RAMANUJAN'S PAPER ON RIEMANN'S FUNCTIONS $\xi(s)$ AND $\Xi(t)$ AND A TRANSFORMATION FROM THE LOST NOTEBOOK

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1. Introduction

Shortly after going to England, Ramanujan wrote several influential papers that have greatly impacted mathematics of the 20^{th} and 21^{st} centuries. One of these papers, namely [24], is the one in which he obtains new representations for certain integrals containing the Riemann Ξ-function under the sign of integration. The Riemann Ξ-function is defined by [27, p. 16]

$$\Xi(t) := \xi\left(\tfrac{1}{2} + it\right),$$

where $\xi(s)$ is another function of Riemann given by

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Here $\Gamma(s)$ and $\zeta(s)$ are the Euler Gamma and the Riemann zeta functions respectively.

In his review of Ramanujan's work in [17], Hardy lists [24] as one of the four most important papers of Ramanujan (that appeared at the time of writing [17]). In this current review of Ramanujan's paper [24], we will concentrate on certain aspects of Ramanujan's results from the paper that were not stated either by Ramanujan or by Hardy in [17] as well as the connection of one of the results with a beautiful formula in Ramanujan's Lost Notebook [25]. We will then briefly look at the developments in connection with [24] that have happened since. We hope to convince the reader that Ramanujan's paper [24] is truly remarkable!

Ramanujan commences [24] by saying that the principal object of the paper is to prove a certain identity [24, Equation (1)] which gives a new representation for the expression

$$\frac{\pi^{-3/4}}{4t}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\sin\left(\frac{t}{8}\log\frac{\beta}{\alpha}\right),$$

where $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, and which he derives by equating the imaginary parts on both sides of another identity [24, Equation (8)]. However, there is no application given of the former identity which justifies the claim of it being the principal object of the paper.

One of the two main results of his paper is [24, Equation (12)]

$$\int_0^\infty \left\{e^{-z} - 4\pi\int_0^\infty \frac{xe^{-3z-\pi x^2 e^{-4z}}}{e^{2\pi x}-1}\,dx\right\}\cos(tz)\,dz = \frac{1}{8\sqrt{\pi}}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right), \tag{1.1}$$

for $z > 0$, which, through Fourier inversion, leads to [24, Equation (13)]

$$e^{-n} - 4\pi e^{-3n}\int_0^\infty \frac{xe^{-\pi x^2 e^{-4n}}}{e^{2\pi x}-1}\,dx = \frac{1}{4\pi\sqrt{\pi}}\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\cos(nt)\,dt \tag{1.2}$$

for $n \in \mathbb{R}^+$. Now the right-hand side of (1.2) is obviously an even function of n. Using a result communicated by him in his first letter to Hardy [24, Equation (14)]

$$\alpha^{-1/4}\left\{1+4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right\}=\beta^{-1/4}\left\{1+4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right\}, \tag{1.3}$$

where $\alpha, \beta > 0, \alpha\beta = \pi^2$, Ramanujan concludes that the left-hand side of (1.2) is also an even function of n, thereby showing that (1.2) holds for all real[1] values of n.

Before proceeding to discuss other results of Ramanujan, several comments are in order. First of all, letting $n = \frac{1}{2}\log\alpha$ and then noting that the integral on the right-hand side of (1.2) is invariant under the replacement of α by β, where this time $\alpha\beta = 1$, we have

$$\begin{aligned}\alpha^{\frac{1}{2}}-4\pi\alpha^{\frac{3}{2}}\int_0^\infty \frac{xe^{-\pi\alpha^2x^2}}{e^{2\pi x}-1}\,dx &= \beta^{\frac{1}{2}}-4\pi\beta^{\frac{3}{2}}\int_0^\infty \frac{xe^{-\pi\beta^2x^2}}{e^{2\pi x}-1}\,dx\\ =\frac{1}{4\pi^{3/2}}\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)&\Xi\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt. \end{aligned}\tag{1.4}$$

Obviously, the first equality in the above equation is equivalent to (1.3).

