now consider $I_1(s,\alpha)$. Split the inner integral as

$$\begin{aligned} I_1(s,\alpha) = \int_0^1 x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right| \Bigg\{ &\int_0^x \frac{t^{1+\sigma/2}}{x^2+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt\\ &+\int_x^\infty \frac{t^{1+\sigma/2}}{x^2+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt\Bigg\}\,dx\\ =: I_3(s,\alpha)+I_4(s,\alpha).& \end{aligned}\tag{15.66}$$

Using (15.64), we see that

$$\int_0^x \frac{t^{1+\sigma/2}}{x^2+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt = O_s\left(\frac{1}{x^2}\int_0^x t^{1+\sigma/2}\left(t^{\sigma/2}+t^{-\sigma/2}\right)dt\right). \tag{15.67}$$

Since $0<t<x<1$, if $0\leq\sigma<1$, then $t^{\sigma/2}+t^{-\sigma/2}=O\left(t^{-\sigma/2}\right)$, and so

$$\int_0^x \frac{t^{1+\sigma/2}}{x^2+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt = O_s\left(\frac{1}{x^2}\int_0^x t\,dt\right) = O_s(1), \tag{15.68}$$

whereas if $-1<\sigma<0$, then $t^{\sigma/2}+t^{-\sigma/2}=O\left(t^{\sigma/2}\right)$, and hence

$$\int_0^x \frac{t^{1+\sigma/2}}{x^2+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt = O_s\left(\frac{1}{x^2}\int_0^x t^{1+\sigma}\,dt\right) = O_s(x^\sigma).$$

Thus, from (15.67) and (15.68),

$$I_3(s,\alpha) = \int_0^1 x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right| \int_0^x \frac{t^{1+\sigma/2}}{x^2+t^2}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right| dt\,dx$$

$$= O_s\left(\int_0^1 \left(x^{\sigma/2}+x^{-\sigma/2}\right)\left|K_{s/2}(2\pi\alpha x)\right|\,dx\right). \tag{15.69}$$

Lastly, it remains to consider $I_4(s,\alpha)$. Since $\dfrac{1}{x^2+t^2}\leq\dfrac{1}{t^2}$,

$$I_4(s,\alpha)\leq\int_0^1 x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right|\int_x^\infty t^{\sigma/2-1}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right|\,dt\,dx. \tag{15.70}$$

Next, split the latter integral into two integrals, one having the limits of its inner integral from $t=x$ to $t=1$, and the other from $t=1$ to $t=\infty$. Since $\sigma<1$, it is easy to see that

$$\int_1^\infty t^{\sigma/2-1}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right|\,dt=O_s(1),$$

and thus

$$\begin{aligned}&\int_0^1 x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right|\int_1^\infty t^{\sigma/2-1}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right|\,dt\,dx\\&=O_s\left(\int_0^1 x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right|\,dx\right)\\&=O_s\left(\int_0^1\left(x^{\sigma/2}+x^{-\sigma/2}\right)\left|K_{s/2}(2\pi\alpha x)\right|\,dx\right).\end{aligned} \tag{15.71}$$

Now

$$\begin{aligned}&\int_0^1 x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right|\int_x^1 t^{\sigma/2-1}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right|\,dt\,dx\\&=\int_0^1 x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right|\int_x^{1/2} t^{\sigma/2-1}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right|\,dt\,dx\\&\quad+\int_0^1 x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right|\int_{1/2}^1 t^{\sigma/2-1}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right|\,dt\,dx.\end{aligned}$$

It is easy to see that

$$\int_{1/2}^1 t^{\sigma/2-1}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right|\,dt=O_s(1). \tag{15.72}$$

Observe that another application of (15.64) yields

$$\begin{aligned}\int_x^{1/2} t^{\sigma/2-1}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right|\,dt&=O_s\left(\int_x^{1/2} t^{\sigma/2-1}\left(t^{\sigma/2}+t^{-\sigma/2}+t^{1+\sigma/2}\right)\,dt\right)\\&=O_s(x^\sigma)+O_s\left(|\log x|\right)+O_s(x^{\sigma+1}).\end{aligned} \tag{15.73}$$

However, since $0<x<1$ and $-1<\sigma<1$, $O_s(x^{\sigma+1})=O_s(x^\sigma)$. Thus, combining this with (15.72) and (15.73), we arrive at

$$\int_x^1 t^{\sigma/2-1}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right|\,dt=O_s(x^\sigma)+O_s(1)+O_s\left(|\log x|\right).$$

This in turn gives

$$\int_0^1 x^{-\sigma/2}\left|K_{s/2}(2\pi\alpha x)\right|\int_x^1 t^{\sigma/2-1}\left|\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right|\,dt\,dx$$

$$= O_s \left(\int_0^1 \left(x^{\sigma/2} + x^{-\sigma/2} + x^{-\sigma/2} \left| \log x \right| \right) \left| K_{s/2}(2\pi\alpha x) \right| \, dx \right). \tag{15.74}$$

