RAMANUJAN-HARDY-LITTLEWOOD-RIESZ PHENOMENA FOR HECKE FORMS

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Abstract. We generalize a result of Ramanujan, Hardy and Littlewood to the setting of primitive Hecke forms, which interestingly exhibits faster convergence than in the initial case of the Riemann zeta function. We also provide a criterion in the spirit of Riesz for the Riemann Hypothesis for the associated L-functions.

1. Introduction

Ramanujan's Notebooks [27, 3, 4] and his Lost Notebook [28, 2] are filled with many striking results. An example of this is the following formula discussed in [4, p. 470] involving infinite series of the Möbius function:

For any real number $p > 0$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)e^{-p/n^2}}{n} = \sqrt{\frac{\pi}{p}} \sum_{n=1}^{\infty} \frac{\mu(n)e^{-\pi^2/(n^2 p)}}{n}. \tag{1.1}$$

However, this formula is incorrect as it stands. During his stay in Cambridge, Ramanujan told Hardy and Littlewood about this identity, and later in [13, p. 156, Section 2.5] they corrected it as follows:

Let α and β be two positive numbers such that $\alpha\beta = \pi$. Assume that the series $\sum_\rho \left(\Gamma\left(\frac{1-\rho}{2}\right)/\zeta'(\rho)\right) a^\rho$ converges, where ρ runs through the non-trivial zeros of $\zeta(s)$ and a denotes a positive real number, and that the non-trivial zeros of $\zeta(s)$ are simple. Then

$$\sqrt{\alpha}\sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-\alpha^2/n^2} - \sqrt{\beta}\sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-\beta^2/n^2} = -\frac{1}{2\sqrt{\beta}} \sum_{\rho} \frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)} \beta^{\rho}. \tag{1.2}$$

The corrected identity (1.2) is discussed in detail in Berndt [4, p. 470], Paris and Kaminski [24, p. 143] and Titchmarsh [34, p. 219, Section 9.8]. Bhaskaran [5] has drawn connections of this formula with Fourier reciprocity and Wiener's Tauberian theory. Some related additional results have been recently obtained in [10]. In (1.2), one does not actually need to assume convergence of the series on the right-hand side, one may instead bracket the terms satisfying

$$|\text{Im } \rho - \text{Im } \rho'| < \exp\left(-c \text{ Im } \rho / \log(\text{Im } \rho)\right) + \exp\left(-c \text{ Im } \rho' / \log(\text{Im } \rho')\right),$$

where c is a positive constant (see [13, p. 158] and [34, p. 220]). The local spacing distribution of zeros exhibits strong repulsion between consecutive zeros. After the pioneering work of

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Montgomery [22] on the pair correlation of zeros of the Riemann zeta function, higher level correlations for general L-functions have been explained by Rudnick and Sarnak [31], and Katz and Sarnak [16], [17].

In (1.2), one still assumes simplicity of the zeros. This is widely believed to be true. The first 1.5×10^9 non-trivial zeros of the Riemann zeta function are on the critical line and are simple (see van de Lune, te Riele and Winter [20]). Also, from the work of Bui, Conrey and Young [6], who improved on earlier results by Selberg [32], Levinson [19], Heath-Brown [14] and Conrey [7], we know that at least 40.58% non-trivial zeros of the Riemann zeta function lie on the critical line and are simple. Actually, the simplicity conjecture is not really needed in the Ramanujan-Hardy-Littlewood result, in the sense that by an appropriate modification of the right-hand side of (1.2), one can prove an unconditional result:

For any positive numbers α and β such that $\alpha\beta = \pi$,

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\alpha^2/n^2}-\sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\beta^2/n^2}$$
$$=-\frac{1}{\sqrt{\beta}}\sum_{\rho}\frac{1}{(m_\rho-1)!}\left.\frac{d^{m_\rho-1}}{ds^{m_\rho-1}}(s-\rho)^{m_\rho}\frac{\Gamma\left(\frac{1}{2}-s\right)}{\zeta(2s)}\beta^{2s}\right|_{s=\rho/2}, \tag{1.3}$$

where the summation on the right-hand side is understood in the sense of bracketing [34, p. 220]*, and m_ρ is the multiplicity of the zero ρ.*

In the present paper, we obtain an analogue of the Ramanujan-Hardy-Littlewood conjecture for normalized primitive Hecke forms. Introducing a new parameter z, we also obtain a one-variable generalization which involves Bessel functions. Lastly, we provide a general criterion in the spirit of Riesz for the Riemann Hypothesis for L-functions attached to primitive Hecke forms.

Let χ be a primitive character modulo q. Let $M_k(q,\chi)$ (respectively $S_k(q,\chi)$) denote the space of modular forms (respectively cusp forms) of weight k, level q and nebentypus χ. Let $f \in S_k(q,\chi)$ be a primitive Hecke form, normalized by $a_f(1) = 1$, so that the Fourier coefficients are the same as the Hecke eigenvalues (see for example [15, p. 372-373]). Let $\bar{f}$ denote the dual of f having Fourier expansion $\bar{f}(z) := \sum_{n=1}^{\infty}\overline{a_f(n)}e^{2\pi i n z} \in S_k(q,\overline{\chi})$. Consider the associated Hecke L-function,

$$L(f,s)=\prod_{p}\left(1-a_f(p)p^{-s}+\chi(p)p^{k-1-2s}\right)^{-1}. \tag{1.4}$$

It has an analytic continuation to an entire function. The completed L-function $\Lambda(f,s) := \left(\frac{\sqrt{q}}{2\pi}\right)^s\Gamma(s)L(f,s)$ satisfies the functional equation [15, p. 375, Theorem 14.17] $\Lambda(f,s) = i^k\overline{\eta}\Lambda(\overline{f},k-s)$. Here $\eta := G(\overline{\chi})a_f(q)q^{-k/2}$, where $G(\chi)$ is the Gauss sum $G(\chi) := \sum_{m=1}^{q}\chi(m)e^{2\pi i m/q}$. We work with the normalized Dirichlet series

$$F(f,s) := L(f,s+\tfrac{k-1}{2}) = \sum_{n=1}^{\infty}\tilde{a}_f(n)n^{-s}, \tag{1.5}$$

where

$$\tilde{a}_f(n) := a_f(n)n^{-(k-1)/2}, \tag{1.6}$$

which converges absolutely for Re $s>1$ by Deligne's bound [9] $a_f(n) \ll_\epsilon n^{\frac{k-1}{2}+\epsilon}$ for any $\epsilon>0$. $F(f,s)$ has an Euler product,

$$F(f,s)=\prod_p \left(1-\tilde{a}_f(p)p^{-s}+\chi(p)p^{-2s}\right)^{-1}, \tag{1.7}$$

and satisfies the functional equation

$$\frac{\Gamma\left(s+\frac{k-1}{2}\right)}{F(\bar{f},1-s)}=i^k\bar{\eta}\left(\frac{\sqrt{q}}{2\pi}\right)^{1-2s}\frac{\Gamma\left(\frac{k+1}{2}-s\right)}{F(f,s)}. \tag{1.8}$$

The non-trivial zeros of $F(f,s)$ lie in the critical strip $0<\text{Re } s<1$, and by GRH are conjectured to be on the critical line Re $s=\frac{1}{2}$.

Let ${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;w)$ denote the generalized hypergeometric function [1, p. 62], [25, p. 73] defined by

$${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;w)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{w^n}{n!}, \tag{1.9}$$

where $(a)_n:=a(a+1)\cdots(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)}$. The series in (1.9) converges for all w in the complex plane if $p\le q$. If $p=q+1$, it converges for $|w|<1$. Kummer's first transformation [1, p. 191, Equation (4.1.11)], [25, p. 125, Equation (2)] states that

$${}_1F_1(a;c;w)=e^w{}_1F_1(c-a;c;-w). \tag{1.10}$$

An important special function, the Bessel function $J_\nu(w)$ of order ν, is defined in terms of ${}_0F_1$ by [1, p. 200, Equation (4.5.2)], [12, p. 910, formula 8.402]

$$J_\nu(w):=\frac{(w/2)^\nu}{\Gamma(\nu+1)}{}_0F_1\left(-;\nu+1;-\frac{w^2}{4}\right). \tag{1.11}$$

The modified Bessel function of imaginary argument, denoted by $I_\nu(w)$, is defined by [12, p. 911, formula 8.406, nos. 1-2]

$$I_\nu(w)=\begin{cases} e^{-\frac{\pi}{2}\nu i}J_\nu(e^{\frac{\pi i}{2}}w) & \text{if } -\pi<\arg w\le\frac{\pi}{2},\\ e^{\frac{3}{2}\pi\nu i}J_\nu(e^{-\frac{3}{2}\pi i}w) & \text{if } \frac{\pi}{2}<\arg w\le\pi.\end{cases} \tag{1.12}$$

For a positive integer n, the formula reduces to [12, p. 911, formula 8.406, no. 3]

$$I_n(w)=i^{-n}J_n(iw). \tag{1.13}$$

We are now ready to state our first result.