One very interesting aspect of (1.4) is that its first equality is an *integral analogue* of the transformation for the Jacobi theta function! Note that for $\mathrm{Re}(\alpha^2) > 0$, the latter is given by

$$\sum_{n=-\infty}^{\infty} e^{-\pi\alpha^2n^2}=\frac{1}{\alpha}\sum_{n=-\infty}^{\infty} e^{-\pi n^2/\alpha^2}.$$

If we let $\alpha\beta = 1$ with $\mathrm{Re}(\beta^2) > 0$, it is easy to see that the above transformation can be rephrased in the form [1, p. 43, Entry 27(i)]

$$\sqrt{\alpha}\bigg(\frac{1}{2\alpha}-\sum_{n=1}^{\infty}e^{-\pi\alpha^2n^2}\bigg)=\sqrt{\beta}\bigg(\frac{1}{2\beta}-\sum_{n=1}^{\infty}e^{-\pi\beta^2n^2}\bigg). \tag{1.5}$$

Now Hardy [15] showed that both sides of (1.5) also equal

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(t/2)}{1+t^2}\cos\left(\frac{1}{2}t\log\alpha\right)dt. \tag{1.6}$$

Hardy used the equality of the three expressions in (1.5) and (1.6) in one of the proofs of his striking result that there are infinitely many zeros of $\zeta(s)$ on the critical line $\mathrm{Re}(s) = 1/2$.

Ramanujan's formula (1.4) is clearly an analogue of the identity obtained from the equality of the three expressions in (1.5) and (1.6)! This tells us that Ramanujan obtained new representations for the integral on the extreme right-hand side of (1.4) with an eye on its possible application for studying the zeros of the Riemann zeta function. This speculation is further corroborated by the fact that the integrand of the extreme right-hand side of (1.4) involves $\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)$ which never vanishes. Thus the zeros of $\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)$ are the same as those of $\Xi\left(\frac{t}{2}\right)$. Indeed, in a short note [16] following Ramanujan's paper [24], Hardy says,

'*The integral has properties similar to those of the integral by means of which I proved recently that $\zeta(s)$ has an infinity of zeros on the line $\sigma = 1/2$, and may be used for the same purpose.*'

Moreover, Hardy's paper [15] appeared just a year before Ramanujan's!

[1]Actually the case $n = 0$ has to considered separately. However, (1.2) is seen to be true for $n = 0$ by integrating both sides of Equation (11) of [24] with respect to t from 0 to ∞.

The second half of Ramanujan's paper is even more interesting and deeper as compared to the first. In it, he obtains new representations for the product $\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)$ where $t \in \mathbb{R}^+$ and $\mathrm{Re}(s) > -1$. In the last section, he obtains curious integral representations for

$$F(n,s) := \int_0^\infty \Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\frac{\cos nt}{(s+1)^2+t^2}\,dt, \tag{1.7}$$

where $n \in \mathbb{R}$ and s lies in a vertical strip or in the half-plane $\mathrm{Re}(s) > 1$. It should be noted, however, that the expressions for this integral when $\mathrm{Re}(s) > 1$ and $-3 < \mathrm{Re}(s) < -1$, namely, Equations (19) and (21) in [24], need to be corrected in that the second expressions in the curly braces on their corresponding right-hand sides should not be present. See [7] for the corrected versions. On the other hand, the one [24, Equation (20)] for $-1 < \mathrm{Re}(s) < 1$ is correct and is as follows.

$$F(n,s) = \frac{1}{8}(4\pi)^{-\frac{1}{2}(s-3)}\int_0^\infty x^s\left(\frac{1}{\exp(xe^n)-1} - \frac{1}{xe^n}\right)\left(\frac{1}{\exp(xe^{-n})-1} - \frac{1}{xe^{-n}}\right)dx. \tag{1.8}$$

Regarding the special case $s = 0$ of the above integral, Hardy [16] says, *"the properties of this integral resemble those of one which Mr. Littlewood and I have used, in a paper to be published shortly in Acta Mathematica to prove that*[2]

$$\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+ti\right)\right|^2 dt \sim \frac{2}{\pi}T\log T \quad (T\to\infty)\text{''}.$$

This special case $s = 0$ of the integral in (1.7) also appears on page 220 of the Lost Notebook [25] where Ramanujan gives an exquisitely beautiful modular relation associated with it. To state this relation, let $\phi(x) := \psi(x) + \dfrac{1}{2x} - \log x$, where $\psi(x)$ denotes the digamma function. Let γ denote Euler's constant. Then if $\alpha > 0, \beta > 0$ and $\alpha\beta = 1$, Ramanujan claims that

$$\sqrt{\alpha}\left\{\frac{\gamma-\log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\phi(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma-\log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\phi(n\beta)\right\}$$
$$= -\frac{1}{\pi^{3/2}}\int_0^\infty\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt, \tag{1.9}$$

The first equality in (1.9) was rediscovered by Guinand [14].

In a recent preprint [6], Darses and Hillion evaluate an integral similar in appearance to the $s = 0$ case of (1.8), namely,

$$\int_0^\infty\left(\frac{1}{e^{mx}-1} - \frac{1}{mx}\right)\left(\frac{1}{e^{kx}-1} - \frac{1}{kx}\right)dx,$$

where m and k are coprime. The evaluation is in terms of cotangent sums and arises in their study of a probabilistic version of the Nyman-Beurling criterion for the Riemann hypothesis (RH).