From (15.66), (15.69), (15.70), (15.71), and (15.74), we deduce that

$$I_1(s, \alpha) = O_s \left(\int_0^1 \left(x^{\sigma/2} + x^{-\sigma/2} + x^{-\sigma/2} \left| \log x \right| \right) \left| K_{s/2}(2\pi\alpha x) \right| \, dx \right). \tag{15.75}$$

From [1, p. 375, equations (9.6.9), (9.6.8)], as $y \to 0$, $K_\nu(y) \sim 2^{\nu-1}\Gamma(\nu)y^{-\nu}$, when $\operatorname{Re} \nu > 0$, and $K_0(y) \sim -\log y$. Hence,

$$K_{s/2}(2\pi\alpha x) = \begin{cases} O_{s,\alpha}\left(x^{-|\sigma|/2}\right), & \text{if } s \neq 0, \\ O_\alpha(\log x), & \text{if } s = 0. \end{cases}$$

If $s \neq 0$, then

$$\begin{aligned} &\int_0^1 \left(x^{\sigma/2} + x^{-\sigma/2} + x^{-\sigma/2} \left| \log x \right| \right) \left| K_{s/2}(2\pi\alpha x) \right| \, dx \\ &= O_{s,\alpha} \left(\int_0^1 \left(1 + x^{-\sigma} + x^{-\sigma} \left| \log x \right| \right) \, dx \right) = O_{s,\alpha}(1), \end{aligned}$$

since $\sigma < 1$. If $s = 0$, then

$$\begin{aligned} &\int_0^1 \left(x^{\sigma/2} + x^{-\sigma/2} + x^{-\sigma/2} \left| \log x \right| \right) \left| K_{s/2}(2\pi\alpha x) \right| \, dx \\ &= O_\alpha \left(\int_0^1 \left(x^{\sigma/2} + x^{-\sigma/2} + x^{-\sigma/2} \left| \log x \right| \right) \left| \log x \right| \, dx \right) \\ &= O_{s,\alpha}(1), \end{aligned}$$

as $\sigma > -1$.

Along with (15.75), this finally implies that $I_1(s, \alpha)$ converges. Hence, the double integral

$$\int_0^\infty x^{-s/2} K_{s/2}(2\pi\alpha x) \int_0^\infty \frac{t^{1+s/2}}{x^2 + t^2} \left(\varphi(t, s) - \frac{\zeta(s)}{2\pi} t^{s/2-1} \right) dt \, dx$$

converges absolutely. Similarly, it can be shown that

$$\int_0^\infty t^{1+s/2} \left(\varphi(t, s) - \frac{\zeta(s)}{2\pi} t^{s/2-1} \right) \int_0^\infty \frac{x^{-s/2} K_{s/2}(2\pi\alpha x)}{x^2 + t^2} \, dx \, dt$$

converges absolutely. This allows us to apply Fubini's theorem and interchange the order of integration in (15.63).

Now from [69, p. 346, equation **2.16.3.16**], for $\operatorname{Re} u > 0$, $\operatorname{Re} y > 0$, and $\pm \operatorname{Re} \nu > -\frac{1}{2}$,

$$\int_0^\infty \frac{x^{\pm\nu}}{x^2 + y^2} K_\nu(ux) \, dx = \frac{\pi^2 y^{\pm\nu-1}}{4\cos(\nu\pi)} \left(\mathbf{H}_{\mp\nu}(uy) - Y_{\mp\nu}(uy) \right), \tag{15.76}$$

where $H_\nu(w)$ is the first Struve function defined by [41, p. 942, formula **8.550.1**]

$$\mathbf{H}_\nu(w) := \sum_{m=0}^\infty (-1)^m \frac{(w/2)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right) \Gamma\left(\nu + m + \frac{3}{2}\right)}.$$

Also from [37, p. 42, formula (84)],

$$\mathbf{H}_\nu(w) = Y_\nu(w) + \frac{2^{1-\nu}\pi^{-1/2}}{\Gamma\left(\nu + \frac{1}{2}\right)} S_{\nu,\nu}(w). \tag{15.77}$$

Hence, from (15.63), (15.76), (15.77), and (2.3),

$$\sqrt{\alpha}\int_0^\infty K_{s/2}(2\pi\alpha x)f(x,s)dx$$
$$=\frac{\Gamma\left(\frac{1}{2}(1-s)\right)\sqrt{\alpha}}{2^{s/2}\sqrt{\pi}}\int_0^\infty S_{s/2,s/2}(2\pi\alpha t)\left(\varphi(t,s)-\frac{\zeta(s)}{2\pi}t^{s/2-1}\right)dt. \tag{15.78}$$