Theorem 1.1. *Let k be an even positive integer and χ be a primitive Dirichlet character modulo q. Let $f\in S_k(q,\chi)$ be a normalized primitive Hecke form. Let $F(f,s)$ and $\tilde{a}_f(n)$ be defined as in* (1.5) *and* (1.6) *respectively. Let $J_\nu(z)$ and $I_\nu(z)$ denote the Bessel functions defined in* (1.11) *and* (1.12)*. Let $\mu(n)$ denote the Möbius function. If ρ runs through the non-trivial zeros of $F(f,s)$, and α and β are positive numbers such that $\alpha\beta=4/q$, then*

$$\alpha^{\frac{k}{2}}e^{\frac{z^2}{8}}\sum_{\substack{d,D=1\\(d,D)=1\\D \text{ squarefree}}}^{\infty}\frac{\mu(d)\chi(D)\tilde{a}_{\bar{f}}(d)}{d^{\frac{k+1}{2}}D^{k+1}}\frac{e^{-\frac{\pi\alpha}{dD^2}}}{\left(\sqrt{\frac{\pi\alpha}{dD^2}}z\right)^{\frac{k}{2}-1}}J_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi\alpha}{dD^2}}z\right)$$

$$- i^k\bar{\eta}\beta^{\frac{k}{2}}e^{\frac{-z^2}{8}}\sum_{\substack{d,D=1\\(d,D)=1\\D\ squarefree}}^{\infty}\frac{\mu(d)\chi(D)\tilde{a}_f(d)}{d^{\frac{k+1}{2}}D^{k+1}}\frac{e^{-\frac{\pi\beta}{dD^2}}}{\left(\sqrt{\frac{\pi\beta}{dD^2}}z\right)^{\frac{k}{2}-1}}$$

$$\times\sum_{m=0}^{\frac{k}{2}}(-1)^m\binom{\frac{k}{2}}{m}I_{m+\frac{k}{2}-1}\left(\sqrt{\frac{\pi\beta}{dD^2}}z\right)\left(\frac{4\pi\beta}{dD^2z^2}\right)^{-\frac{m}{2}}$$

$$= -\frac{i^k\bar{\eta}\pi^{-\frac{(k+1)}{2}}e^{\frac{-z^2}{8}}}{\sqrt{\beta}2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\sum_{\rho}\frac{1}{(m_\rho-1)!}\frac{d^{m_\rho-1}}{ds^{m_\rho-1}}(s-\rho)^{m_\rho}\frac{\Gamma\left(\frac{k+1}{2}-s\right)}{F(f,s)}{}_1F_1\left(\frac{1}{2}-s;\frac{k}{2};\frac{z^2}{4}\right)(\pi\beta)^s\Bigg|_{s=\rho}, \tag{1.14}$$

where m_ρ is the multiplicity of the zero $\rho := \delta + i\gamma$ and the sum over ρ involves bracketing the terms so that the terms for which

$$|\gamma-\gamma'| < \exp\left(-A_1|\gamma|/\log(|\gamma|+3)\right) + \exp\left(-A_1|\gamma'|/\log(|\gamma'|+3)\right),$$

where A_1 is a positive constant, are included in the same bracket.

We expect the pairs of zeros $\{\rho, \rho'\}$ that need to be bracketed together in (1.14) to occur very rarely.

The Ramanujan tau function $\tau(n)$ is defined through its generating function $\Delta(\zeta) = e^{2\pi i\zeta}(e^{2\pi i\zeta}; e^{2\pi i\zeta})_\infty^{24}$ (known as the modular discriminant or the Ramanujan Delta function) by $\Delta(\zeta) := \sum_{n=1}^{\infty}\tau(n)e^{2\pi in\zeta}$. The Delta function is a cusp form of level 1 and weight 12. The tau function enjoys many beautiful properties. The deepest of the conjectures made by Ramanujan on the tau function [26], namely that, $|\tau(n)| \leq d(n)n^{11/2}$, where $d(n)$ denotes the divisor function, was proved by Deligne [9]. A still unsolved problem on Ramanujan's tau function is the Lehmer conjecture [18] which states that the tau function never vanishes, i.e., $\tau(n) \neq 0$ for any $n > 0$.

Let $f = \Delta$ in Theorem 1.1 so that $f = \bar{f}$. Moreover, $\tilde{a}_f(n) = \tilde{a}_\Delta(n) = \tau(n)n^{-11/2}$, $q = 1$, $\chi(n) = 1$, $\eta = \bar{\eta} = 1$ and $k = 12$. Then let $z \to 0$. Then Theorem 1.1 reduces to the following analogue of (1.2).

Corollary 1.2. *Let α and β be two positive numbers such that $\alpha\beta = \pi$. Let ρ run through the non-trivial zeros of $F(\Delta, s)$ and assume that these non-trivial zeros are simple. Then,*

$$\alpha^6\sum_{\substack{d,D=1\\(d,D)=1\\D\ squarefree}}^{\infty}\frac{\mu(d)\tau(d)}{d^{12}D^{13}}e^{-\frac{2\sqrt{\pi}\alpha}{dD^2}} - \beta^6\sum_{\substack{d,D=1\\(d,D)=1\\D\ squarefree}}^{\infty}\frac{\mu(d)\tau(d)}{d^{12}D^{13}}e^{-\frac{2\sqrt{\pi}\beta}{dD^2}}$$

$$= -\frac{(2\sqrt{\pi})^{-\frac{13}{2}}}{\sqrt{\beta}}\sum_{\rho}\frac{\Gamma\left(\frac{13}{2}-\rho\right)}{F'(\Delta,\rho)}\left(2\sqrt{\pi}\beta\right)^{\rho}, \tag{1.15}$$

where the sum over $\rho := \delta + i\gamma$ involves bracketing the terms so that the terms for which

$$|\gamma-\gamma'| < \exp\left(-A_1|\gamma|/\log(|\gamma|+3)\right) + \exp\left(-A_1|\gamma'|/\log(|\gamma'|+3)\right),$$

where A_1 is a positive constant, are included in the same bracket.

Similarly, if we let $z = 0$ in Theorem 1.1 and further assume that the non-trivial zeros ρ of $F(f, s)$ are simple, then we obtain a more general analogue of (1.2) of which (1.15) is a special case.

Some comments are in order. Our goal was to obtain an identity which relates the coefficients of a primitive Hecke form, the non-trivial zeros of the associated L-functions, and the Möbius function, and this naturally involves working with $1/F(f,s)$. We also wanted a one-variable generalization, achieved in Theorem 1.1, which for instance, leads to an infinite sequence of identities by successively differentiating both sides with respect to z and then letting $z=0$.

Perhaps the nicest feature of Theorem 1.1 above is that the identity is built in such a way that all sums converge quickly in practice. The series on the left-hand side of (1.15) certainly converge faster than the series on the left-hand side of (1.2). This has not been done at the expense of the convergence on the right-hand side; both right-hand sides of (1.2) and (1.15) are rapidly convergent. In an appendix below, we collect some numerical data. Tables 2 and 3 compare the rates of convergence of the sums on each of the two sides of (1.2) and (1.15). For $f=\Delta$, Table 1 compares both sides of (1.14), where α takes on some positive integer values and z runs over a few small Gaussian integers. For the right-hand side of (1.14), we employ the list of zeros of the L-function associated with the Ramanujan tau function provided by Michael Rubinstein [30]. The list contains the first 284410 zeros with positive imaginary part, but it turns out that already the contribution of the first 100 zeros matches the left- hand side with an error less than 10^{-4}.

The identity (1.2) led Hardy and Littlewood to obtain the following equivalence criterion for the Riemann Hypothesis for $\zeta(s)$ by adopting an approach similar to the one considered by Riesz [29]:

Consider the function $P(\beta):=\sum_{m=1}^{\infty}\frac{(-\beta)^m}{m!\zeta(2m+1)}$. *Then, the estimate* $P(\beta)=O_\delta\left(\beta^{-\frac{1}{4}+\delta}\right)$ *as* $\beta\to\infty$ *for all positive values of* δ *is equivalent to the Riemann Hypothesis.*

As an application of Theorem 1.1, we note that it leads one to a criterion for L-functions attached to primitive Hecke forms, which involves the values of the given L-function at half integers in the half plane of absolute convergence. Assuming the Riemann Hypothesis for $F(f,s)$, assuming the non-trivial zeros are simple, assuming the absolute convergence of the series $\sum_\rho \frac{\pi^\rho\Gamma\left(\frac{k+1}{2}-\rho\right)}{F'(f,\rho)}\, {}_1F_1\left(\frac{1}{2}-\rho;\frac{k}{2};\frac{z^2}{4}\right)$ over the non-trivial zeros ρ, and letting $\alpha\to\infty$ (or equivalently $\beta\to 0$, where $\alpha\beta=4/q$), it follows from (1.14) that

$$\sum_{\substack{d,D=1\\(d,D)=1\\D\text{ squarefree}}}^{\infty}\frac{\mu(d)\chi(D)\tilde{a}_{\bar{f}}(d)}{d^{\frac{k+1}{2}}D^{k+1}}\frac{e^{-\frac{\pi\alpha}{dD^2}}}{\left(\sqrt{\frac{\pi\alpha}{dD^2}}z\right)^{\frac{k}{2}-1}}J_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi\alpha}{dD^2}}z\right)=O_{f,z}\left(\alpha^{-\frac{k}{2}}\right). \tag{1.16}$$

We will see below that the assumption of the Riemann Hypothesis for $F(f,s)$ alone is already enough in order to establish estimates such that (1.16) above. For a technical reason coming from the fact that unlike in the case of $\zeta(s)$, our L-function $F(f,s)$ does not have a pole at $s=1$, in what follows, we will be working with the derivative of the left-hand side of (1.16) with respect to α rather than the left-hand side itself. As we shall see later, this derivative equals the function $Q(\alpha,f,z)$ defined in our next theorem, which establishes the desired criterion.