2. Importance of studying integrals involving the Riemann Ξ-function

Regarding Ramanujan's results from [24], Hardy [17] says,

[2] Note that there is a typo in this formula in that π should not be present.

"It is difficult at present to estimate the importance of these results. The unsolved problems concerning the zeros of $\zeta(s)$ or of $\Xi(t)$ are among the most obscure and difficult in the whole range of Pure Mathematics. Any new formulae involving $\zeta(s)$ or $\Xi(t)$ are of very great interest, because of the possibility that they may throw light on some of these outstanding questions. It is, as I have shown in a short note attached to Mr. Ramanujan's paper, certainly possible to apply his formulae in this direction; but the results which can be deduced from them do not at present go beyond those obtained already by Mr. Littlewood and myself in other ways. But I should not be at all surprised if still more important applications were to be made of Mr. Ramanujan's formulae in the future".

Consider the well-known result [27, p. 36, Equation (2.16.1)]

$$\int_0^\infty \Xi(t)\cos(xt)\,dt = 2\pi^2\sum_{n=1}^\infty\left(2\pi n^4 e^{-9x/2} - 3n^2 e^{-5x/2}\right)\exp\left(-n^2\pi e^{-2x}\right),$$

which, by Fourier inversion, leads to

$$\Xi(t) = 2\int_0^\infty\left(2\sum_{n=1}^\infty\left(2\pi^2 n^4 e^{-9u/2} - 3n^2\pi e^{-5u/2}\right)\exp\left(-n^2\pi e^{-2u}\right)\right)\cos(ut)\,du. \tag{2.1}$$

By analytic continuation, the above identity is valid for complex values of t. Let $H_0 : \mathbb{C} \to \mathbb{C}$ be defined by $H_0(z) := \frac{1}{8}\Xi\left(\frac{z}{2}\right)$. Then (2.1) takes the form

$$H_0(z) = \int_0^\infty \Phi(u)\cos(zu)\,du, \tag{2.2}$$

where

$$\Phi(u) := \sum_{n=1}^\infty\left(2\pi^2 n^4 e^{-9u} - 3\pi n^2 e^{-5u}\right)\exp\left(-\pi n^2 e^{-4u}\right). \tag{2.3}$$

(Note that Φ is an even function of u.) For $v \in \mathbb{R}$, de Bruijn [5] more generally considered

$$H_v(z) := \int_0^\infty e^{vu^2}\Phi(u)\cos(zu)\,du, \tag{2.4}$$

and Polya [23] proved that if H_v has purely real zeros for some v, then $H_{v'}$ has purely real zeros for all $v' > v$. For $v \geq 1/2$, de Bruijn showed that the zeros of H_v are purely real. Newman [22] showed that there exists a constant Λ (now known as the de Bruijn-Newman constant) with $-\infty < \Lambda \leq 1/2$ with the property that H_v has purely real zeros iff $v \geq \Lambda$.

It is well known that RH is equivalent to saying all zeros of $H_0(z)$ are real. In other words, RH holds provided $\Lambda \leq 0$. Very recently, Rodgers and Tao [26] proved that $\Lambda \geq 0$. Thus RH will hold provided $\Lambda = 0$.

Viewed from this perspective, the integrals involving the Riemann Ξ-function under the sign of integration, for example, those of Ramanujan from [24], may provide alternate avenues toward the study on RH. Indeed, note that the exponential function in the numerator of the integrand on the left-hand side of (1.1) has an uncanny resemblance to that in the definition of Φ (see (2.3)) occurring in the integrand of (2.2). This again corroborates the fact that Ramanujan's results fall under the purview of an integral analogue of the Jacobi theta function. Yet further evidence for this is given in the next section.

3. The transformation formula (1.9)

The transformation formula in (1.9) occurring on page 220 of the Lost Notebook is actually among the loose papers that were published along with the Lost Notebook in [25], and is in the hand-writing of G. N. Watson. It occurs amidst some results by Ramanujan on a pair of reciprocal functions in the theory of Fourier cosine transforms. Considering Ramanujan's extraordinary abilities, it is not surprising that he immediately saw the connection of the integral on the extreme right of (1.9) with the expressions in its first equality. Most likely Ramanujan obtained the first equality through his general theorems on reciprocal functions.