Using (15.11) and (15.78), we obtain (15.62). □

Remark: From (15.10), (15.58), and the evaluation [38, p. 345, formula **(14)**]

$$\int_0^\infty \frac{t^s}{x^2+t^2}\,dt=\frac{\pi x^{s-1}}{2\cos\left(\frac{1}{2}\pi s\right)},\quad -1<\sigma<1,$$

we find that

$$\frac{2}{\pi}x^{-s/2}\int_0^\infty \frac{t^{1+s/2}\varphi(t,s)}{x^2+t^2}\,dt=f(x,s)+\frac{x^{s/2-1}\zeta(s)}{2\pi\cos\left(\frac{1}{2}\pi s\right)}. \tag{15.79}$$

Koshliakov's formula [55, equation (7)][6], namely,

$$\int_0^\infty \frac{t\varphi(t,0)}{1+t^2}\,dt=\gamma^2-2\gamma_1-\frac{1}{4}+\frac{\pi^2}{6}-\sum_{n=1}^\infty d(n)\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

is a special case of (15.79) when $s=0$ and $x=1$.

16. Some Results Associated with the Second Koshliakov Transform

In Section 15.1, we studied a modular transformation associated with the function $f(x,s)$ defined in (15.10), which in turn, was found by choosing the function $F(z,s)$ given in (15.12). Then in (15.58) of Section 15.3, we found an integral representation for $f(x,s)$. This, however, completely obscures the discovery of this choice of $F(z,s)$, and in fact, it was discovered by first considering (15.59) and asking ourselves, what function needs to be multiplied with $1/\left(2\cos\left(\frac{\pi}{2}\left(z+\frac{s}{2}\right)\right)\right)$ in order to form a function $F(z,s)$ that satisfies $F(z,s)=F(1-z,s)$ and hence to be able to use Theorem 15.2. This motivated us to consider the function defined by the integral in the second equality in (15.61), which we then evaluated in two different ways, leading to $f(x,s)$ and its aforementioned integral representation.

With the same intention of constructing an $F(z,s)$ satisfying $F(z,s)=F(1-z,s)$ towards applying Theorem 15.4 and obtaining a modular transformation, we introduce a function defined by means of the integral

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{2\sin\left(\frac{1}{2}\pi\left(z-\frac{1}{2}s\right)\right)}{(2\pi)^{2z}}\Gamma\left(z-\frac{s}{2}\right)\Gamma\left(z+\frac{s}{2}\right)\zeta\left(z-\frac{s}{2}\right)\zeta\left(z+\frac{s}{2}\right)x^{-z}\,dz$$

for $\pm\frac{1}{2}\sigma<c'=\text{Re } z<1\pm\frac{1}{2}\sigma$. We first evaluate this integral in the half-plane $\text{Re } z>1\pm\frac{1}{2}\sigma$ as well as in the vertical strip $\pm\frac{1}{2}\sigma<c'=\text{Re } z<1\pm\frac{1}{2}\sigma$. It is easy to see, using (3.14), (3.15), and (2.5), that this integral converges in both of these regions.

Theorem 16.1. *For $c'=\text{Re } z>1\pm\frac{1}{2}\sigma$,*

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{2\sin\left(\frac{1}{2}\pi\left(z-\frac{1}{2}s\right)\right)}{(2\pi)^{2z}}\Gamma\left(z-\frac{s}{2}\right)\Gamma\left(z+\frac{s}{2}\right)\zeta\left(z-\frac{s}{2}\right)\zeta\left(z+\frac{s}{2}\right)x^{-z}\,dz$$
$$=\eta(x,s), \tag{16.1}$$

[6]Note that there is a misprint in Koshliakov's formulation of this identity, namely, there is a plus sign in front of the infinite sum, which actually should be a minus sign.

where

$$\eta(x,s) := 2i\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}\left(e^{\pi is/4}K_s\left(4\pi e^{\pi i/4}\sqrt{nx}\right) - e^{-\pi is/4}K_s\left(4\pi e^{-\pi i/4}\sqrt{nx}\right)\right).$$

Moreover, if

$$\kappa(x,s) := \eta(x,s) - \frac{1}{2\pi^2 x}\left(\frac{\Gamma(1+s)\zeta(1+s)}{(2\pi\sqrt{x})^s} + \frac{\Gamma(1-s)\zeta(1-s)\cos\left(\frac{\pi s}{2}\right)}{(2\pi\sqrt{x})^{-s}}\right), \tag{16.2}$$

then for $\pm\frac{1}{2}\sigma < c = \text{Re } z < 1 \pm \frac{1}{2}\sigma$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{2\sin\left(\frac{1}{2}\pi\left(z-\frac{1}{2}s\right)\right)}{(2\pi)^{2z}}\Gamma\left(z-\frac{s}{2}\right)\Gamma\left(z+\frac{s}{2}\right)\zeta\left(z-\frac{s}{2}\right)\zeta\left(z+\frac{s}{2}\right)x^{-z}\,dz = \kappa(x,s). \tag{16.3}$$

Remark: Note that $\eta(x,s)$ is merely i times the series in (1.19).