Theorem 1.3. *Let k be an even positive integer. Let f be a primitive Hecke form of weight k, level q and nebentypus χ. Let $z\in\mathbb{C}$. Let $F(f,s)$ be defined as in* (1.5). *Consider the* *function*

$$Q(\alpha, f, z) := \frac{1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} \sum_{m,t=0}^{\infty} \frac{(m+t)(-\pi)^{m+t}\alpha^{m+t-1}(z^2/4)^t}{m!t!\left(\frac{k}{2}\right)_t F\left(f, m+t+\frac{k+1}{2}\right)}, \tag{1.17}$$

defined for all $\alpha \in \mathbb{R}^+$. *Then we have the following:*

(1) *The Riemann Hypothesis for* $F(f,s)$ *implies* $Q(\alpha, f, z) = O_{f,\delta,z}\left(\alpha^{-\frac{k}{2}-1+\delta}\right)$ *as* $\alpha \to \infty$ *for all positive values of* δ.

(2) (a) *If* $z = 0$, *the estimate* $Q(\alpha, f, z) = O_{f,\delta,z}\left(\alpha^{-\frac{k}{2}-1+\delta}\right)$ *as* $\alpha \to \infty$ *for all positive values of* δ *implies the Riemann Hypothesis for* $F(f,s)$.

(b) *If* $z \neq 0$, *the estimate* $Q(\alpha, f, z) = O_{f,\delta,z}\left(\alpha^{-\frac{k}{2}-1+\delta}\right)$ *as* $\alpha \to \infty$ *for all positive values of* δ *implies that* $F(f,s)$ *has at most finitely many non-trivial zeros off the critical line.*

2. Inverse Mellin Transforms involving confluent hypergeometric function

The following lemma is of independent interest in itself. This lemma and the corollaries that follow may be added as new entries in the existing tables of Mellin transforms, for example, in [11] and in [23].

Lemma 2.1. *For* $a, b \in \mathbb{C}$, $n \in \mathbb{Z}^+$, *Re* $\nu > 0$, *and* $c = \text{Re } s > 0$, *we have*

$$\begin{aligned} &\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} a^{-s}e^{-\frac{b^2}{4a}}\Gamma(s)\,{}_1F_1\left(n\nu - s; \nu; \frac{b^2}{4a}\right)x^{-s}\,ds \\ &= e^{-ax}\sum_{j=0}^{\infty} {}_1F_1\left(-(n-1)\nu; \nu+j; -\frac{b^2}{4a}\right)\frac{(-\frac{b^2x}{4})^j}{(\nu)_j j!}. \end{aligned} \tag{2.1}$$

Proof. Employing the series representation of ${}_1F_1$ and then interchanging the order of summation and integration, we have

$$\begin{aligned} &\int_{c-i\infty}^{c+i\infty} a^{-s}\Gamma(s)\,{}_1F_1\left(n\nu - s; \nu; \frac{b^2}{4a}\right)x^{-s}\,ds \\ &= \sum_{m=0}^{\infty}\frac{(\frac{b^2}{4a})^m}{(\nu)_m m!}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(n\nu - s + m)}{\Gamma(n\nu - s)}(ax)^{-s}\,ds \\ &= (ax)^{-\frac{n\nu}{2}}\sum_{m=0}^{\infty}\frac{(\frac{b^2}{4a})^m}{(\nu)_m m!}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(n\nu/2 + w)\Gamma(n\nu/2 - w + m)}{\Gamma(n\nu/2 - w)}(ax)^{-w}\,dw, \end{aligned} \tag{2.2}$$

where in the last step, we made the change of variable $s = w + n\nu/2$. Let $c' = $Re w so that $c' > -$ Re $\frac{n\nu}{2}$. Letting $\alpha = \frac{n\nu}{2}$ and $\beta = \frac{n\nu}{2} + m$, with $m \in \mathbb{Z}, m \geq 0$, in the formula [23, p. 197, (5.45)],

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha + w)\Gamma(\beta - w)}{\Gamma(\alpha - w)}x^{-w}\,dw = \frac{\Gamma(\alpha+\beta)}{\Gamma(2\alpha)}e^{-x/2}M_{\beta,\alpha-\frac{1}{2}}(x), \tag{2.3}$$

valid for $-$ Re $\alpha <$ Re $w <$ Re β, Re $(\alpha+\beta) > 0$, and using the relation [12, p. 1024, formula 9.220, no.2]

$$M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-z/2}\,{}_1F_1\left(\mu - \lambda + \tfrac{1}{2}; 2\mu+1; z\right), \tag{2.4}$$

we obtain

$$\begin{aligned}
&\int_{c-i\infty}^{c+i\infty} a^{-s}\Gamma(s)\,{}_1F_1\left(n\nu-s;\nu;\frac{b^2}{4a}\right)x^{-s}\,ds\\
&=2\pi i e^{-\frac{ax}{2}}(ax)^{-\frac{n\nu}{2}}\sum_{m=0}^{\infty}\frac{(\frac{b^2}{4a})^m}{(\nu)_m m!}\frac{\Gamma(n\nu+m)}{\Gamma(n\nu)}M_{\frac{n\nu}{2}+m,\frac{n\nu-1}{2}}(ax)\\
&=2\pi i e^{-ax}\sum_{m=0}^{\infty}\frac{(\frac{b^2}{4a})^m(n\nu)_m}{(\nu)_m m!}\,{}_1F_1(-m;n\nu;ax). \qquad (2.5)
\end{aligned}$$

Employing the series representation of ${}_1F_1$ again and observing that [33, p. 72]

$$(-m)_t=\begin{cases}0 & \text{if } t>m,\\ \frac{(-1)^t m!}{(m-t)!} & \text{if } t\le m,\end{cases}$$

we derive

$$\begin{aligned}
&\int_{c-i\infty}^{c+i\infty} a^{-s}\Gamma(s)\,{}_1F_1\left(n\nu-s;\nu;\frac{b^2}{4a}\right)x^{-s}\,ds\\
&=2\pi i e^{-ax}\sum_{m=0}^{\infty}\frac{(\frac{b^2}{4a})^m(n\nu)_m}{(\nu)_m m!}\sum_{t=0}^{m}\frac{m!(-ax)^t}{(m-t)!(n\nu)_t t!}\\
&=2\pi i e^{-ax}\sum_{t=0}^{\infty}\sum_{d=0}^{\infty}\frac{(\frac{b^2}{4a})^{t+d}(n\nu)_{t+d}(-ax)^t}{(\nu)_{t+d}(n\nu)_t d!t!}\\
&=2\pi i e^{-ax}\sum_{t=0}^{\infty}\frac{(-\frac{b^2x}{4})^t}{(\nu)_t t!}\sum_{d=0}^{\infty}\frac{(\frac{b^2}{4a})^d(n\nu+t)_d}{(\nu+t)_d d!}\\
&=2\pi i e^{-ax}\sum_{t=0}^{\infty}{}_1F_1\left(n\nu+t;\nu+t;\frac{b^2}{4a}\right)\frac{(-\frac{b^2x}{4})^t}{(\nu)_t t!}\\
&=2\pi i e^{-ax+\frac{b^2}{4a}}\sum_{t=0}^{\infty}{}_1F_1\left(-(n-1)\nu;\nu+t;-\frac{b^2}{4a}\right)\frac{(-\frac{b^2x}{4})^t}{(\nu)_t t!}, \qquad (2.6)
\end{aligned}$$

where in the last step we used (1.10). This completes the proof of the lemma. ☐

Corollary 2.2. *Let $J_\nu(z)$ be defined as before. For $c = Re\ s > 0$, $Re\ \nu > 0$, and $a, b \in \mathbb{C}$, we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} a^{-s}e^{-\frac{b^2}{4a}}\Gamma(s)\,{}_1F_1\left(\nu-s;\nu;\frac{b^2}{4a}\right)x^{-s}\,ds=\frac{e^{-ax}\Gamma(\nu)J_{\nu-1}(b\sqrt{x})}{\left(\frac{b\sqrt{x}}{2}\right)^{\nu-1}}. \qquad (2.7)$$

Remark. Note that replacing x by x^2 and substituting $\nu = 1/2$ in (2.7), and then using the fact [1, p. 202, (4.6.4)] $J_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\cos z$, we obtain the well-known result [23, p. 47, Equation 5.30]

$$e^{-ax^2}\cos bx=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2}a^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)e^{-\frac{b^2}{4a}}\,{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};\frac{b^2}{4a}\right)x^{-s}\,ds. \qquad (2.8)$$

Corollary 2.3. *Let k be an even positive integer and let $I_k(z)$ be defined as before. Then,*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{z^2}{4}}\Gamma(s){}_1F_1\left(k-s;\frac{k}{2};-\frac{z^2}{4}\right)x^{-s}\,ds$$
$$=\frac{e^{-x}2^{\frac{k}{2}-1}\Gamma(\frac{k}{2})}{(z\sqrt{x})^{\frac{k}{2}-1}}\sum_{m=0}^{k/2}\binom{k/2}{m}I_{m+\frac{k}{2}-1}(z\sqrt{x})\left(-\frac{z}{2\sqrt{x}}\right)^{m}. \tag{2.9}$$