The first proof of (1.9) was given in [4] along with another proof of the first equality along the lines envisaged by Guinand [14]. There exist two other equivalent forms of the first equality of (1.9). The first one is due to Koshliakov [19, Equation (6)], [20, Equations (21), (27)] who obtained a modular relation involving Laplace transform of an infinite series consisting of modified Bessel function of the second kind. The second equivalent form is discussed in the next section.

4. Recent developments

As anticipated by Hardy in the last line of his quote from Section 2, there have been many important developments connected with Ramanujan's results from [24]. To begin with, the following one-variable generalization of Ramanujan's identity (1.4) was recently obtained in [13] by Roy and the present authors for $\alpha\beta = 1$ and $w \in \mathbb{C}$:

$$\begin{aligned}
&\sqrt{\alpha}e^{\frac{w^2}{8}}\left(\operatorname{erf}\left(\frac{w}{2}\right) - 4\int_0^\infty \frac{e^{-\pi\alpha^2x^2}\sin(\sqrt{\pi}\alpha x w)}{e^{2\pi x}-1}\,dx\right)\\
&= \sqrt{\beta}e^{\frac{-w^2}{8}}\left(\operatorname{erfi}\left(\frac{w}{2}\right) - 4\int_0^\infty \frac{e^{-\pi\beta^2x^2}\sinh(\sqrt{\pi}\beta x w)}{e^{2\pi x}-1}\,dx\right)\\
&= \frac{w}{8\pi^2}\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\Delta\left(\alpha, w, \frac{1+it}{2}\right)dt, \qquad (4.1)
\end{aligned}$$

where

$$\begin{aligned}
\Delta(x,w,s) &:= \omega(x,w,s) + \omega(x,w,1-s),\\
\omega(x,w,s) &:= x^{\frac{1}{2}-s}e^{-\frac{w^2}{8}}\,{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{w^2}{4}\right).
\end{aligned}$$

Here ${}_1F_1(a;c;w)$ is the confluent hypergeometric function given by

$${}_1F_1(a;c;w) := \sum_{n=0}^\infty \frac{(a)_n w^n}{(c)_n n!},$$

where $(a)_n := a(a+1)\cdots(a+n-1)$. Also, $\operatorname{erf}(w)$ and $\operatorname{erfi}(w)$ are the error and imaginary error functions respectively defined by

$$\begin{aligned}
\operatorname{erf}(w) &:= \frac{2}{\sqrt{\pi}}\int_0^w e^{-t^2}\,dt,\\
\operatorname{erfi}(w) &:= \frac{2}{\sqrt{\pi}}\int_0^w e^{t^2}\,dt.
\end{aligned}$$

Dividing all sides of (4.1) by w and then letting $w \to 0$ clearly gives (1.4). A transformation complementary to (4.1) was also derived in [13], namely,

$$\sqrt{\alpha}e^{\frac{w^2}{8}}\left(\operatorname{erf}\left(\frac{w}{2}\right)+4\int_{-\infty}^{0}\frac{e^{-\pi\alpha^2x^2}\sin(\sqrt{\pi}\alpha xw)}{e^{2\pi x}-1}\,dx\right)$$
$$=\sqrt{\beta}e^{\frac{-w^2}{8}}\left(\operatorname{erfi}\left(\frac{w}{2}\right)+4\int_{-\infty}^{0}\frac{e^{-\pi\beta^2x^2}\sinh(\sqrt{\pi}\beta xw)}{e^{2\pi x}-1}\,dx\right). \tag{4.2}$$

The reason for stating the above two transformations is that when one subtracts the expressions in the first equality in (4.1) with the corresponding ones from (4.2), one gets an equivalent form of a result on page 198 of the Lost Notebook which is an integral analogue of Jacobi's imaginary transformation. See [3] for more on Ramanujan's results on the integral analogue of the Jacobi theta function and [13] for more details. This provides a further evidence that Ramanujan had the theory of the integral analogue of theta function in mind while discovering (1.2) and (1.3).

The one- and two-variable generalizations of (1.9) involving respectively Hurwitz's zeta function $\zeta(s,a)$ and an interesting new generalization of $\zeta(s,a)$ are given in [7] and [11]. Character analogues of (1.9) are given in [9, Corollaries 5.1, 5.2]. An equivalent form of (1.9) in terms of infinite series of Lommel functions can be obtained by letting $s = 1$ in Corollary 3.2 of [10]. Several analogues and generalizations of (1.9) are given in [2], [8] and [12]. A simple proof of (1.8) and the corrected versions of [24, Equations (19), (21)] were obtained by Kim [18] who also generalized (1.9) in the context of Lerch's transcendent. The entire last chapter of Koshliakov's monograph [21] is devoted to generalizing the results from Sections 1, 2 and 3 of Ramanujan's paper [24] in a completely different direction.

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