Proof. The proof of (16.1) is similar to that given by Koshliakov in [54, equation (11)] for the special case $s = 0$ of the series (6.6), and employs (15.7) along with (15.14). Then (16.3) follows from (16.1) by shifting the line of integration from $c' = \text{Re } z > 1 \pm \frac{1}{2}\sigma$ to $\pm\frac{1}{2}\sigma < c = \text{Re } z < 1 \pm \frac{1}{2}\sigma$, considering the contribution of the poles of the integrand at $1 \pm \frac{1}{2}s$, and employing the residue theorem. □

Given below is an analogue of Theorem 15.11.

Theorem 16.2. *Let* $\kappa(x,s)$ *be defined in* (16.2)*, and let*

$$\Phi(x,s) := 2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_s(4\pi\sqrt{nx}) - \frac{\Gamma(1+s)\zeta(1+s)x^{-1-s/2}}{(2\pi)^{2+s}} - \frac{\Gamma(1-s)\zeta(1-s)x^{-1+s/2}}{(2\pi)^{2-s}}$$

Then, for $-1 < \sigma < 1$,

$$\Phi(x,s) = \frac{x^{-s/2}}{\pi}\int_0^{\infty}\frac{t^{1+s/2}\kappa(t,s)}{t^2+x^2}\,dt.$$

Proof. The proof simply uses the identity [30, equation (5.26)]

$$\Phi(x,s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma\left(z-\frac{s}{2}\right)\Gamma\left(z+\frac{s}{2}\right)\zeta\left(z-\frac{s}{2}\right)\zeta\left(z+\frac{s}{2}\right)\frac{x^{-z}}{(2\pi)^z}\,dz,$$

valid for $\pm\frac{1}{2}\sigma < c = \text{Re } z < 1 \pm \frac{1}{2}\sigma$, (15.60), which is valid for $\frac{1}{2}\sigma < d' = \text{Re } z < 2 + \frac{1}{2}\sigma$, and (2.13). □

As of now, we have been unable to use (16.3) to construct an $f(x,s)$ in (15.5) to produce a modular transformation of the form (15.6).

17. The Second Identity on Page 336

On page 336 of his lost notebook, Ramanujan claims the following:

Let $\sigma_s(n)=\sum_{d|n}d^s$ *and let* $\zeta(s)$ *denote the Riemann zeta function. If* α, *and* β *are positive numbers such that* $\alpha\beta=4\pi^2$, *then*

$$\alpha^{(s+1)/2}\left\{\frac{1}{\alpha}\zeta(1-s)+\frac{1}{2}\zeta(-s)\tan\left(\tfrac{1}{2}\pi s\right)+\sum_{n=1}^{\infty}\sigma_s(n)\sin n\alpha\right\}$$
$$=\beta^{(s+1)/2}\left\{\frac{1}{\beta}\zeta(1-s)+\frac{1}{2}\zeta(-s)\tan\left(\tfrac{1}{2}\pi s\right)+\sum_{n=1}^{\infty}\sigma_s(n)\sin n\beta\right\}. \tag{17.1}$$

As remarked in [13], this formula is easily seen to be false because the series are divergent.

Fix an s. If a correct version of Ramanujan's identity (17.1) exists, we believe that it should be a special case of (15.27), where $G(x,s)=H(x,s)=f(x,s)$, and $f(x,s)$ is self-reciprocal with respect to the kernel

$$-2\pi\sin\left(\tfrac{1}{2}\pi s\right)J_s(4\pi\sqrt{xt})-\cos\left(\tfrac{1}{2}\pi s\right)\Big(2\pi Y_s(4\pi\sqrt{xt})-4K_s(4\pi\sqrt{xt})\Big);$$

in other words, f is equal to its first Koshliakov transform.

The appearance of $\tan\left(\frac{1}{2}\pi s\right)$ in Theorem 15.6 of Section 15 is pleasing when compared with (17.1). Thus, a series analogue of this theorem (as attempted in Section 15.2) or a series transformation through Guinand's formula (15.27) with the choice of $G(x,s)=H(x,s)=f(x,s)$, with $f(x,s)$ defined in (15.10), may shed some light on (17.1), provided, of course, a correct version of (17.1) does exist.

Acknowledgements

A significant part of this research was accomplished when the second author visited the University of Illinois at Urbana-Champaign in March, July and November of 2013, and October, 2014. He is grateful to the University of Illinois, and especially to Professor Bruce C. Berndt, for hospitality and support.

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