3. A Ramanujan-Hardy-Littlewood type identity for primitive Hecke forms

We now present a proof of Theorem 1.1. Let $F(f,s)^{-1}=\sum_{n=1}^{\infty}\tilde{b}_f(n)n^{-s}$ for Re $s>1$. Then from (1.7),

$$\sum_{n=1}^{\infty}\frac{\tilde{b}_f(n)}{n^s}=\prod_p\left(1-\frac{\tilde{a}_f(p)}{p^s}+\frac{\chi(p)}{p^{2s}}\right). \tag{3.1}$$

Here $p^m|n$ implies $m\le 2$, thus, only those terms for which n can be written in the form $n=dD^2$, where d and D are both squarefree and relatively prime, appear on the left-hand side of (3.1). Since both $\tilde{a}_f(n)$ and $\chi(n)$ are multiplicative, we can write

$$\tilde{b}_f(n)=\begin{cases}\mu(d)\chi(D)\tilde{a}_f(d), & \text{if } n=dD^2,\ (d,D)=1 \text{ and } d,\ D \text{ squarefree}\\ 0, & \text{otherwise.}\end{cases} \tag{3.2}$$

The first infinite sum on the left hand side of (1.14) can be rephrased as

$$\sum_{\substack{d,D=1\\(d,D)=1\\D\text{ squarefree}}}^{\infty}\frac{\mu(d)\chi(D)\tilde{a}_{\bar{f}}(d)}{d^{\frac{k+1}{2}}D^{k+1}}\frac{e^{-\frac{\pi\alpha}{dD^2}}J_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi\alpha}{dD^2}}z\right)}{\left(\sqrt{\frac{\pi\alpha}{dD^2}}z\right)^{\frac{k}{2}-1}}=\sum_{n=1}^{\infty}\frac{\tilde{b}_{\bar{f}}(n)}{n^{\frac{k+1}{2}}}\frac{e^{-\frac{\pi\alpha}{n}}J_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi\alpha}{n}}z\right)}{\left(\sqrt{\frac{\pi\alpha}{n}}z\right)^{\frac{k}{2}-1}}. \tag{3.3}$$

Invoking Corollary 2.2 with $a=1$, $b=z$, $\nu=\frac{k}{2}$ and $x=\frac{\pi\alpha}{n}$, we have

$$\sum_{\substack{d,D=1\\(d,D)=1\\D\text{ squarefree}}}^{\infty}\frac{\mu(d)\chi(D)\tilde{a}_{\bar{f}}(d)}{d^{\frac{k+1}{2}}D^{k+1}}\frac{e^{-\frac{\pi\alpha}{dD^2}}}{\left(\sqrt{\frac{\pi\alpha}{dD^2}}z\right)^{\frac{k}{2}-1}}J_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi\alpha}{dD^2}}z\right)$$
$$=\frac{e^{-\frac{z^2}{4}}}{2\pi i2^{\frac{k}{2}-1}\Gamma(\frac{k}{2})}\sum_{n=1}^{\infty}\frac{\tilde{b}_{\bar{f}}(n)}{n^{\frac{k+1}{2}}}\int_{c-i\infty}^{c+i\infty}\Gamma(s){}_1F_1\left(\frac{k}{2}-s;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{\pi\alpha}{n}\right)^{-s}ds \tag{3.4}$$

where $c>0$. Assume $0<c<\frac{k-1}{2}$. Employing the change of variable $w=s-\frac{(k-1)}{2}$ in the integral on the right-hand side of (3.4), interchanging the order of summation and integration, representing the Dirichlet series inside the integral by $F(\bar{f},1-w)^{-1}$ where $-\frac{(k-1)}{2}<\lambda=$ Re $w<0$, and then using (1.8), we find that

$$\sum_{n=1}^{\infty}\frac{\tilde{b}_{\bar{f}}(n)}{n^{\frac{k+1}{2}}}\int_{c-i\infty}^{c+i\infty}\Gamma(s){}_1F_1\left(\frac{k}{2}-s;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{\pi\alpha}{n}\right)^{-s}ds$$
$$=(\pi\alpha)^{-\frac{(k-1)}{2}}\int_{\lambda-i\infty}^{\lambda+i\infty}\sum_{n=1}^{\infty}\frac{\tilde{b}_{\bar{f}}(n)}{n}\Gamma\left(w+\frac{k-1}{2}\right){}_1F_1\left(\frac{1}{2}-w;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{\pi\alpha}{n}\right)^{-w}dw$$
$$=(\pi\alpha)^{-\frac{(k-1)}{2}}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma(w+\frac{k-1}{2})}{F(\bar{f},1-w)}{}_1F_1\left(\frac{1}{2}-w;\frac{k}{2};\frac{z^2}{4}\right)(\pi\alpha)^{-w}\,dw.$$

$$= \frac{i^k \bar{\eta}\sqrt{q}}{2\pi}(\pi\alpha)^{-\frac{(k-1)}{2}} \int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\Gamma\left(\frac{k+1}{2}-w\right)}{F(f,w)} {}_1F_1\left(\frac{1}{2}-w;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{q\alpha}{4\pi}\right)^{-w} dw \tag{3.5}$$

In order to represent $F(f,w)^{-1}$ as a Dirichlet series, we shift the line of integration from Re $w=\lambda$ to Re $w=\lambda'$, $\lambda'>1$. Let $1<\lambda'<\frac{k+1}{2}$. Consider a positively oriented rectangular contour with sides $[\lambda-iT,\lambda',-iT],[\lambda',-iT,\lambda',+iT],[\lambda',+iT,\lambda+iT]$ and $[\lambda+iT,\lambda-iT]$, where T is any positive real number. Let $R_g(a)$ denote the residue of the function $g(w):=\frac{\Gamma\left(\frac{k+1}{2}-w\right)}{F(f,w)}{}_1F_1\left(\frac{1}{2}-w;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{q\alpha}{4\pi}\right)^{-w}$ at $w=a$. Since the first simple pole of $\Gamma\left(\frac{k+1}{2}-w\right)$ occurs at $w=\frac{k+1}{2}$, which is to the right of the line Re $w=\lambda'$, and $F(f,w)^{-1}$ has poles at the non trivial zeros ρ of $F(f,w)$ in the critical strip $0<\lambda<1$, invoking the residue theorem, we get

$$\begin{aligned}
&\int_{\lambda-iT}^{\lambda+iT} \frac{\Gamma\left(\frac{k+1}{2}-w\right)}{F(f,w)} {}_1F_1\left(\frac{1}{2}-w;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{q\alpha}{4\pi}\right)^{-w} dw \\
&= \left[\int_{\lambda-iT}^{\lambda'-iT} + \int_{\lambda'-iT}^{\lambda'+iT} + \int_{\lambda'+iT}^{\lambda+iT}\right] \frac{\Gamma\left(\frac{k+1}{2}-w\right)}{F(f,w)} {}_1F_1\left(\frac{1}{2}-w;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{q\alpha}{4\pi}\right)^{-w} dw \\
&\quad - 2\pi i \sum_{|\mathrm{Im}\rho|<T} R_g(\rho),
\end{aligned} \tag{3.6}$$

where

$$R_g(\rho) = \frac{1}{(m_\rho-1)!}\frac{d^{m_\rho-1}}{ds^{m_\rho-1}}(s-\rho)^{m_\rho}\frac{\Gamma\left(\frac{k+1}{2}-s\right)}{F(f,s)}{}_1F_1\left(\frac{1}{2}-s;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{4\pi}{q\alpha}\right)^s\Bigg|_{s=\rho}. \tag{3.7}$$

By Stirling's formula in a vertical strip, for $\alpha\le\sigma\le\beta$, $s=\sigma+it$,

$$|\Gamma(s)| = (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|}\left(1+O\left(\frac{1}{|t|}\right)\right) \tag{3.8}$$

uniformly as $|t|\to\infty$. Using an approach similar to that in [34, Section 9.8, p. 219] and (3.8), it can be seen that the integrals along the horizontal segments $[\lambda-iT,\lambda-iT]$ and $[\lambda+iT,\lambda+iT]$ tend to zero as $T\to\infty$ through values such that $|T-\gamma|>\exp\left(-A_1\gamma/\log\gamma\right)$. For formulas necessary for showing that the integrals along the horizontal segments indeed go to zero, the reader is referred to [15, pp. 94–102]. Now letting $T\to\infty$ in (3.6), representing $F(f,w)^{-1}$ by its Dirichlet series $\sum_{n=1}^{\infty}\tilde{b}_f(n)n^{-w}$ and then interchanging the order of summation and integration, we see that

$$\begin{aligned}
&\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\Gamma\left(\frac{k+1}{2}-w\right)}{F(f,w)} {}_1F_1\left(\frac{1}{2}-w;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{q\alpha}{4\pi}\right)^{-w} dw \\
&= \sum_{n=1}^{\infty}\tilde{b}_f(n)\int_{\lambda'-i\infty}^{\lambda'+i\infty}\Gamma\left(\frac{k+1}{2}-w\right){}_1F_1\left(\frac{1}{2}-w;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{q\alpha n}{4\pi}\right)^{-w} dw - 2\pi i\sum_\rho R_g(\rho) \\
&= e^{\frac{z^2}{4}}\left(\frac{q\alpha}{4\pi}\right)^{-\frac{(k+1)}{2}}\sum_{n=1}^{\infty}\frac{\tilde{b}_f(n)}{n^{\frac{k+1}{2}}}\int_{c-i\infty}^{c+i\infty}\Gamma(s){}_1F_1\left(k-s;\frac{k}{2};-\frac{z^2}{4}\right)\left(\frac{q\alpha n}{4\pi}\right)^s ds - 2\pi i\sum_\rho R_g(\rho),
\end{aligned} \tag{3.9}$$

where in the last step, we again made a change of the variable $s=\frac{k+1}{2}-w$, so that $0<c=\text{Re }s<\frac{k-1}{2}$, and then applied (1.10). Invoking Corollary 2.3 on the extreme right-hand side of (3.9) and making use of the fact that $\alpha\beta = 4/q$, we find that

$$\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma\left(\frac{k+1}{2}-w\right)}{F(f,w)}{}_1F_1\left(\frac{1}{2}-w;\frac{k}{2};\frac{z^2}{4}\right)\left(\frac{q\alpha}{4\pi}\right)^{-w}dw$$
$$=2\pi i(\pi\beta)^{\frac{k+1}{2}}2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)\sum_{n=1}^{\infty}\frac{\tilde{b}_f(n)}{n^{\frac{k+1}{2}}}\frac{e^{-\frac{\pi\beta}{n}}}{\left(\sqrt{\frac{\pi\beta}{n}}z\right)^{\frac{k}{2}-1}}\sum_{m=0}^{k/2}\binom{k/2}{m}I_{m+\frac{k}{2}-1}\left(\sqrt{\frac{\pi\beta}{n}}z\right)\left(-\frac{z}{2\sqrt{\frac{\pi\beta}{n}}}\right)^m$$
$$-2\pi i\sum_{\rho}R_g(\rho). \tag{3.10}$$

From (3.4), (3.5) and (3.10), we obtain upon simplification

$$\sum_{\substack{d,D=1\\(d,D)=1\\D\text{ squarefree}}}^{\infty}\frac{\mu(d)\chi(D)\tilde{a}_{\bar{f}}(d)}{d^{\frac{k+1}{2}}D^{k+1}}\frac{e^{-\frac{\pi\alpha}{dD^2}}}{(\sqrt{\frac{\pi\alpha}{dD^2}}z)^{\frac{k}{2}-1}}J_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi\alpha}{dD^2}}z\right)$$
$$=i^k\bar{\eta}\beta^{\frac{k}{2}}\alpha^{-\frac{k}{2}}e^{-\frac{z^2}{4}}\sum_{n=1}^{\infty}\frac{\tilde{b}_f(n)}{n^{\frac{k+1}{2}}}\frac{e^{-\frac{\pi\beta}{n}-z^2/4}}{\left(z\sqrt{\frac{\pi\beta}{n}}\right)^{k/2-1}}\sum_{m=0}^{k/2}\binom{k/2}{m}I_{m+\frac{k}{2}-1}\left(z\sqrt{\frac{\pi\beta}{n}}\right)\left(-\frac{z}{2\sqrt{\frac{\pi\beta}{n}}}\right)^m$$
$$-\frac{i^k\bar{\eta}\alpha^{-\frac{k}{2}}\pi^{-\frac{(k+1)}{2}}e^{-\frac{z^2}{4}}}{\sqrt{\beta}2^{\frac{k}{2}-1}\Gamma(\frac{k}{2})}\sum_{\rho}R_g(\rho). \tag{3.11}$$

Multiplying both sides by $\alpha^{\frac{k}{2}}e^{\frac{z^2}{8}}$ and simplifying, we arrive at (1.14). This completes the proof of Theorem 1.1.

4. A Riesz-type criterion for Hecke L-functions

As we remarked before, Theorem 1.1 leads one to consider the function $Q(\alpha, f, z)$ defined in (1.17). With the assumptions on bracketing as described in Theorem 1.1, this theorem can be rephrased in the following form:

$$\alpha^{\frac{k}{2}}e^{\frac{z^2}{8}}\sum_{n=1}^{\infty}\frac{\tilde{b}_{\bar{f}}(n)}{n^{\frac{k+1}{2}}}\frac{e^{-\frac{\pi\alpha}{n}}}{(\sqrt{\frac{\pi\alpha}{n}}z)^{\frac{k}{2}-1}}J_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi\alpha}{n}}z\right)$$
$$-i^k\bar{\eta}\beta^{\frac{k}{2}}e^{-\frac{z^2}{8}}\sum_{n=1}^{\infty}\frac{\tilde{b}_f(n)}{n^{\frac{k+1}{2}}}\frac{e^{-\frac{\pi\beta}{n}}}{(\sqrt{\frac{\pi\beta}{n}}z)^{\frac{k}{2}-1}}\sum_{m=0}^{\frac{k}{2}}\binom{\frac{k}{2}}{m}I_{m+\frac{k}{2}-1}\left(\sqrt{\frac{\pi\beta}{n}}z\right)\left(\frac{4\pi\beta}{nz^2}\right)^{-\frac{m}{2}}$$
$$=-\frac{i^k\bar{\eta}\pi^{-\frac{(k+1)}{2}}e^{\frac{-z^2}{8}}}{\sqrt{\beta}2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\sum_{\rho}\frac{1}{(m_\rho-1)!}\frac{d^{m_\rho-1}}{ds^{m_\rho-1}}(s-\rho)^{m_\rho}\frac{\Gamma\left(\frac{k+1}{2}-s\right)}{F(f,s)}{}_1F_1\left(\frac{1}{2}-s;\frac{k}{2};\frac{z^2}{4}\right)(\pi\beta)^s\Bigg|_{s=\rho}, \tag{4.1}$$

where $\tilde{b}_f(n)$ is defined in (3.2). Assuming the Riemann Hypothesis for $F(f,s)$, assuming all non-trivial zeros are simple, and assuming the absolute convergence of the series $\sum_\rho \frac{\pi^\rho\Gamma\left(\frac{k+1}{2}-\rho\right)}{F'(f,\rho)}{}_1F_1\left(\frac{1}{2}-\rho;\frac{k}{2};\frac{z^2}{4}\right)$ over the non-trivial zeros ρ, and then letting $\beta\to 0$ (or equivalently, $\alpha \to \infty$, since $\alpha\beta = 4/q$) in (4.1), we see that

$$\sum_{n=1}^{\infty} \frac{\tilde{b}_{\bar{f}}(n)}{n^{\frac{k+1}{2}}} \frac{e^{-\frac{\pi\alpha}{n}}}{\left(\sqrt{\frac{\pi\alpha}{n}}z\right)^{\frac{k}{2}-1}} J_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi\alpha}{n}}z\right) = O_{f,z}(\alpha^{-k/2}), \tag{4.2}$$

since the second sum on the left-hand side goes to zero as $\beta \to 0$. Define

$$P(\alpha, f, z) := \sum_{n=1}^{\infty} \frac{\tilde{b}_f(n)}{n^{\frac{k+1}{2}}} \left(\frac{e^{-\frac{\pi\alpha}{n}} J_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi\alpha}{n}}z\right)}{\left(\sqrt{\frac{\pi\alpha}{n}}z\right)^{\frac{k}{2}-1}} - \frac{1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} \right). \tag{4.3}$$

By (1.11), the definition of ${}_0F_1$, and interchanging the order of summation, we have

$$P(\alpha, f, z) = \frac{1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} \sum_{\substack{m,t=0 \\ (m,t)\neq(0,0)}}^{\infty} \frac{(-\pi\alpha)^{m+t}(z^2/4)^t}{m!t!\left(\frac{k}{2}\right)_t F\left(f, m+t+\frac{k+1}{2}\right)}, \tag{4.4}$$

where the interchange can be justified by first letting the upper limits of the infinite sums be finite and then showing that the tail goes to zero as they go to infinity. Now from (1.17), one can see that $Q(\alpha, f, z) = \frac{\partial}{\partial\alpha}P(\alpha, f, z)$. Note that $P(\alpha, f, z)$, and hence $Q(\alpha, f, z)$, are entire functions of α.

Now we show that the Riemann Hypothesis for $F(f, s)$ implies the bound $Q(\alpha, f, z) = O_{\epsilon,f,z}(\alpha^{-k/2-1+\epsilon})$ as $\alpha \to \infty$. Since $Q(\alpha, f, z)$ is an even function of z, we restrict ourselves to Re $z \geq 0$. From [15, p. 114], we know that the Riemann Hypothesis for $F(f, s)$ implies $M(x) := \sum_{n\leq x} \tilde{b}_f(n) = O_{f,\epsilon}(x^{\frac{1}{2}+\epsilon})$, for all $\epsilon > 0$. By partial summation,

$$M(\nu, n) := \sum_{m=\nu}^{n} \frac{\tilde{b}_f(m)}{m^{\frac{k+3}{2}}} = O_{f,\epsilon}(\nu^{-\frac{k}{2}-1+\epsilon}). \tag{4.5}$$

for all $\epsilon > 0$ and uniformly on n. From (4.3), we see that

$$\begin{aligned} Q(f, \alpha, z) &= -\frac{\pi}{2k} \sum_{n=1}^{\infty} \frac{\tilde{b}_f(n)}{n^{\frac{k+3}{2}}} e^{\frac{-\pi\alpha}{n}} \left(z^2 {}_0F_1\left(-; 1+\frac{k}{2}; \frac{-\pi\alpha z^2}{4n}\right) + 2k {}_0F_1\left(-; \frac{k}{2}; \frac{-\pi\alpha z^2}{4n}\right)\right) \\ &=: Q_1 + Q_2, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} Q_1 := Q_1(\alpha, f, z) &= -\frac{\pi}{2k} \sum_{n=1}^{\nu-1} \frac{\tilde{b}_f(n)}{n^{\frac{k+3}{2}}} e^{\frac{-\pi\alpha}{n}} \left(z^2 {}_0F_1\left(-; 1+\frac{k}{2}; \frac{-\pi\alpha z^2}{4n}\right) + 2k {}_0F_1\left(-; \frac{k}{2}; \frac{-\pi\alpha z^2}{4n}\right)\right), \\ Q_2 := Q_2(\alpha, f, z) &= -\frac{\pi}{2k} \sum_{n=\nu}^{\infty} \frac{\tilde{b}_f(n)}{n^{\frac{k+3}{2}}} e^{\frac{-\pi\alpha}{n}} \left(z^2 {}_0F_1\left(-; 1+\frac{k}{2}; \frac{-\pi\alpha z^2}{4n}\right) + 2k {}_0F_1\left(-; \frac{k}{2}; \frac{-\pi\alpha z^2}{4n}\right)\right), \end{aligned} \tag{4.7}$$

and $\nu = [\alpha^{1-\epsilon}]$. From [35, p. 199], we have

$$J_\nu(w) = \left(\frac{2}{\pi w}\right)^{1/2} \cos\left(w - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\right)\left(1 + O(|w|^{-1})\right), \tag{4.8}$$

for $|w|$ large, provided that $|\arg w| < \pi$. Combining (4.8) with (1.11), we obtain

$$\begin{aligned} {}_0F_1\left(-; \nu+1; -\frac{w^2}{4}\right) &= \frac{2^{\nu+1/2}\Gamma(\nu+1)}{\sqrt{\pi}w^{\nu+1/2}} \cos\left(w - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\right)\left(1 + O(|w|^{-1})\right) \\ &= O_\nu\left(w^{-\nu-1/2}e^{|w|}\right). \end{aligned} \tag{4.9}$$

Consider the difference operator $\triangle$ given by, $\triangle(h(n)) := h(n) - h(n+1)$. Note that

$$\begin{aligned}
Q_2 &= -\frac{\pi}{2k}\sum_{n=\nu}^{\infty} M(\nu,n) \triangle \left(e^{\frac{-\pi\alpha}{n}}\left(z^2{}_0F_1\left(-;1+\frac{k}{2};\frac{-\pi\alpha z^2}{4n}\right)+2k{}_0F_1\left(-;\frac{k}{2};\frac{-\pi\alpha z^2}{4n}\right)\right)\right),\\
&= \frac{\pi}{2k}\sum_{n=\nu}^{\infty} M(\nu,n)\frac{\pi\alpha e^{\frac{-\pi\alpha}{\lambda_n}}}{\lambda_n^2}\Bigg(4(2+k)z^2{}_0F_1\left(-;1+\frac{k}{2};\frac{-\pi\alpha z^2}{4\lambda_n}\right)\\
&\qquad + 4k(2+k){}_0F_1\left(-;\frac{k}{2};\frac{-\pi\alpha z^2}{4\lambda_n}\right)+z^4{}_0F_1\left(-;2+\frac{k}{2};\frac{-\pi\alpha z^2}{4\lambda_n}\right)\Bigg),\\
&= \frac{\pi}{2k}\left[\sum_{\nu\leq n\leq c\alpha}+\sum_{n>c\alpha}\right] M(\nu,n)\frac{\pi\alpha e^{\frac{-\pi\alpha}{\lambda_n}}}{\lambda_n^2}\Bigg(4(2+k)z^2{}_0F_1\left(-;1+\frac{k}{2};\frac{-\pi\alpha z^2}{4\lambda_n}\right)\\
&\qquad + 4k(2+k){}_0F_1\left(-;\frac{k}{2};\frac{-\pi\alpha z^2}{4\lambda_n}\right)+z^4{}_0F_1\left(-;2+\frac{k}{2};\frac{-\pi\alpha z^2}{4\lambda_n}\right)\Bigg),\\
&=: Q_3(\alpha,f,z)+Q_4(\alpha,f,z), \qquad (4.10)
\end{aligned}$$

say, where in the first step, the existence of the above $n < \lambda_n < n+1$ is assured by the mean value theorem. Also we choose $c > 0$ in the above computations to be a small constant depending only on z. Then by (4.5) and (4.9), for a large α, we have

$$\begin{aligned}
Q_3 := Q_3(\alpha,f,z) &= O_{f,\epsilon,z}\left(\nu^{-\frac{k}{2}-1+\epsilon}\sum_{\nu\leq n<c\alpha}\frac{\alpha e^{\frac{-\pi\alpha}{\lambda_n}+\sqrt{\frac{\pi\alpha}{\lambda_n}}|z|}}{\lambda_n^2\left(\frac{\pi\alpha}{\lambda_n}\right)^{\frac{k-1}{4}}}\right) = O_{f,\epsilon,z}\left(\nu^{-\frac{k}{2}-1+\epsilon}\sum_{\nu\leq n<c\alpha}\frac{\alpha}{n^2}\right)\\
&= O_{f,\epsilon,z}\left(\nu^{-\frac{k}{2}-2+\epsilon}\alpha\right)\\
&= O_{f,\epsilon,z}\left(\alpha^{-\frac{k}{2}-1+\delta}\right). \qquad (4.11)
\end{aligned}$$

Using the fact ${}_0F_1(-;\nu;w) = O\left(e^{|w|}\right)$ together with (4.5), we find that

$$Q_4 := Q_4(\alpha,f,z) = O_{f,\epsilon,z}\left(\nu^{-\frac{k}{2}-1+\epsilon}\sum_{n>c\alpha}\frac{\alpha}{n^2}\right) = O_{f,\epsilon,z}\left(\alpha^{-\frac{k}{2}-1+\delta}\right). \qquad (4.12)$$

By Deligne's bound and (4.9),

$$Q_1 = O_{f,\epsilon,z}\left(\sum_{n=1}^{\nu-1}\frac{n^\epsilon}{n^{\frac{k+3}{2}}}\frac{e^{-\frac{\pi\alpha}{n}+\sqrt{\frac{\pi\alpha}{n}}|z|}}{\left(\frac{\pi\alpha}{n}\right)^{\frac{k-1}{4}}}\right). \qquad (4.13)$$

For $\alpha > \left(\frac{\pi}{(\pi-1)^2}|z^2|\right)^{1/\epsilon}$, we conclude from (4.13) that

$$Q_1 = O_{f,\epsilon,z}\left(\nu^{1+\epsilon}e^{-\frac{\alpha}{\nu}}\right) = O_{f,\epsilon,z}\left(\alpha^{1-\epsilon^2}e^{-\alpha^\epsilon}\right). \qquad (4.14)$$

By (4.6), (4.11), (4.12) and (4.14) it follows that $Q(\alpha,f,z) = O_{\epsilon,f,z}(\alpha^{-k/2-1+\epsilon})$. This proves part (1) of Theorem 1.3.

Now we prove part (2) of Theorem 1.3. We need the following key lemma.

Lemma 4.1. *Let $Q(\alpha, f, z)$ be defined as in* (1.17)*. For all s with $\frac{1-k}{2} < Re\ s < 1$,*

$$\int_0^\infty \alpha^{-s} Q(\alpha, f, z)\, d\alpha = -\frac{\pi^s \Gamma(1-s)\, {}_1F_1\left(-s; \frac{k}{2}; -\frac{z^2}{4}\right)}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right) F\left(f, s+\frac{k+1}{2}\right)}. \tag{4.15}$$

Proof. First we prove the result for $0 < \text{Re } s < 1$. Then we extend it to $\frac{1-k}{2} < \text{Re } s < 1$ by using analytic continuation. Let $0 < \text{Re } s < 1$ and

$$\phi(s, f, z) := \int_0^\infty \alpha^{-s-1} P(\alpha, f, z)\, d\alpha.$$

Writing $\alpha = x/n$ and multiplying both sides by $\tilde{a}_f(n) n^{-\frac{(k+1)}{2}}$, we obtain

$$\tilde{a}_f(n) n^{-s-\frac{(k+1)}{2}} \phi(s, f, z) = \int_0^\infty x^{-s-1} \frac{\tilde{a}_f(n)}{n^{\frac{k+1}{2}}} P\left(\frac{x}{n}, f, z\right) dx. \tag{4.16}$$

Now we sum over n from 1 to ∞ and invert the order of summation and integration. We justify this interchange below. By using Deligne's bound for $\tilde{b}_f(n)$, for all $\epsilon > 0$,

$$M(x) = \sum_{n \leq x} \tilde{b}_f(n) = O_{f,\epsilon}(x^{1+\epsilon}), \tag{4.17}$$

unconditionally. By partial summation,

$$M^*(\nu, n) := \sum_{m=\nu}^{n} \frac{\tilde{b}_f(m)}{m^{\frac{k+1}{2}}} = O_{f,\epsilon}\left(\nu^{-\frac{k}{2}+\frac{1}{2}+\epsilon}\right), \tag{4.18}$$

uniformly on n and for all $\epsilon > 0$. We use (4.18) with $\nu = [\alpha^{1-\epsilon}]$ and carry out the same argument as in part (1) of Theorem 1.3 for $P(\alpha, f, z)$ instead of $Q(\alpha, f, z)$. We have

$$P(\alpha, f, z) = O_{f,\epsilon,z}\left(\alpha^{-\frac{k}{2}+\frac{1}{2}+\epsilon}\right). \tag{4.19}$$

Observe that using (4.19) and Weierstras-M test, $\sum_{n=1}^\infty \frac{\tilde{a}_f(n)}{n^{\frac{k+1}{2}}} P\left(\frac{x}{n}, f, z\right)$ is uniformly convergent on any compact subinterval of $(0, \infty)$. Next, we show that the integral

$$\int_0^\infty \sum_{n=1}^\infty \left| x^{-s-1} \frac{\tilde{a}_f(n)}{n^{\frac{k+1}{2}}} P\left(\frac{x}{n}, f, z\right) \right| dx \tag{4.20}$$

is finite. To see this, split the above integral into two, with limits from 0 to 1, and from 1 to ∞ respectively. For the first integral, observe that $\sum_{n=1}^\infty \left| \frac{\tilde{a}_f(n)}{n^{\frac{k+1}{2}}} P\left(\frac{x}{n}, f, z\right) \right| = O_f(x)$. Thus the integral from 0 to 1 is finite, since $\text{Re } s < 1$. Using the bound in (4.19) and the fact that $\text{Re } s > 0$, we see that the second integral is also finite. Thus, by Lebesgue's dominated convergence theorem [33, p. 30, Theorem 2.1], we see that

$$\sum_{n=1}^\infty \int_0^\infty x^{-s-1} \frac{\tilde{a}_f(n)}{n^{\frac{k+1}{2}}} P\left(\frac{x}{n}, f, z\right) dx = \int_0^\infty x^{-s-1} \sum_{n=1}^\infty \frac{\tilde{a}_f(n)}{n^{\frac{k+1}{2}}} P\left(\frac{x}{n}, f, z\right) dx. \tag{4.21}$$

From (4.16) and (4.21),

$$F\left(f, s+\frac{k+1}{2}\right) \phi(s, f, z) = \int_0^\infty x^{-s-1} \sum_{n=1}^\infty \frac{\tilde{a}_f(n)}{n^{\frac{k+1}{2}}} P\left(\frac{x}{n}, f, z\right) dx. \tag{4.22}$$

Using (4.4) and then interchanging the order of summation, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\tilde{a}_f(n)}{n^{\frac{k+1}{2}}}P\left(\frac{x}{n},f,z\right)&=\frac{1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\sum_{\substack{m,t=0\\(m,t)\neq(0,0)}}^{\infty}\frac{(-\pi x)^{m+t}(z^2/4)^t}{m!t!\left(\frac{k}{2}\right)_t}\\&=\frac{1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\left(\sum_{m=1}^{\infty}\frac{(-\pi x)^m}{m!}+\sum_{t=1}^{\infty}\frac{(-\pi x)^t(z^2/4)^t}{t!\left(\frac{k}{2}\right)_t}+\sum_{m,t=1}^{\infty}\frac{(-\pi x)^{m+t}(z^2/4)^t}{m!t!\left(\frac{k}{2}\right)_t}\right)\\&=\frac{e^{-\pi x}{}_0F_1\left(-;\frac{k}{2};\frac{-\pi xz^2}{4}\right)-1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}.\end{aligned}\tag{4.23}$$

Substituting this in (4.22), we have

$$F\left(f,s+\frac{k+1}{2}\right)\phi(s,f,z)=\frac{1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\int_0^{\infty}x^{-s-1}\left(e^{-\pi x}{}_0F_1\left(-;\frac{k}{2};\frac{-\pi xz^2}{4}\right)-1\right)dx.\tag{4.24}$$

By an integration by parts, for $0<\text{Re } s<1$,

$$\int_0^{\infty}x^{-s-1}\left(e^{-\pi x}{}_0F_1\left(-;\frac{k}{2};\frac{-\pi xz^2}{4}\right)-1\right)dx=\frac{1}{s}\int_0^{\infty}x^{-s}\frac{d}{dx}\left(e^{-\pi x}{}_0F_1\left(-;\frac{k}{2};\frac{-\pi xz^2}{4}\right)-1\right)dx,\tag{4.25}$$

since $\lim_{x\to\infty}e^{-\pi x}{}_0F_1\left(-;\frac{k}{2};\frac{-\pi xz^2}{4}\right)=0$. Next, since $\text{Re } s<1$,

$$\begin{aligned}&\int_0^{\infty}x^{-s}\frac{d}{dx}\left(e^{-\pi x}{}_0F_1\left(-;\frac{k}{2};\frac{-\pi xz^2}{4}\right)-1\right)\\&=-\pi\int_0^{\infty}x^{-s}e^{-\pi x}\left(\frac{z^2}{2k}{}_0F_1\left(-;\frac{k}{2}+1;\frac{-\pi xz^2}{4}\right)+{}_0F_1\left(-;\frac{k}{2};\frac{-\pi xz^2}{4}\right)\right)dx\\&=-\pi\left(\frac{z^2}{2k}\sum_{n=0}^{\infty}\frac{(-\pi z^2/4)^n}{n!\left(\frac{k}{2}+1\right)_n}\int_0^{\infty}x^{n-s}e^{-\pi x}\,dx+\sum_{n=0}^{\infty}\frac{(-\pi z^2/4)^n}{n!\left(\frac{k}{2}\right)_n}\int_0^{\infty}x^{n-s}e^{-\pi x}\,dx\right)\\&=-\pi^s\left(\frac{z^2}{2k}\sum_{n=0}^{\infty}\frac{(-z^2/4)^n\Gamma(1+n-s)}{n!\left(\frac{k}{2}+1\right)_n}+\sum_{n=0}^{\infty}\frac{(-z^2/4)^n\Gamma(1+n-s)}{n!\left(\frac{k}{2}\right)_n}\right)\\&=-\pi^s\Gamma(1-s)\left(\frac{z^2}{2k}\sum_{n=0}^{\infty}\frac{(1-s)_n(-z^2/4)^n}{n!\left(\frac{k}{2}+1\right)_n}+\sum_{n=0}^{\infty}\frac{(1-s)_n(-z^2/4)^n}{n!\left(\frac{k}{2}\right)_n}\right)\\&=-\pi^s\Gamma(1-s){}_1F_1\left(-s;\frac{k}{2};\frac{-z^2}{4}\right).\end{aligned}\tag{4.26}$$

From (4.24), (4.25) and (4.26), we arrive at

$$\phi(s,f,z)=\frac{\pi^s\Gamma(-s){}_1F_1\left(-s;\frac{k}{2};-\frac{z^2}{4}\right)}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)F\left(f,s+\frac{k+1}{2}\right)}.\tag{4.27}$$

Finally, upon an integration by parts, we see that, for $0<\text{Re } s<1$,

$$\phi(s,f,z)=\frac{1}{s}\int_0^{\infty}\alpha^{-s}Q(\alpha,f,z)\,d\alpha.\tag{4.28}$$

Hence from (4.27) and (4.28), we conclude that, for $0 < \text{Re } s < 1$,

$$\int_0^\infty \alpha^{-s} Q(\alpha, f, z)\, d\alpha = -\frac{\pi^s \Gamma(1-s) {}_1F_1\left(-s; \frac{k}{2}; -\frac{z^2}{4}\right)}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right) F\left(f, s+\frac{k+1}{2}\right)}. \tag{4.29}$$

Since ${}_1F_1\left(-s; \frac{k}{2}; -\frac{z^2}{4}\right)$ is an entire function of s, $\Gamma(1-s)$ is an analytic function for $\text{Re } s < 1$, and $F\left(f, s+\frac{k+1}{2}\right)$ is a nonvanishing analytic function for $\frac{1-k}{2} < \text{Re } s < 1$, we see that the right hand side of (4.29) is an analytic function for $\frac{1-k}{2} < \text{Re } s < 1$.

We now show that the left hand side of (4.29) is also analytic in the above range. In the proof of part (1) of Theorem 1.3, if we do not assume the Riemann Hypothesis for $F(f, s)$, by using (4.17), we get

$$M(\nu, n) = \sum_{m=\nu}^{n} \frac{\tilde{b}_f(m)}{m^{\frac{k+3}{2}}} = O_{f,\epsilon}\left(\nu^{-\frac{k}{2}-\frac{1}{2}+\epsilon}\right), \tag{4.30}$$

uniformly on n, for all $\epsilon > 0$. By the same argument, and using (4.30), we get

$$Q(\alpha, f, z) = O_{f,\epsilon,z}\left(\alpha^{-\frac{k}{2}-\frac{1}{2}+\epsilon}\right). \tag{4.31}$$

Split the left hand side of (4.29) into two integrals, from 0 to 1 and from 1 to ∞ respectively. For the first integral, observe that $Q(\alpha, f, z) = O_f(1)$. Since $\text{Re } s < 1$, this integral is finite. Using the bound in (4.31), and the fact that $\text{Re } s > \frac{1-k}{2}$, we see that the second integral is also finite. Hence the integral $\int_0^\infty \alpha^{-s} Q(\alpha, f, z)\, d\alpha$ is finite and analytic in $\frac{1-k}{2} < \text{Re } s < 1$. By analytic continuation, we conclude that (4.29) holds for $\frac{1-k}{2} < \text{Re } s < 1$. This proves the lemma. □

Lemma 4.1 implies that

$$F\left(f, s+\frac{k+1}{2}\right) \int_0^\infty \alpha^{-s} Q(\alpha, f, z)\, d\alpha = -\frac{\pi^s \Gamma(1-s) {}_1F_1\left(-s; \frac{k}{2}; -\frac{z^2}{4}\right)}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}. \tag{4.32}$$

We now show that if we assume $Q(\alpha, f, z) = O_{\epsilon,f,z}(\alpha^{-k/2-1+\epsilon})$ for $\alpha \geq \alpha_0$, then (4.32) holds for $-\frac{k}{2} < \text{Re } s \leq \frac{1-k}{2}$ as well. Since $F\left(f, s+\frac{k+1}{2}\right)$ is an entire function of s, it is analytic in $-\frac{k}{2} < \text{Re } s \leq \frac{1-k}{2}$. Also,

$$\begin{aligned}\int_0^\infty \alpha^{-s} Q(\alpha, f, z)\, d\alpha &= \int_0^{\alpha_0} \alpha^{-s} Q(\alpha, f, z)\, d\alpha + \int_{\alpha_0}^\infty \alpha^{-s} Q(\alpha, f, z)\, d\alpha \\ &= I_1(s, f, z) + I_2(s, f, z),\end{aligned} \tag{4.33}$$

say. Since $Q(\alpha, f, z)$ is an entire function of α, $I_1(s, f, z) = O_{\epsilon,f,z}(1)$. Since by assumption, $Q(\alpha, f, z) = O_{\epsilon,f}(\alpha^{-k/2-1+\epsilon})$, we have

$$I_2(s, f, z) = O_{\epsilon,f,z}\left(\int_{\alpha_0}^\infty \alpha^{-s-\frac{k}{2}-1+\epsilon}\, d\alpha\right) = O_{\epsilon,f,z}(1).$$

This shows that $\int_0^\infty \alpha^{-s} Q(\alpha, f, z)\, d\alpha$ is analytic in $-\frac{k}{2} < \text{Re } s \leq \frac{1-k}{2}$, so that the left-hand side of (4.32) is analytic in $-\frac{k}{2} < \text{Re } s \leq \frac{1-k}{2}$. Since $\pi^s {}_1F_1\left(-s; \frac{k}{2}; -\frac{z^2}{4}\right)$ is an entire function of s and $\Gamma(1-s)$ is also analytic in $-\frac{k}{2} < \text{Re } s \leq \frac{1-k}{2}$, we conclude that (4.32) also holds in $-\frac{k}{2} < \text{Re } s \leq \frac{1-k}{2}$. Now $\Gamma(1-s)$ has no zeros; hence the only zeros of the right-hand side of (4.32) are those of ${}_1F_1\left(-s; \frac{k}{2}; -\frac{z^2}{4}\right)$. If $z = 0$, then ${}_1F_1\left(-s; \frac{k}{2}; -\frac{z^2}{4}\right) = 1$, and hence in this special case, the left-hand side of (4.32) does not have any zeros in $-\frac{k}{2} < \text{Re } s < 1$. This implies the Riemann Hypothesis for $F(f, s)$. Let us now consider the case when $z \neq 0$. For $|\lambda| \to \infty$ and $|\arg(\lambda z)| < 2\pi$, the following estimate for $M_{\lambda,\mu}(z)$, where $M_{\lambda,\mu}(z)$ is defined in (2.4), holds [21, p. 318]:

$$M_{\lambda,\mu}(z) = \pi^{-1/2} z^{1/4} \lambda^{-\mu-1/4} \Gamma(2\mu+1) \cos\left(2\sqrt{\lambda z} - \frac{\pi}{4} - \mu\pi\right) + O\left(|\lambda|^{-\mu-3/4}\right). \tag{4.34}$$

From (2.4) and (4.34), as $|\lambda| \to \infty$,

$${}_1F_1\left(\mu - \lambda + \frac{1}{2}; 2\mu+1; z\right) = \pi^{-1/2} (\lambda z)^{-\mu-1/4} e^{z/2} \Gamma(2\mu+1) \cos\left(2\sqrt{\lambda z} - \frac{\pi}{4} - \mu\pi\right) + O_{z,\mu}\left(|\lambda|^{-\mu-3/4}\right). \tag{4.35}$$

This gives for $|s| \to \infty$,

$$\begin{aligned} {}_1F_1\left(-s; \frac{k}{2}; -\frac{z^2}{4}\right) &= \frac{\pi^{-\frac{1}{2}} e^{-\frac{z^2}{8}}}{\left(\frac{-z^2}{4}\left(s + \frac{k}{4}\right)\right)^{\frac{k-1}{4}}} \Gamma\left(\frac{k}{2}\right) \cos\left(\sqrt{-z^2\left(s + \frac{k}{4}\right)} - \frac{k\pi}{4} + \frac{\pi}{4}\right) \\ &\quad + O_{z,k}\left(\left|s + \frac{k}{4}\right|^{-\frac{k}{4}-\frac{1}{4}}\right). \end{aligned} \tag{4.36}$$

For $s = \sigma + it$, since $-\frac{k}{2} < \sigma < 1$ and $|s| \to \infty$, we have $|t| \to \infty$. Since $z \neq 0$, this implies that the main term on the right-hand side of (4.36) tends to ∞ in absolute value as $|s| \to \infty$, implying that for t large enough, we have $\left|{}_1F_1\left(-s; \frac{k}{2}; -\frac{z^2}{4}\right)\right| > 0$, i.e., for a fixed non-zero z, there exists a number T_z such that for $t > T_z$, we have ${}_1F_1\left(-s; \frac{k}{2}; -\frac{z^2}{4}\right) \neq 0$. Hence, the left-hand side of (4.32) has zeros at most up to a fixed height T_z depending on z. But $F\left(f, s + \frac{k+1}{2}\right)$ has only finitely many zeros up to any fixed height T [15, p. 104, Theorem 5.8]. This proves Theorem 1.3.

Appendix

In this appendix we present three tables. Table 1 shows numerical evaluation of both sides of (1.14) for $f = \Delta$, the Ramanujan Delta function. Table 2 does the same but for $z = 0$, which is then compared with the evaluation of (1.2) in Table 3 with regards to the rate of convergence. That the rate in (1.15) is faster than that in (1.2) is seen from Tables 2 and 3.

Table 1. Left and right-hand sides (using the first 100 zeros) of (1.14) with $f = \Delta$.

α	z	Left-hand side	Right-hand side (200 terms)
1	$2+4i$	$-0.000024958.. + 3.468281.. \times 10^{-6}i$	$-0.000024958.. + 3.46828.. \times 10^{-6}i$
2	$4+6i$	$0.0015894085.. + 0.0071204039i$	$0.00158941.. + 0.0071204..i$
3	10	$0.0003427702..$	$0.00034277.. - 1.5816..10^{-23}i$
4	-15	$789.45005..$	$789.45.. + 3.18962..10^{-15}i$
6	$-13i$	$-15701.52123..$	$-15701.5.. + 4.30767..10^{-16}i$
7	$1+i$	$-1.74653.. \times 10^{-7} + 1.24457.. \times 10^{-7}i$	$-1.74654.. \times 10^{-7} + 1.24457.. \times 10^{-7}i$
9	$2+i$	$1.37740.. \times 10^{-7} - 2.65521.. \times 10^{-7}i$	$1.37741.. \times 10^{-7} - 2.65521.. \times 10^{-7}i$
10	$10+20i$	$-2.2551.. \times 10^{15} - 3.84651.. \times 10^{15}i$	$-2.26.. \times 10^{15} - 3.86.. \times 10^{15}i$
11	$3+5i$	$0.000272321.. - 0.0003106155..i$	$0.000272322.. - 0.000310616..i$
15	$15+15i$	$4.80644.. \times 10^{17} + 9.70191.. \times 10^{18}i$	$4.80645.. \times 10^{17} + 9.70192.. \times 10^{18}i$

Table 2. Left and right-hand sides of (1.15).

	Left-hand side			Right-hand side	
$\alpha/\sqrt{\pi}$	250000 terms	1000000 terms	2250000 terms	50 zeros	100 zeros
1	0	0	0	0	0
2	0.000077992..	0.000077992..	0.000077992..	0.000077992..	0.000077992..
3	−0.000462863..	−0.000462863..	−0.000462863...	−0.000462864..	−0.000462864..
4	0.000153353..	0.000153353..	0.000153353..	0.000153354..	0.000153354..
30	−0.000034332..	−0.000034320..	−0.000034322..	−0.000034321..	−0.000034321..
40	0.000362260..	0.000362310..	0.000362303..	0.000362304..	0.000362304..

Table 3. Left and right-hand sides of (1.2).

	Left-hand side			Right-hand side	
$\alpha/\sqrt{\pi}$	250000 terms	1000000 terms	2250000 terms	50 zeros	100 zeros
1	0	0	0	0	0
2	−0.0001414884..	0.0001995784..	−0.0000881286..	0.00001072..	0.00001072..
3	−0.0002538673..	0.0003030926..	−0.0001667311..	−0.00000529..	−0.00000529..
4	−0.0003428087..	0.0003807035..	−0.0002296153..	−0.0000199..	−0.0000199..
30	−0.0011159031..	0.0014379270..	−0.0007163570..	0.00002387..	0.00002387..
40	−0.0013552796..	0.0016190509..	−0.0008899463..	−0.0000278..	−0.0000278..

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